STOCHASTIC HYBRID MODELS: AN OVERVIEW

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Abstract: An overview of Stochastic Hybrid Models developed in the literature is presented. Attention is concentrated on three classes of models: Piecewise Deterministic Markov Processes, Switching Diffusion Processes and Stochastic Hybrid Systems. The descriptive power of the three classes is compared and conditions under which the classes coincide are developed. The theoretical analysis is motivated by modelling problems in Air Traffic Management.

Keywords: Stochastic hybrid model, Piecewise Deterministic Markov Process, Switching Diffusion Process, Stochastic Hybrid System, Air Traffic Management.

1. INTRODUCTION

Deterministic hybrid systems have been the topic of intense research in recent years. By contrast, relatively few classes of stochastic hybrid processes have been studied in detail. Even though deterministic hybrid models can capture a wide range of behaviors encountered in practice, stochastic features are very important in modelling, because of the uncertainty inherent in most real world applications. Certain classes of stochastic hybrid models have been proposed in the literature. For example, Levy processes and jump processes that have been studied extensively in the stochastic processes literature (Ethier and Kurtz, 1986) can be considered as stochastic hybrid systems, even though the hybrid aspect of the dynamics is fairly weak. Closer to the framework developed in recent years for deterministic hybrid systems are the Piecewise Deterministic Markov Processes (PDMP) of (Davis, 1993), the Switched Diffusion Processes (SDP) of (Ghosh *et al.*, 1997) and the so called Stochastic Hybrid Systems (SHS) of (Hu *et al.*, 2000). Stochastic extensions of timed automata have been studied in (Kwiatkowska *et al.*, 1999; Baier *et al.*, 1999; Kwiatkowska *et al.*, 2000). The most important difference among the models lies in where the randomness is introduced. Some models allow diffusions to model continuous evolution (Ghosh *et al.*, 1997; Hu *et al.*, 2000), while others do not (Ethier and Kurtz, 1986; Davis, 1993). Likewise, some models force transitions to take place from certain states (e.g. (Hu *et al.*, 2000)), others only allow transitions to take place at random times (e.g. using a generalized Poisson process (Ghosh *et al.*, 1997)), while others allow both (Davis, 1993). This paper attempts to precisely characterize the similarities and differences of three classes of stochastic hybrid processes: PDMP, SDP and SHS. We start by giving a brief overview of the autonomous versions of the three classes (Section 2). We then proceed to characterize the differences in descriptive power between them (Section 3). The analysis is based on the stochastic properties of the models: their Markov properties, extended generators, etc. For PDMP and SDP these properties have been studied in detail in the literature (Davis, 1993), (Ghosh *et al.*, 1997). The Markov property for SHS was established in (Hu *et al.*, 2000). To complete the comparison, we also derive the expression of the infinitesimal generator for SHS. Our work is motivated by the necessity to develop an appropriate stochastic hybrid framework for modelling Air Traffic Management (ATM) systems. The ultimate goal of our work (under the European Commission's HYBRIDGE project) is to use theoretical tools developed for stochastic hybrid models as a basis for designing and analyzing advanced ATM concepts for the European airspace.

2. AUTONOMOUS STOCHASTIC HYBRID MODELS

In this section we review stochastic hybrid models introduced by Davis (Davis, 1993), Ghosh et. al. (Ghosh *et al.*, 1997) and Hu et. al. (Hu *et al.*, 2000). In order to study the similarities and differences among these models we use a common formalism inspired by the formalism developed for deterministic hybrid systems in (Lygeros *et al.*, 2003).

2.1 Notation

For a subset A, of a topological space X, 2^A denotes the power set (set of all subsets) of A, ∂A denotes the boundary of A in X. $\mathcal{B}(A)$ denotes the Borel σ-algebra of A (the smallest σ-algebra containing the open subsets of A). For a complete, separable metric space Y, $\mathcal{P}(Y)$ denotes the space of probability measures endowed with the topology of weak convergence. Let $\mathcal{M}(Y)$ be the set of all nonnegative, integer-valued, σ-finite measures on $\mathcal{B}(Y)$. Let $\mathcal{M}_\sigma(Y)$ be the smallest σ-field on $\mathcal{M}(Y)$ with respect to which all the maps from $h_B : \mathcal{M}(Y) \to \mathbb{N} \cup \{\infty\}$ with $B \in \mathcal{B}(Y)$ of the form $h_B(\mu) \to \mu(B)$ are measurable. Notice that $(\mathcal{M}(Y), \mathcal{M}_\sigma(Y))$ is a measurable space. $\mathcal{C}^k(\mathbb{R}^n, \mathbb{R}^m)$ denotes the class of functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ which are differentiable k times, with continuous k-order derivatives. Given a function $\theta \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ and a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$, we use $L_f\theta$ to denote the Lie derivative of θ along f, i.e. $L_f\theta(x) = \sum_{i=1}^n \frac{\partial\theta}{\partial x_i}(x) f_i(x)$. Given a function $f \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$, we use $\mathbb{H}^f$ to denote the Hamiltonian operator applied to f, i.e. $\mathbb{H}^f(x) = (h_{ij}(x))_{i,j=1\dots n} \in \mathbb{R}^{n\times n}$, where $h_{ij}(x) = \frac{\partial^2 f}{\partial x_i \partial x_j}(x)$. Given a matrix $A = (a_{ij})_{i,j=1\dots n} \in \mathbb{R}^{n\times m}$, A^T denotes the traspose matrix of A and $Tr(A)$ denotes its trace, i.e. $Tr(A) = \sum_{i=1}^n a_{ii}$.

2.2 Piecewise Deterministic Markov Process

Piecewise Deterministic Markov Processes (PDMP) involve a hybrid state space, with both continuous and discrete states. Randomness appears only in the discrete transitions; between two consecutive transitions the continuous state evolves according to a nonlinear ordinary differential equation. Transitions occur either when the state hits the state space boundary, or in the interior of the state space, according to a generalized Poisson process. Whenever a transition occurs, the hybrid state is reset instantaneously according to a probability distribution which depends on the hybrid state before the transition. We introduce formally PDPM following the notation of (Bujorianu and Lygeros, 2003; Lygeros *et al.*, 2003). Let Q be a countable set of discrete states, and let $d : Q \to \mathbb{N}$ and $X : Q \to \mathbb{R}^{d(.)}$ be two maps assigning to each discrete state $i \in Q$ a subset of $\mathbb{R}^{d(i)}$. We call the set $\mathcal{D} = \bigcup_{i\in Q} \{i\} \times X(i)$ the hybrid state space of the PDMP and $\alpha = (i, x) \in \mathcal{D}$ the hybrid state. We define the boundary of the hybrid state space as $\partial\mathcal{D} = \bigcup_{i\in Q} \{i\} \times \partial X(i)$. A vector field f on the hybrid state space $\mathcal{D}$ is a function $f : \mathcal{D} \to \mathbb{R}^{d(.)}$ assigning to each hybrid state $\alpha = (i, x)$ a direction $f(\alpha) \in \mathbb{R}^{d(i)}$. The flow of f is a function $\Phi : \mathcal{D} \times \mathbb{R} \to \mathcal{D}$ with $\Phi(\alpha, t) = \begin{bmatrix} \Phi_Q(\alpha, t) \\ \Phi_X(\alpha, t) \end{bmatrix}$, where $\Phi_Q(\alpha, t) \in Q$ and $\Phi_X(\alpha, t) \in X(i)$, such that for $\alpha = (i, x)$, $\Phi(\alpha, 0) = \alpha$ and for all $t \in \mathbb{R}$, $\Phi_Q(\alpha, t) = i$ and $\frac{d}{dt}\Phi_X(\alpha, t) = f(\Phi(\alpha, t))$. Let $\Gamma = \{\alpha \in \partial\mathcal{D} \mid \exists(\alpha', t) \in \mathcal{D} \times \mathbb{R}^+, \alpha = \Phi(\alpha', t)\}$; $\mathcal{D}^* = \mathcal{D} \cup \Gamma$. Define $\mathcal{B}(\mathcal{D}^\infty) = \sigma\left(\bigcup_{i\in Q} \{i\} \times \mathcal{B}(X(i))\right)$ where $\mathcal{D}^\infty = Q \times \mathbb{R}^\infty$. The space $(\mathcal{D}^\infty, \mathcal{B}(\mathcal{D}^\infty))$ is a Borel Space and $\mathcal{B}(\mathcal{D}^\infty)$ is a sub-σ-algebra of its Borel σ-algebra. We can now introduce the following definition.

Definition 1. (**PDMP**). A Piecewise Deterministic Markov Process is a collection $H = ((Q, d, X), f, Init, \lambda, R)$ where

- Q is a countable set of discrete variables;
- $d : Q \to \mathbb{N}$ is a map giving the dimensions of the continuous state spaces;
- $X : Q \to \mathbb{R}^{d(.)}$ maps each $i \in Q$ into a subset $X(i)$ of $\mathbb{R}^{d(i)}$;
- $f : \mathcal{D} \to \mathbb{R}^{d(.)}$ is a vector field;
- $Init : \mathcal{B}(\mathcal{D}^\infty) \to [0, 1]$ is an initial probability measure on $(\mathcal{D}^\infty, \mathcal{B}(\mathcal{D}^\infty))$,with $Init(\mathcal{D}^c) = 0$;
- $\lambda : \mathcal{D}^* \to \mathbb{R}^+$ is a transition rate function;
- $R : \mathcal{B}(\mathcal{D}^\infty) \times \mathcal{D}^* \to [0, 1]$ is a transition measure, with $R(\mathcal{D}^c, .) = 0$.

To ensure the process is well-defined, the following assumption is introduced in (Davis, 1993).

Assumption A: *The sets $X(i)$ are open. For all $i \in Q$, $f(i, .)$ is globally Lipschitz continuous.*

$\lambda : \mathcal{D}^* \to \mathbb{R}^+$ *is measurable. For all* $\alpha \in \mathcal{D}$ *there exists* $\varepsilon > 0$ *such that the function* $t \to \lambda(\Phi(\alpha, t))$ *is integrable for all* $t \in [0, \varepsilon)$. *For all* $A \in \mathcal{B}(\mathcal{D}^*)$, $R(A, \cdot)$ *is measurable.*

To define the PDMP executions we introduce the notion of exit time $t^* : \mathcal{D} \to \mathbb{R}^+ \cup \{\infty\}$, as $t^*(\alpha) = \inf\{t > 0 : \Phi(\alpha, t) \notin \mathcal{D}\}$ and of survivor function $F : \mathcal{D} \times \mathbb{R}^+ \to [0, 1]$, as $F(\alpha, t) = \exp\left(-\int_0^t \lambda(\Phi(\alpha, \tau))d\tau\right)$ if $t < t^*(\alpha)$ and $F(\alpha, t) = 0$ if $t \geq t^*(\alpha)$. The executions of the PDMP can be thought of as being generated by the following algorithm.

Algorithm 2.2 : (Generation of PDMP Executions)

set $T = 0$
select $\mathcal{D}$-valued random variable $\hat{\alpha}$ according to $Init$
repeat
select $\mathbb{R}^+$-valued random variable $\hat{T}$ such that $P(\hat{T} > t) = F(\hat{\alpha}, t)$
set $\alpha_t = \Phi(\hat{\alpha}, t - T)$ for all $t \in [T, T + \hat{T})$
select $\mathcal{D}$-valued random variable $\hat{\alpha}$ according to $R(., \Phi(\hat{\alpha}, \hat{T}))$
set $T = T + \hat{T}$
until true

All random extractions in Algorithm 2.2 are assumed to be independent. To ensure that α_t is defined on the entire $\mathbb{R}^+$ it is necessary to exclude Zeno executions (Lygeros *et al.*, 2003). The following assumption is introduced in (Davis, 1993) to accomplish this.

Assumption B: *Let* $N_t = \sum_i I_{(t \geq T_i)}$ *be the number of jumps in* $[0, t]$. *Then* $\mathbb{E}[N_t] < \infty$ *for all* t.

Under Assumptions A and B, Algorithm 2.2 defines a strong Markov process (Davis, 1993). The expression of the infinitesimal generator of this process is given by (see (Davis, 1993)),

$$L^{PDMP}\theta\ (\beta) = \mathcal{L}_f\theta(\beta) + \lambda(\beta)\int_{\mathcal{D}^*} R(d\alpha, \beta)\,(\theta(\alpha) - \theta(\beta))$$

where θ belongs to the domain of generator as defined in (Davis, 1993).

2.3 Switching Diffusion Process

Switching Diffusion Processes involve a hybrid state space, with both continuous and discrete states. The continuous state evolves according to a stochastic differential equation (SDE), while the discrete state is a controlled Markov chain. Both the dynamics of the SDE and the transition matrix of the Markov Chain depend on the hybrid state. The continuous hybrid state evolves without jumps, i.e. the evolution of the continuous state can be assumed to be a continuous function of time. In the following we introduce formally SDP following (Ghosh *et al.*, 1997). To allow a comparison with PDMP and SHS we restrict our attention to autonomous SDP.

Definition 2. (**SDP**). A Switching Diffusion Process is a collection $H = (Q, X, f, Init, \sigma, \lambda_{ij})$ where

- $Q = \{1, 2, ..., N\}$ is a finite set of discrete variables, $N \in \mathbb{N}$;
- $X = \mathbb{R}^n$ is the continuous state space;
- $f : Q \times X \to \mathbb{R}^n$ is a vector field;
- $Init : \mathcal{B}(Q \times X) \to [0, 1]$ is an initial probability measure on $(Q \times X, \mathcal{B}(Q \times X))$;
- $\sigma : Q \times X \to \mathbb{R}^{n \times n}$ is a state dependent real-valued matrix;
- $\lambda_{ij} : X \to \mathbb{R}$, $i, j \in Q$ are a set of x-dependent transition rates, with $\lambda_{ij}(.) \geq 0$ if $i \neq j$ and $\sum_{j \in Q} \lambda_{ij}(x) = 0$ for all $i \in Q$, $x \in X$.

As for PDMP, we will use $\alpha = (q, x)$ to denote the hybrid state of an SDP. To ensure the SDP model is well-defined (Ghosh *et al.*, 1997) introduce the following assumption.

Assumption C: *The functions* $f(i, x)$, $\sigma_{kj}(i, x)$ *and* $\lambda_{kj}(x)$ *are bounded and Lipschitz continuous in* x.

Assumption C ensures that for any $i \in Q$, the solution to the SDE $dx(t) = f(i, x(t))dt + \sigma(i, x(t))dW_t$, where W_t is an n-dimensional standard Wiener process, exists and is unique (see, for example, Theorem 6.2.2. in (Arnold, 1974)).

For $i, j \in Q$ and $x \in \mathbb{R}^n$ let $\Delta(i, j, x)$ be consecutive, with respect to lexicographic ordering on $Q \times Q$, left closed, right open intervals of the real line, each having length $\lambda_{ij}(x)$ (for details see (Ghosh *et al.*, 1997)). Now define a function $h : \mathbb{R}^n \times Q \times \mathbb{R} \to \mathbb{R}$ by setting $h(x, i, z) = j - i$, if $z \in \Delta(i, j, x)$; $h(x, i, z) = 0$ otherwise. SDP executions can be defined using h.

Definition 3. (**SDP execution**). A stochastic process $\alpha_t = (q(t), x(t))$ is called an SDP execution if it is the solution of the following stochastic differential equation and stochastic integral:

$$dx(t) = f(q(t), x(t))dt + \sigma(q(t), x(t))dW_t,$$
$$dq(t) = \int_{\mathbb{R}} h(x(t), q(t^-), z)\varphi(dt, dz)$$

for $t \geq 0$ with $x(0) = x_0$, $q(0) = q_0$, where $\alpha_0 = (q(0), x(0))$ is a random variable extracted according to the probability measure $Init$; W_t is a n-dimensional standard Wiener process; $\varphi(dt, dz)$ is an $\mathcal{M}(\mathbb{R}^+ \times \mathbb{R})$-valued Poisson random measure with intensity $dt \times m(dz)$, where m is the Lebesgue measure on $\mathbb{R}$ (see (Jacod and Shiryayev, 1987)); $\varphi(.,.)$, W_t, and $(q(0), x(0))$ are independent.

It can be shown (Ghosh *et al.*, 1997) that the SDP defines a Markov process, whose infinitesimal generator is given by

$$L^{SDP}\theta\ (i,x) = L_c^{SDP}\theta\ (i,x) + \left(\sum_{j=1}^{N}\lambda_{ij}(x)\left(\theta(j,x)-\theta(i,x)\right)\right) \tag{1}$$

where $L_c^{SDP}\theta\ (i,x) = \mathcal{L}_f\theta(i,x) + \frac{1}{2}Tr(\sigma(i,x)\sigma(i,x)^T \mathbb{H}^{\theta}(i,x))$. θ is assumed to belong to the domain of generator defined in (Ghosh *et al.*, 1997).

2.4 Stochastic Hybrid System

Stochastic Hybrid Systems (SHS), involve a hybrid state space, with both continuous and discrete states. The continuous state obeys an SDE that depends on the hybrid state. Transitions occur when the continuous state hits the boundary of the state space. Whenever a transition occurs the hybrid state is reset instantly to a new value. The value of the discrete state after the transition is determined deterministically by the hybrid state before the transition. The new value of the continuous state, on the other hand, is governed by a probability law which depends on the last hybrid state. We introduce formally SHS following (Hu *et al.*, 2000). To make the comparison with SDP and PDMP easier we make a minor change to the definition of (Hu *et al.*, 2000): we allow probabilistic choice of the initial condition.

Definition 4. (**SHS**). A Stochastic Hybrid System is a collection $H = (Q, X, Dom, f, g, Init, G, R)$ where

- Q is a countable set of discrete variables;
- $X = \mathbb{R}^n$ is the continuous state space;
- $Dom : Q \to 2^X$ assigns to each $i \in Q$ an open subset of X;
- $f, g : Q \times X \to \mathbb{R}^n$ are vector fields;
- $Init : \mathcal{B}(Q \times X) \to [0,1]$ is an initial probability measure on $(Q\times X, \mathcal{B}(Q\times X))$ concentrated on $\cup_{i\in Q}\{i\} \times Dom(i)$;
- $G : Q\times Q \to 2^X$ assigns to each $(i,j) \in Q\times Q$ a guard $G(i,j) \subset X$ such that
  - · For each $(i,j) \in Q \times Q$, $G(i,j)$ is a measurable subset of $\partial Dom(i)$ (possibly empty);
  - · For each $i \in Q$, the family $\{G(i,j) \mid j \in Q\}$ is a disjoint partition of $\partial Dom(i)$;
- $K : Q \times Q \times X \to \mathcal{P}(X)$ assigns to each $(i,j) \in Q \times Q$ and $x \in G(i,j)$ a reset probability kernel on X concentrated on $Dom(j)$.

We again use $\alpha = (q,x)$ to denote the hybrid state of an SHS. To ensure that the model is well-defined (Hu *et al.*, 2000), we impose the following assumption:

Assumption D: *The functions $f(i,x)$ and $g(i,x)$ are bounded and Lipschitz continuous in x. For all $i,j \in Q$ and for any measurable set $A \subset Dom(j)$, $K(i,j,x)(A)$ is a measurable function in x.*

The first part of Assumption D ensures that for any $i \in Q$, the solution of the SDE $dx(t) = f(i,x(t))dt + g(i,x(t))dW_t$, where W_t is a 1-dimensional standard Wiener process, exists and is unique (see Theorem 6.2.2 in (Arnold, 1974)). Moreover, the assumption on K ensures that events we encounter later are measurable w.r.t. the underlying σ-field, hence their probabilities make sense. We can introduce the SHS execution.

Definition 5. (**SHS Execution**). A stochastic process $\alpha_t = (q(t), x(t))$ is called a SHS execution if there exists a sequence of stopping times $T_0 = 0 \le T_1 \le T_2 \le \ldots$ such that for each $j \in \mathbb{N}$,

- $\alpha_0 = (q(0), x(0))$ is a $Q \times X$-valued random variable extracted according to the probability measure $Init$;
- For $t \in [T_j, T_{j+1})$, $q(t) = q(T_j)$ is constant and $x(t)$ is a (continuous) solution of the *SDE*:
$$dx(t) = f(q(T_j), x(t))dt + g(q(T_j), x(t))dW_t \tag{2}$$
  where W_t is a 1-dimensional standard Wiener process;
- $T_{j+1} = \inf\{t \ge T_j : x(t) \notin Dom(q(T_j))\}$;
- $x(T_{j+1}^-) \in G(q(T_j), q(T_{j+1}))$, where $x(T_{j+1}^-)$ denotes $\lim_{t\uparrow T_{j+1}} x(t)$;
- The probability distribution of $x(T_{j+1})$ is governed by the law $K\left(q(T_j), q(T_{j+1}), x(T_{j+1}^-)\right)$.

To highlight the relation of SHS to PDMP and SDP, let $\mathcal{D} = \bigcup_{i\in Q}\{i\} \times Dom(i)$; $\partial\mathcal{D} = \bigcup_{i\in Q}\{i\}\times\partial Dom(i) = \bigcup_{i\in Q}\left(\{i\} \times \bigcup_{j\in Q, j\neq i} G(i,j)\right)$. As for the PDMP, let $\mathcal{B}(\mathcal{D}^*)$ denote the σ-algebra on the set $\mathcal{D}^* = Q \times \mathbb{R}^n$ generated by the sets $\{\{i\} \times \mathcal{B}(\mathbb{R}^n)\}$. One can then consider the reset probability kernel K of an SHS as a transition measure $R : \mathcal{B}(\mathcal{D}^*) \times \mathcal{D}^* \to [0,1]$ such that $R(\cdot, (i,x)) = 0$, for all $(i,x) \in \mathcal{D}^*\backslash\partial\mathcal{D}$; for all $(i,x) \in \partial\mathcal{D}$ the function $R(\cdot,(i,x))$ is a probability measure concentrated on $\{j\} \times Dom(j)$ where j is the unique value of the discrete state such that $x \in G(i,j)$. An SHS is a homogenous Markov process whose sample paths are the SHS stochastic executions. The state space of this process is $\mathcal{D}^*$, but clearly it can be restricted to $\mathcal{D} \cup \partial\mathcal{D}$. From the definition of SHS it is easy to see that the càdlàg property is fulfilled. The underlying probability space can be defined in a canonical way. The jump times $T_1 < T_2 < T_3 < ...$ are random variables which are defined for each stochastic execution as exit times (see Definition 5). We can now introduce the following result where the expression of SHS infinitesimal generator is given.

Theorem 1. Let (α_t) be an SHS. Then the domain $D(L^{SHS})$ of the extended generator L^{SHS} of (α_t) consists of those measurable functions θ on $\mathcal{D} \cup \partial\mathcal{D}$ satisfying: (1) $\theta \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$; (2) (Boundary condition) $\theta(\alpha) = \int_{\mathcal{D}} \theta(\beta) R(d\beta, \alpha))$, $\alpha \in \partial\mathcal{D}$; (3) $B\theta \in L_1^{loc}(p)$, where $B\theta(\alpha, s, \omega) := \theta(\alpha) - \theta(\alpha_{s-}(\omega))$. For $\theta \in D(L)$, $L^{SHS}\theta$ is given by

$$L^{SHS}\theta\ (\alpha) = L_c^{SHS}\theta(\alpha) + \int_{\mathcal{D}^*} (\theta(\beta) - \theta(\alpha) R(d\beta, \alpha)) \tag{3}$$

where $L_c^{SHS}\ \theta(\alpha) = \mathcal{L}_f\theta(\alpha) + \frac{1}{2}Tr(g(\alpha)g(\alpha)^T \mathbb{H}^\theta(\alpha))$.

3. MODEL COMPARISON

Randomness enters in different places for the the three classes of stochastic hybrid processes discussed in Section 2. In this section we highlight the similarities and differences among these classes. It is simple to check that the only stochastic processes that can be executions of all three models (PDMP, SDP and SHS) can be trivially represented a finite family of ODE (parametrized by q). The particular ODE and its initial condition are determined according to a probability distribution and no discrete transitions are permitted from them on. Pairwise comparisons, however, provide some more insight into the differences in descriptive power between the three classes of model. To formalize the pairwise comparisons we introduce the concept of modification.

Definition 6. (**Modification**) Given two stochastic processes α_t and $\hat{\alpha}_t$ defined on the same underlying probability space $(\Omega, \mathcal{P})$, we say that α_t is a modification of $\hat{\alpha}_t$ if $\mathcal{P}(\alpha_t = \hat{\alpha}_t) = 1$ for all t.

A comparison between PDMP and SDP. To find a subclass of PDMP and a subclass of SDP that coincide in the sense of modification we have to assume that the number of discrete states of the PDMP is finite, to eliminate the diffusion element of the SDP. Moreover, it is necessary to assume for all discrete states i of the PDMP, $X(i) = \mathbb{R}^{d(i)}$ and a relation between the SDP transition matrix (λ_{ij}) and the PDMP transition rate and transition measure has to be established. These qualitative remarks are formalized in the following lemmas.

Lemma 2. (**SDP→PDMP**) Let $H^{SDP} = (Q, X, f, Init, \sigma, \lambda_{ij})$ be a SDP with $X = \mathbb{R}^n$. Suppose that $\sigma(i, x) = 0$ for all $(i, x) \in Q \times X$. Then, there exists a PDMP H^{PDMP} which is a modification of H^{SDP}.

Lemma 3. (**PDMP→SDP**) Let $H^{PDMP} = ((Q, d, X), f, Init, \lambda, R)$ be a PDMP, and assume that

(1) $|Q| = N \in \mathbb{N}$;
(2) $X(i) = \mathbb{R}^{d(i)}$, $\forall i \in Q$;
(3) For any $i, j \in Q$, $R(\{(j, x), j \in Q\}, (i, x)) = 1$.

Then there exists a SDP H^{SDP} which is a modification of H^{PDMP}.

Condition (3) effectively implies that the evolution of the continuous state of H^{PDMP} is continuous as a function of time. Condition (2) can be weakened to $t^*(\alpha) = \infty$, $\forall \alpha \in \mathcal{D}$. Lemmas 2 and 3 indicate that the common model for SDP and PDMP is a stochastic hybrid model with deterministic continuous evolution between two consecutive jumps (as in PDMP) and with discrete state switchings governed by a transition rate depending on the last hybrid state and on the next discrete state (as in SDP). Finally no jumps on the continuous state are allowed when a switching occurs (as in SDP).

A comparison between SHS and SDP. To find a subclass of SHS and a subclass of SDP that coincide in the sense of modification we have to assume that the number of discrete states of the SHS is finite, the diffusion process for the SDP is governed by a standard 1-dimensional Wiener process. It is also necessary to assume that $Dom(i) = \mathbb{R}^n$, since the continuous motion of the SDP is unconstrained. The last one together with Assumption D implies that the guards G have to be empty. Finally, since the evolution of the continuous state of an SDP is continuous in time, the reset relation R of the SHS also has to be trivial. These remarks are formalized in the following lemmas.

Lemma 4. (**SHS→SDP**) Let $H^{SHS} = (Q, X, Dom, f, g, Init, G, R)$ be a SHS with $X = \mathbb{R}^n$. Assume that the cardinality of the set Q is finite, $Dom(i) = \mathbb{R}^n$, for all $i \in Q$, $G(i, j) = \emptyset$ for all $i, j \in Q$ and $R(., (., x)) = 0$if $x \in \mathbb{R}^n$; $R(., (., x)) = 1$ if $x = \Delta$ where Δ is the compactification point of $\mathbb{R}^n$. Then, there exists an SDP H^{SDP} which is a modification of H^{SHS}.

Lemma 5. (**SDP→SHS**) Let $H^{SDP} = (Q, X, f, Init, \sigma, \lambda_{ij})$ be a SDP with $X = \mathbb{R}^n$. Assume that $rank(\sigma(i, x)) \leq 1$ for all $(i, x) \in Q \times X$ and $\lambda_{ij}(x) = 0$ for all $i, j \in Q$, $x \in X$. Then, there exists a SHS H^{SHS} which is a modification of H^{SDP}.

Lemmas 4 and 5 indicate that the common model between SHS and SDP are a finite family of SDE (parametrized by q) driven by a 1-dimensional Wiener process. The particular SDE and its initial condition are determined according to a probability distribution and no discrete transitions are permitted from them on.

A comparison between SHS and PDMP. To find a subclass of SHS and a subclass of PDMP that coincide in the sense of modification we have to eliminate the diffusion component of the SHS, since PDMP are deterministic processes between two consecutive jumps. We also have to assume that the dimension of the continuous state space is bounded. We have to eliminate jumps governed by the transition rate λ and establish a relation between the transition measures of SHS and PDMP. These qualitative remarks are formalized in the following Lemmas 6 and 7.

Lemma 6. (**SHS→PDMP**) Consider a SHS, $H^{SHS} = (Q, X, Dom, f, g, Init, G, R)$, with $X = \mathbb{R}^n$. Assume that $g(i,x) = 0$, for all $(i,x) \in Q \times X$. Then, there exists a PDMP H^{PDMP} which is a modification of H^{SHS}.

Lemma 7. (**PDMP→SHS**) Consider a PDMP, $H^{PDMP} = ((Q, d, X), f, Init, \lambda, R)$. Assume that

(1) There exists $n \in \mathbb{N}$ such that $d(i) \leq n$ for all $i \in Q$;
(2) $\lambda(i,x) = 0$ for all $(i,x) \in \mathcal{D}(Q, d, X)$;
(3) For all $i, j \in Q$ the set $G(i,j) = \{x \in \partial X(i) : R(\{j\} \times X(j), (i,x)) \neq 0\}$ is measurable.

Then, there exists a SHS H^{SHS} which is a modification of H^{PDMP}.

Lemmas 6 and 7 suggest that the common model between SHS and PDMP is a stochastic hybrid process where continuous evolution between two consecutive jumps is deterministic and where switchings between two discrete states occur only when the continuous state hits the hybrid state space boundary. Whenever a switching occurs, the hybrid state jumps, according to a probability law depending on the last hybrid state.

4. CONCLUSIONS AND FURTHER WORKS

In this paper we gave an overview of Stochastic Hybrid Models developed in the literature. We described Piecewise Deterministic Markov Process, Switching Diffusion Process and Stochastic Hybrid System. We developed a comparison among them, underlying the assumptions under which they coincide. We developed the expression of generator for Stochastic Hybrid Systems. Further theoretical investigations will be concentrated on the analysis of a possible general stochastic hybrid model that could include PDMP, SDP and SHS as special cases. This work can be considered as a base for the building of an ATM mathematical model; this is the aim of our further applied work in the context of HYBRIDGE.

Acknowledgement: Work supported by the European Commission under project HYBRIDGE, IST-2001-32460.

REFERENCES

Arnold, L. (1974). *Stochastic Differential Equations: Theory and Application.* John Wiley & Sons.

Baier, C., J.-P. Katoen and H. Hermanns (1999). Approximate symbolic model checking of continous-time markov chains. In: *Concurrency Theory* (J.C.M. Baeten and S. Mauw, Eds.). pp. 146–162. Number 1664 In: *LNCS*. Springer Verlag.

Bujorianu, M.L. and J. Lygeros (2003). Reachability questions in piecewise deterministic markov processes. In: *Submitted to HSCC*.

Davis, M.H.A. (1993). *Markov Processes and Optimization.* Chapman & Hall. London.

Ethier, S.N. and T.G. Kurtz (1986). *Markov Processes: Characterisation and Convergence.* Willey. New York.

Ghosh, M.K., A. Arapostathis and S.I. Marcus (1997). Ergodic control of switching diffusions. *SIAM Journal on Control Optimization* **35**(6), 1952–1988.

Hu, J., J. Lygeros and S. Sastry (2000). Towards a theory of stochastic hybrid systems. In: *Hybrid Systems: Computation and Control* (Nancy Lynch and Bruce H. Krogh, Eds.). pp. 160–173. Number 1790 In: *LNCS*. Springer Verlag.

Jacod, J. and A.N. Shiryayev (1987). *Limit Theorems for Stochastic Processes.* Springer-Verlag. New York.

Kwiatkowska, M., G. Norman, R. Segala and J. Sprosto (1999). Automatic verification of real-time systems with discrete probability distributions. In: 5^{th} *International AMAST Workshop on Real Time and Probabilistic Systems*. pp. 79–95. Number 1601 In: *LNCS*. Springer Verlag.

Kwiatkowska, M., G. Norman, R. Segala and J. Sprosto (2000). Verifying quantitative properties of continuous probabilistic timed automata. In: *CONCUR'00*. pp. 123–137. Number 1877 In: *LNCS*. Springer Verlag.

Lygeros, J., K. H. Johansson, S. N. Simić, J. Zhang and S. Sastry (2003). Dynamical properties of hybrid automata. *IEEE Transactions on Automatic Control* **48**(1), 2–17.