STOCHASTIC EQUIVALENCE OF CPDP-AUTOMATA AND PIECEWISE DETERMINISTIC MARKOV PROCESSES

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Abstract: CPDP is a class of automata designed for compositional specification/analysis of certain stochastic hybrid processes. We prove equivalence of the stochastic behaviors of CPDPs (newly defined here) and PDPs. With this result we obtain a clear stochastic processes semantics for CPDPs and we obtain the opportunity to use the powerful PDP analysis techniques in the context of the compositional framework CPDP.

Keywords: (Communicating) Piecewise Deterministic Markov Processes, hybrid systems, compositional specification.

1. INTRODUCTION

CPDP, which stands for Communicating Piecewise Deterministic Markov Processes, is a class of automata that represent stochastic hybrid jump processes of the PDP-type (for PDPs, see (Davis 1984, Davis 1993)). The CPDP framework (introduced in (Strubbe *et al.* 2003)) was/is developed for compositional specification and analysis of PDP-type systems. Because labelled transition systems like automata can easily be composed by means of a composition operator, we chose to develop CPDP as an automata framework.

In (Everdij and Blom 2003) the Petri net formalism DCPN (Dynamically Colored Petri Nets) is developed for compositional specification of PDP systems. However, we believe that from a compositional analysis point of view, automata are more suitable than Petri nets. Furthermore, for a rigorous proof of stochastic equivalence (which we give in this paper in the case of CPDP) of DCPN/CPDP and PDP, it seems to be more convenient to use automata.

Compared to PDP, the CPDP framework can be regarded as compositional in two different ways: First, a CPDP has labelled transitions between the locations. With these labels communication can be established with other CPDPs. This communication mechanism is formalized in (Strubbe *et al.* 2003) by defining a composition operator (via structural operational rules). Thus, a complex CPDP system can be built in a compositional way as a network of CPDPs which are running simultaneously and interact with each other.

Second, we could say that the transition mechanism (from one hybrid state to another) of the PDP is for CPDPs decomposed into several transitions of different kinds and the PDP state space is decomposed into different variables (having there own state space). For example, in CPDP we can specify several Poisson processes (via spontaneous transitions) which are running at the same time and are in a race (concerning which process generates the first point and will therewith determine the next transition). For PDPs, this race of Poisson processes cannot not be specified compositionally (with the Poisson processes as components), but should be specified as one Poisson process that expresses this race of different Poisson processes.

It is clear that the advantage of a compositional framework is present in the context of complex systems. If a system is not very complex, it is probably possible to specify the PDP directly.

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However, if the system is complex (i.e. consists of many interacting components), then it might be hard to specify the PDP directly and specifying the network of interacting CPDPs (representing the same complex system) might be much easier with a more comprehensive specification as result. Related to PDPs are continuous time Markov chains (CTMC). In (Hermanns 2002) a compositional formalism called IMC (Interactive Markov Chains) is developed for CTMCs. We can regard CPDP more or less as a generalization of IMC.

The reason that we want to consider especially PDP systems is that, first, almost all stochastic hybrid processes that do not have diffusions can be modelled as a PDP and second, the PDP process has very nice properties (as the strong Markov property) when it comes to stochastic analysis. (In (Davis 1993) powerful analysis techniques have been developed).

What we therefore want, is that the stochastic behavior of the (composite) CPDP (with its multiple transitions and variables) is equivalent with the stochastic behavior of the PDP. In this paper we prove that this is the case for a notion of equivalence that we call first jump equivalence (from which will follow that execution/simulation of first jump equivalent processes will show no difference). This result is possible mainly because of two facts. First, the class of Poisson processes is closed under composition (in the sense of the minimum of two processes). Second, Borel sets in euclidean space behave very well on product spaces (which come into play when we have different variables simultaneously evolving): The Borel sets of a (euclidean) product space are generated by the products of Borel sets of the individual spaces.

The paper is organized as follows. In Sections two and three we give the definitions of the PDP and the CPDP and we show (by means of stochastic execution procedures) how they represent stochastic processes. In this definition of CPDP we allow separate variables in one location, where the CPDP model of (Strubbe *et al.* 2003) does not consider variables with their own state spaces, but considers (as it is done for PDPs) one combined state space which is a subset of $\mathbb{R}^n$. These variables will ask some extra efforts from us for the equivalence proof, but for specification and composition purposes, this use of variables will turn out to be more convenient because the set of variables of a composite CPDP location is then equal to the union of the sets of variables of the corresponding locations of the component-CPDPs. (This advantage in the context of composition can not be seen in this paper since we do not treat composition here. See (Strubbe *et al.* 2003) for composition of CPDPs).

Section four contains the main result of this paper. We observe there that both the PDP and the CPDP can also be characterized via the assignment of so called 'first jump probability measures' to each hybrid state. We then prove (by construction) that for each CPDP a corresponding PDP can be found such that they define the same first jump probabilities. This shows that executing a CPDP is stochastically equivalent to executing its corresponding PDP.

Section five closes the paper by drawing conclusions and giving an outlook for future research concerning CPDPs.

2. DEFINITION OF THE PDP

The state space and the dynamics of a PDP are defined as follows: K is a countable set of locations. For each $\nu \in K$, $d(\nu) \in \mathbb{N}$ denotes the dimension of the continuous state space of location ν. For each $\nu \in K$, let E_ν be an open subset of $\mathbb{R}^{d(\nu)}$ and let $g_\nu : \mathbb{R}^{d(\nu)} \to \mathbb{R}^{d(\nu)}$ be a locally Lipschitz continuous function on E_ν. The flow $\phi_\nu(t,\zeta)$ is uniquely determined by the differential equation $\dot{\zeta} = g_\nu(\zeta)$ and equals the state at time t if the state equals ζ at time $t = 0$. The hybrid state space of the PDP is now defined as

$$E = \{(\nu,\zeta) | \nu \in K, \zeta \in E_\nu\}.$$

Remark 1. In fact, the state space E of the PDP is in (Davis 1993) extended such that E also contains the boundary points that are backward reachable (via flow ϕ) but not forward reachable from the interior of E.

For $x = (\nu,\zeta) \in E$ define

$$t_*(x) = \begin{cases} \inf\{\mathrm{t} > 0 | \phi_\nu(\mathrm{t},\zeta) \in \partial \mathrm{E}_\nu\} \\ \infty \text{ if no such time exists.} \end{cases},$$

where $\partial E_\nu = \bar{E}_\nu \backslash E_\nu$ is the boundary of E_ν, $\bar{E}_\nu$ is the closure of E_ν.

The jump mechanism of the PDP is determined by a jump rate function λ and a transition measure Q. The jump rate $\lambda : E \to \mathbb{R}_+$ is a measurable function such that for each $x = (\nu,\zeta) \in E$, there exists $\epsilon(x) > 0$ such that the function $s \to \lambda(\nu, \phi_\nu(s,\zeta))$ is integrable on $[0,\epsilon(x)[$. With Γ^* we denote the boundary of E that is reachable from the interior of E. The transition measure Q maps $E \cup \Gamma^*$ into the set $\mathcal{P}(E)$ of probability measures on the Borel space $(E,\mathcal{E})$, where $\mathcal{E}$ is the set containing all Borel sets of E (according to a 'natural' topology, defined in (Davis 1993)), with the properties that for each fixed $A \in \mathcal{E}$ the map $x \to Q(A,x)$ is measurable, and $Q(\{x\},x) = 0$ for all $x \in E \cup \Gamma^*$.

A PDP process, starting from initial state $x_0 = (\nu_0, \zeta_0)$, can be 'executed' as follows: The dynamics at $t = 0$ is determined by the vectorfield g_{ν_0} until either the boundary is hit at time $t_*(x_0)$ or until a point is generated by the Poisson process that has density $\lambda(x)$. In either case, a jump takes place and the target hybrid state is determined by the probability measure $Q(\cdot, (\nu_0, \phi_{\nu_0}(\hat{t}, \zeta_0)))$, where $\hat{t}$ is the jump time. From the target state this execution procedure can be repeated.

For a PDP it is assumed that there are no explosions (i.e. $|\phi_\nu(t, \zeta)| \not\to \infty$ if $t \not\to \infty$) and that there is no Zeno behavior (i.e. for every starting point $x \in E$, $\mathrm{E}N_t < \infty$ for all $t \in \mathbb{R}_+$, where N_t is a random variable 'counting' the number of jumps up to time t).

3. DEFINITION OF THE CPDP

A CPDP is a 10-tuple $(L, V, v, Inv, G, \Sigma, A, P, S, C)$, where

- L is a countable set of locations
- V is a set of variables. With $d(y)$ for $y \in V$ we denote the dimension of variable y. $y \in V$ takes its values in $\mathbb{R}^{d(y)}$. We also say that $\mathbb{R}^{d(y)}$ is the valuation space of y.
- $v : L \to 2^V$ maps each location to a subset of V, which is the set of active variables of the corresponding location
- Inv assigns to each location l and each variable $y \in v(l)$ an open subset of $\mathbb{R}^{d(y)}$, i.e. $Inv(l, y) \subset \mathbb{R}^{d(y)}$. With Inv_l we denote the subset of the valuation space of $v(l)$ that is built from (or loosely speaking: is the product of) the invariants of the individual variables. With ∂Inv_l we denote the set of boundary points of l, which is equal to the set of valuations of $v(l)$ where each $y \in v(l)$ takes value in $\overline{Inv(l, y)}$ and at least one $y \in v(l)$ takes value in $\partial Inv(l, y) := \overline{Inv(l, y)} \backslash Inv(l, y)$.
- G assigns to each location l and each $y \in v(l)$ a locally Lipschitz continuous function from $\mathbb{R}^{d(y)}$ to $\mathbb{R}^{d(y)}$, i.e. $G(l, y) : \mathbb{R}^{d(y)} \to \mathbb{R}^{d(y)}$. This vectorfield uniquely determines a flow $\phi_{l,y}(t, y_0)$ along this vectorfield.
- Σ is the set of communication labels. $\bar{\Sigma}$ denotes the 'passive' mirror of Σ and is defined as $\bar{\Sigma} = \{\bar{a} | a \in \Sigma\}$.
- B is a finite set of boundary hit transitions and consists of 4-tuples (l, a, l', R), denoting a transition from location $l \in L$ to location $l' \in L$ with communication label $a \in \Sigma$ and reset map R. This reset map R assigns to each boundary point of l for each active variable $y \in v(l')$ a probability measure on the invariant (and its Borel sets) of y for location l'. We will denote the measure of R for variable y at boundary point ζ by $R^y(\zeta)$.
- P is a finite set of passive transitions and consists of 4-tuples $(l, \bar{a}, l', R)$, denoting a transition from location $l \in L$ to location $l' \in L$ with passive communication label $\bar{a} \in \bar{\Sigma}$ and reset map R. R assigns to each interior point of location l for each active variable $y \in v(l')$ a probability measure on the invariant (and its Borel sets) of y for location l'.
- S is a finite set of spontaneous (also called Poisson) transitions and consists of 5-tuples (l, λ, a, l', R), denoting a transition from location $l \in L$ to location $l' \in L$ with communication label $a \in \Sigma$, jump-rate function λ and reset map R. The jump rate $\lambda : Inv_l \to \mathbb{R}_+$ is a measurable function such that for each $\zeta \in Inv_l$, there exists $\epsilon(\zeta) > 0$ such that the function $s \to \lambda(\phi_l(s, \zeta))$ is integrable on $[0, \epsilon(\zeta)[$, where ϕ_l denotes the flow of the valuations of variables $v(l)$ for location l. R is defined on all interior points of l as it is done for spontaneous transitions.
- C is the choice function. C assigns to each boundary point (l, ζ) of the CPDP a probability measure on the set of outgoing boundary hit transitions, i.e. $C(l, \zeta)$ (with $\zeta \in \partial Inv_l$) is a probability measure on B_l, where B_l is the set of boundary hit transitions that have l as origin location.

For a reset map R of either a boundary hit, a passive or a spontaneous transition, we assume that for any active variable y of the target location and any fixed Borel set A of the invariant of y of the target location, the map $\zeta \to R^y(A, \zeta)$ is measurable. Measurability of $\zeta \to R^y(A, \zeta)$ here means that for any $B \in \mathcal{B}[0, 1]$ the set $\{(\zeta_1, \zeta_2, \cdots, \zeta_n) | R(A, \{y_1 = \zeta_1, y_2 = \zeta_2, \cdots, y_n = \zeta_n\}) \in B\}$ is a Borel set of $\mathbb{R}^{d(y_1)+d(y_2)+\cdots+d(y_n)}$, where $y_1, y_2, \cdots, y_n$ are the active variables of the origin location. (This fact does not depend on the order of y_1 till y_n).

For reset maps of boundary hit and spontaneous transitions that return to the same location (i.e. origin and target location are the same), we assume that the probability that all variables jump to the same value as before the jump equals zero (i.e. for every ζ in the invariant or on the boundary of the invariant there exists an active variable y such that $R^y(\{\zeta\}, \zeta) = 0$).

We also assume that the choice function C is such that for any fixed boundary transition $\alpha = (l, a, l', R) \in A$ we have that the map $\zeta \to C(l, \zeta)(\alpha)$ is measurable.

Finally we assume that the CPDP does not have explosions and does not exhibit Zeno behavior (see the PDP definition for the definition of explosions and Zeno behavior).

Passive transitions are used to interact with the environment (see (Strubbe *et al.* 2003) for an explanation of the communication mechanism established by the interplay of boundary hit, spontaneous and passive transitions). The environment can activate/trigger these passive transitions. When a CPDP does not have passive transitions, then it can not be influenced by the environment, which means that it is autonomous and can be executed 'on its own'.

Execution of a CPDP $(L, V, v, Inv, G, \Sigma, A, P, S, C)$ without passive transitions (i.e. $P = \emptyset$), starting from initial state $x_0 = (l_0, \zeta_0)$, is done as follows: The dynamics at $t = 0$ is determined by the vectorfield $G(l_0)$ until either the boundary $(\partial Inv(l_0))$ is hit at time $t_*(x_0)$ (which is defined similarly as t_* for the PDP) or until a point is generated by a Poisson process of one of the spontaneous transitions. For each spontaneous transition $\alpha = (l_0, \lambda_\alpha, l', R_\alpha)$ a Poisson process is 'running' with density $\lambda_\alpha(\zeta(t))$. Note that the jump-rate λ_α is not constant, but depends on the current state of the process $\zeta(t)$. As soon as one of these Poisson processes generates a point, the corresponding spontaneous transition will be taken. If the first jump is caused by a boundary-hit at boundary point ζ, a boundary hit transition will be selected according to the probability measure $C(l_0, \zeta)$. The new continuous state in the target location of the active transition, will be selected according to the probability measures of the reset map R of the boundary hit transition. If the first jump is caused by one of the Poisson processes, the reset map of the corresponding spontaneous transition will select the new continuous state in the target location. From the new hybrid state on, this execution procedure can be repeated.

Example 1. In Figure 1 we see a CPDP automaton, where we have between the locations (from top to bottom) a boundary hit, a spontaneous and another boundary hit transition. We specify the reset maps R_i $(i = 1, 2, 3)$ and the jump rate function λ as: R_1 and R_3 assign to each boundary point of location l_1 and l_2 respectively for variable x a uniform distribution on the interval $[0, 1]$. R_2 assigns the same uniform distribution for x to each interior point of l_1. R_1 till R_3 all reset the variable y to zero (with probability one). Thus, all three reset maps are 'the same'. We specify $\lambda(x, y) = 1$ (thus, constant on the interior of l_1 and therefore constant in time).

Execution of the CPDP from initial location l_1 with the variables initially set as $x = 0$ and $y = 0$, goes as follows: x grows exponentially and y has a clock dynamics and has therefore as value the amount of time elapsed. One Poisson process is running with density $\lambda = 1$ (this gives an exponential distribution). If this Poisson

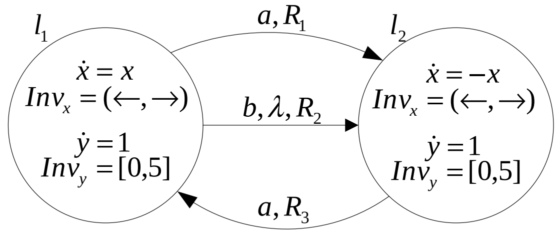

Fig. 1. Example of a CPDP automaton

process generates a point before 5 time units, then the spontaneous transition is taken (and R_2 will stochastically reset x and y in the new location l_2). If the Poisson process does not generate the point before 5 time units, then at $t = 5$ the boundary of the invariant of y is hit, which means that a boundary hit transition must be taken at $t = 5$. Since there is only one boundary hit transition from l_1 (the one with R_1), this one is taken (a choice function is thus not necessary here) and R_1 resets x and y in the new location l_2. In l_2, x will decrease exponentially and y monitors the time spent in this location. A jump to l_1 happens only when y reaches the boundary after 5 time units and then the boundary hit transition with reset map R_3 is taken. (The labels a and b above the transition are used for communication with other CPDPs and do not influence the stochastic behavior here).

4. PDP-SEMANTICS FOR CPDPS

We can say that a PDP or a CPDP determines for each hybrid state $x \in E$ a probability measure on $\mathbb{R} \times E$ and its Borel sets (with $\mathbb{R}$ for the time and with E the hybrid state space) for the next jump time (starting at x) and the (target) hybrid state at this jump time. With these probability measures, an execution of a PDP or CPDP from initial hybrid state x_0 could also be done as follows: Select an element (t_1, x_1) of $\mathbb{R} \times E$ according to the probability measure of x_0. The evolution of the hybrid state between t_0 and t_1 is determined by the differential equation of the initial location. The state at time t_1 is now set to x_1 and from x_1 we can draw a new sample from $\mathbb{R} \times E$ according to the probability measure of x_1, etc.

We will see that this 'first jump' probability measure is well-defined (for all Borel sets of $\mathbb{R} \times E$) for all hybrid states by the characteristics (jump rates, reset maps) of a PDP or a CPDP.

We now look at how a PDP can represent (mimic) the stochastic behavior of a CPDP. In order to do that, we need a PDP-like representation of the state space of a CPDP. For a PDP, the continuous state space for a specific location l is an open subset of $\mathbb{R}^{d(l)}$. For a CPDP this is the case for a variable rather than for a location. To overcome this difference, we assume a specific ordering of the CPDP variables (i.e. $V = \{v_1, v_2, \cdots, v_n\}$). Then the continuous state space of a CPDP location l with active variables $\hat{v}_1$ till $\hat{v}_m$ (in that order) can be seen as $Inv(l, \hat{v}_1) \times Inv(l, \hat{v}_2) \times \cdots \times Inv(l, \hat{v}_m)$ which is an open subset of $\mathbb{R}^{d(\hat{v}_1)+d(\hat{v}_2)+\cdots+d(\hat{v}_m)}$. We will call this the euclidean state space representation of a CPDP.

Suppose we have a CPDP $\mathcal{A} = (L, V, v, Inv, G, \Sigma, A, P, S, C)$ with $P = \emptyset$. We denote the hybrid state space (in euclidean form) by E. With $S_{l\rightarrow}$ ($S_{l\rightarrow l'}$) we denote the subset of transitions of S which have l as origin location (and l' as target location) etc. Functions, measures etc. that use the valuation of variables as an argument, are naturally transformed into functions, measures etc. that use euclidean states as argument (e.g. we write $\lambda(\zeta)$ for the value of λ for the valuation that corresponds with ζ). We will now define a transition measure Q on $E \cup \partial E$ and a jump rate function λ on E as follows: For $x = (l, \zeta) \in E$ and $B \in \mathcal{B}(E \downarrow l')$, where $E \downarrow l' := \{(l, \zeta) \in E | l = l'\}$,

$$\lambda(x) := \sum_{s \in S_{l\rightarrow}} \lambda_s(\zeta),$$

$$Q(B, x) := \sum_{s \in S_{l\rightarrow l'}} \left(\frac{\lambda_s(\zeta)}{\sum_{s' \in S_{l\rightarrow}} \lambda_{s'}(\zeta)} R_s(B, \zeta) \right)$$

if $\sum_{s' \in S_{l\rightarrow}} \lambda_{s'}(\zeta) \neq 0$ and else $Q(B, x) = \tilde{P}(B)$ with $\tilde{P}$ some arbitrary probability measure (this $\tilde{P}(B)$ will not influence the stochastic behavior of the process). Here we have written $R_s(B, \zeta)$ for the probability of jumping into the set of the valuations corresponding to B according to $R_s(\zeta)$ (this probability is uniquely determined by R for all $B \in \mathcal{B}(E \downarrow l')$, see the following paragraph). For $x = (l, \zeta) \in \partial E$ and $B \in \mathcal{B}(E \downarrow l')$ we define

$$Q(B, x) := \sum_{s \in A_{l\rightarrow l'}} C(s, x) R_s(B, \zeta).$$

Now, λ and Q, as defined above, are proper PDP functions, i.e. they satisfy the PDP conditions (measurability etc.): For λ it is evident from the fact that it is a finite sum of proper PDP jump rate functions. For Q it follows from the way it is composed of proper jump rate functions (including the fact that $\{\zeta | \sum_{s' \in S_{l\rightarrow}} \lambda_{s'}(\zeta) = 0\}$ is a measurable set) and from the fact that $\zeta \rightarrow R^s(B, \zeta)$ is measurable for fixed B. The latter statement follows from the fact that any B can be written as a countable sum of products: $B = \cup_i B_i^{y_1} \times B_i^{y_2} \cdots \times B_i^{y_m}$ with y_1 till y_m the active variables of l' and $B_i^{y_j} \in \mathcal{B}(Inv(l', y_j))$. Then $R_s(B, \zeta) = \cup_i R_s^{y_1}(B_i^{y_1}, \zeta) R_s^{y_2}(B_i^{y_2}, \zeta) \cdots R_s^{y_m}(B_i^{y_m}, \zeta)$ and measurability follows from measurability of all the $\zeta \rightarrow R_s^{y_j}(B_i^{y_j}, \zeta)$.

We now calculate the first jump probability measures of the CPDP and of the PDP with the same state space (and boundaries) and vector fields as CPDP $\mathcal{A}$ and with transition measure Q and jump rate function λ as defined above. (It can be easily seen that locally Lipschitz continuity of the vectorfields of the PDP follows from the locally Lipschitz continuity of the CPDP vectorfields of the individual variables). We denote this PDP by $\tilde{\mathcal{A}}$ and call it the corresponding PDP of CPDP $\mathcal{A}$. We show that these first jump probability measures of CPDP $\mathcal{A}$ and of the corresponding PDP $\tilde{\mathcal{A}}$ are equal for all $x \in E$.

(The corresponding PDP of the CPDP of Figure 1, if we ignore the passive transition and order the variables as $\{x, y\}$, has specifications: $K = \{l_1, l_2\}$, $E_{l_1} = E_{l_2} = \mathbb{R} \times [0, 5[$, $\lambda(x) = 1$ for $x \in \{(\nu, \zeta) | \nu = l_1\}$ else $\lambda(x) = 0$. If $A, B \in \mathcal{B}(\mathbb{R})$ then for all $x \in E \cup \Gamma^*$, $Q(A \times B; x)$ equals $l(A \cap [0, 1])$ if $0 \in B$ (where l is the Lebesgue measure) and equals zero otherwise).

Take an arbitrary $x = (l, \zeta) \in E$. We now first look at the survivor functions of the Poisson processes corresponding to the spontaneous transitions of $\mathcal{A}$. We denote the survivor function of transition α at state x by $F_\alpha(t, x)$. $F_\alpha(t, x)$ denotes the probability that the Poisson process of transition α does not generate a point before time t when the CPDP starts at state x. We have that

$$F_\alpha(t, x) = \exp\left(- \int_0^t \lambda_\alpha(\phi_l(t, \zeta)) \mathrm{d}t \right).$$

It can be seen that the survivor function of the process $\mathcal{A}$ at state x, which equals the probability that no jump occurs up till time t, is then given by $F_{\mathcal{A}}(t, x) =$

$$I_{(t < t_*(x))} \exp\left(- \int_0^t \sum_{\alpha \in S_{l\rightarrow}} \lambda_\alpha(\phi_l(t, \zeta)) \mathrm{d}t \right), \quad (1)$$

where $t_*(x)$ is the boundary hitting time from state x and I_A is the indicator function which equals one for points in A and zero for points outside A. Now it can be seen that this survivor function is equal to the survivor function of PDP $\tilde{\mathcal{A}}$.

We denote the first jump probability measure at state x for $\mathcal{A}$ (and $\tilde{\mathcal{A}}$) by $P_x^{\mathcal{A}}$ (and $P_x^{\tilde{\mathcal{A}}}$). Now, take $x = (l, \zeta) \in E$ and $l' \in L$ arbitrary. Suppose we have a set $A = [0, t] \times B$, with $t < t_*(x)$ and B a Borel set contained in the invariant of location l'. If we write $F_\alpha(\tau)$ for $F_\alpha(\phi_l(\tau, \zeta))$, $R_\alpha(B, \tau)$ for $R_\alpha(B, \phi_l(\tau, \zeta))$, etc., then it can be seen that the first jump probability for this set, $P_x^{\mathcal{A}}(A) =$

$$\sum_{\alpha \in S_{l\rightarrow l'}} \int_0^t \left(\prod_{\tilde{\alpha} \in S_{l\rightarrow l'} \setminus \alpha} F_{\tilde{\alpha}}(\tau) \right) R_\alpha(B, \tau) \mathrm{d}F_\alpha(\tau),$$

The above integrals are well-defined because $F_\alpha(\tau)$ is bounded and measurable (and therefore integrable) and $R_\alpha(B, \tau)$, which equals

$R_\alpha(B, \phi_l(\tau, \zeta))$, is bounded and is measurable because $R_\alpha(B, \zeta)$ is measurable for fixed B and ϕ_l is continuous and therefore measurable ($\mathrm{d}F_\alpha(\tau)$ in this integral means integrating over the measure induced by F_α). Then we can derive from (1) that

$$P_x^{\mathcal{A}}(A) = \int_0^t \left(\sum_{\alpha \in S_{l \to l'}} \lambda_\alpha(\tau) R_\alpha(B, \tau) \right) F_{\mathcal{A}}(\tau) \mathrm{d}\tau,$$

which can be rewritten as

$$P_x^{\mathcal{A}}(A) =$$

$$\int_0^t \left(\frac{\sum_{\alpha \in S_{l \to l'}} \lambda_\alpha(\tau)}{\sum_{\alpha \in S_{l \to}} \lambda_\alpha(\tau)} R_\alpha(B, \tau) \right) \mathrm{d}F_{\mathcal{A}}(\tau),$$

which equals

$$\int_0^t Q(B, \tau) \mathrm{d}F_{\tilde{\mathcal{A}}}(\tau),$$

which is exactly the expression for $P_x^{\tilde{\mathcal{A}}}(A)$. Therefore the first jump probabilities for sets of the form A as above coincide for CPDP $\mathcal{A}$ and PDP $\tilde{\mathcal{A}}$.

Now suppose we have a set of the form $A = [t_*(x), t] \times B$, with $t \geq t_*(x)$ and B a Borel set contained in the invariant of l'. Then we can see that

$$P_x^{\mathcal{A}}(A) = \lim_{s \uparrow t_*} F_{\mathcal{A}}(s) \sum_{\alpha \in A_{l \to l'}} C(\alpha, t_*) R_\alpha(B, t_*),$$

which is equal to

$$\int_{[t_*(x), t]} Q(B, \tau) \mathrm{d}F_{\tilde{\mathcal{A}}}(\tau),$$

which is the expression for $P_x^{\tilde{\mathcal{A}}}(A)$. Therefore the first jump probabilities for sets of this form of A also coincide for CPDP $\mathcal{A}$ and PDP $\tilde{\mathcal{A}}$.

It is clear now, by combination of the two cases above, that the first jump probability measures of the CPDP and the PDP also agree on sets of the form $[0, t] \times B$ with $t \in \mathbb{R}_+$ and B a Borel set within the invariant of one location (as above). It can easily be seen that they then also agree when B has points in different locations (i.e. any $B \in \mathcal{B}(E)$). The collection of sets $[0, t] \times B$ (with $t \in \mathbb{R}_+$ and $B \in \mathcal{B}(E)$) is closed under finite intersections and probability measures that agree on such a collection of sets, also agree on the σ-algebra generated by those sets. This means that the first jump probability measures agree on $\mathcal{B}(\mathbb{R}_+ \times E)$, because this is the σ-algebra generated by sets of the form $[0, t] \times B$.

Therefore, instead of executing (or simulating) CPDP $\mathcal{A}$, we could as well execute the PDP $\tilde{\mathcal{A}}$. This would make no difference. In (Davis 1993) the stochastic process that is generated by a PDP is formally defined. With the above observation, it now makes sense to speak about the PDP-semantics (or stochastic process semantics) of CPDP $\mathcal{A}$, expressed by the stochastic process that is generated by the corresponding PDP. In this way we formally defined the stochastic process that is generated by a 'closed' CPDP (i.e. a CPDP that has no passive transitions).

5. CONCLUSIONS

In this paper we have shown that the stochastic behavior of a CPDP automaton can be characterized by a PDP process. In the proof of this result it is shown how such a PDP can be constructed from the CPDP specification.

Stochastic analysis of CPDPs can now be done by using the PDP analysis techniques by transforming the CPDP into its corresponding PDP. In that scenario, CPDPs are used mainly for the compositional specification and as soon as the specification is ready, the complex is transformed into a PDP and can then be analyzed.

Another, more intriguing, scenario is compositional analysis. A complex CPDP consists of multiple CPDPs that are connected via composition operators. The composition structure of the complex CPDP (together with PDP analysis techniques) might then be used for analyzing specific parts or specific behaviors of the CPDP. In this scenario, bisimulation (which is for CPDPs defined in (Strubbe and van der Schaft 2005)) can also be an interesting concept.

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