General Stochastic Hybrid Systems: Modelling and Optimal Control

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Abstract— **We develop a model for General Stochastic Hybrid Systems (GSHS) which is a generalization of Piecewise-Deterministic Markov Processes (PDMP), introduced by Davis and stochastic hybrid systems proposed by Hu, Lygeros and Sastry. This model possesses certain desirable properties, as the strong Markov property and the càdlàg property. Extending results available for PDMP, we develop the extended generator formula and the differential formula for GSHS. Then we investigate the dynamic programming for GSHS, using the differential formula.**

I. Introduction

In the face of growing complexity of control systems, stochastic modelling has got a crucial role. Indeed, stochastic techniques for modelling control and hybrid systems have attracted attention of many researchers and constitute one of the hottest issues in contemporary high level research. As a consequence researchers all over the world have united their efforts in the framework of an international projects like Columbus [6], Hybridge [10] and al. This paper presents some results of research activity in the Columbus project, that aims to form a bridge between the US and the European control research communities. The project scope is to develop a methodology for the design of embedded controllers for safety critical systems, with particular emphasis on air-traffic control, flight control and automotive control.

Some of the most used stochastic processes are the piecewise-deterministic Markov processes (PDMP), introduced in [7], and applied by us to hybrid system modelling in [3]. The other modelling approaches are those presented in [9], [8], see [11] for quick presentation and comparisons. In the paper [4] we have proposed a very general formal model for stochastic hybrid systems (SHS) extending the model from [9], where the deterministic differential equations for the continuous flow are replaced by their stochastic counterparts, and the reset maps are generalized to (state-dependent) distributions that define the probability density of the state after a discrete transition. In this model transitions are always triggered by deterministic conditions (guards) on the state.

In this paper we propose a new model for General Stochastic Hybrid Systems (GSHS), which is a generalization both of PDMP and SHS. The class of GSHS allows: 1. Diffusion processes in the continuous evolution; 2. Spontaneous discrete transitions (according to a transition rate); 3. Forced transitions (driven by a boundary hitting time); 4. Probabilistic reset of the hybrid state as a result of discrete transitions.

The difference between GSHS and PDMP is that for GSHS between two consecutive jumps the process is a diffusion while for PDMP the inter-jumps motion is deterministic, according to a vector field. GSHS are, in fact, a kind of extended SHS for which the transitions between modes are triggered by some stochastic event (boundary hitting time and transition rate).

The paper is structured as follows. The next section introduces the mathematical model, which is required by some safety critical situations in air traffic control (ATC). In section 3 some useful and necessary properties of our model are established. Mainly, we prove the expression of the process generator. Based on this we derive the differential formula for GSHS. In section 4 we put the bases of the dynamic programming for our model. The results in this section are based on the GSHS differential formula obtained in the section 3. The conclusions of our work are drawn in the final section.

II. The Mathematical Model

Motivation

In air traffic management (ATM), the following safety critical situations have been identified [11]: vertical crossings; overtake manoeuvres in unmanaged airspace; ATC sector transitions; missed approaches. The modelling of these situations by different stochastic hybrid system models described in [11] leads to the necessity to develop further a more general class of stochastic hybrid processes than those found in the literature. This is because: 1. Different types of models seem to be needed to capture the different situations. This implies that a number of different techniques and tools must be mastered to be able to deal with all the cases of interest. If a GSHS framework were available the process would be more efficient, since a single set of results, simulation procedures, etc. could be used in all cases. 2. Certain situations, such as vertical crossings during descent and missed approaches due to runway incursions, would be more accurately modelled by GSHS.

The above discussion also suggests that none of the safety critical situations seems to require resetting the continuous state of the system during discrete transitions. It should be

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noted, however, that this may depend on the coordinate frames used in the definition of the continuous state. For example, if the positions of all aircraft are given in a global coordinate frame, then the continuous state will remain constant during discrete transitions. If on, the other hand, the positions of aircraft are given in coordinates relative to their flight plan or to one another, then the continuous state may experience discrete transitions whenever aircraft reach way points, execute turns, etc.

Description

General Stochastic Hybrid Systems (GSHS) are a class of non-linear stochastic continuous-time hybrid dynamical systems. GSHS are characterized by a hybrid state defined by two components: a continuous state (denoted by x) and a discrete state (denoted by i). The continuous state evolves in given modes X^i (open subsets of Euclidean space) and the discrete variable belongs to a countable set Q. The continuous state is governed by a stochastic differential equation (SDE) that depends on the hybrid state. The discrete dynamics produces transitions in both (continuous and discrete) state variables. Transitions occur either when the continuous state hits the boundary of the state space (forced transitions) or according to a probability law (spontaneous transitions). Whenever a transition occurs, the hybrid state is reset instantaneously to a new value, according to a probability law depending on the pre-jump location. A sample trajectory has the form $(q_t, x_t, t \geq 0)$, where $(x_t, t \geq 0)$ is piecewise continuous and $q_t \in Q$ is piecewise constant. Let $0 < T_1 < T_2 < \ldots < T_i < T_{i+1} < \ldots$ be the sequence of jump times.

State space

Let Q be a countable set of discrete states and let $d : Q \to \mathbb{N}$ and $\mathcal{X} : Q \to \mathbb{R}^{d(\cdot)}$ be two maps assigning to each discrete state $i \in Q$ an open subset $X^i \subseteq \mathbb{R}^{d(i)}$. We call the set $X(Q, d, \mathcal{X}) = \bigcup_{i\in Q}\{i\} \times X^i$ the hybrid state space of the GSHS and $x = (i, x^i) \in X(Q, d, \mathcal{X})$ the hybrid state. The closure of the hybrid state space will be $\overline{X} = X \cup \partial X$, where $\partial X = \bigcup_{i\in Q}\{i\} \times \partial X^i$.

It is clear that, for each $i \in Q$, the state space X^i is a Borel space (homeomorphic to a Borel subset of a complete separable metric space). It is possible to define a metric ρ on X in such a way the restriction of ρ to any component X^i is equivalent to the usual Euclidean metric [7]. Then $(X, \mathcal{B}(X))$ is a Borel space. Moreover, X is a homeomorphic with a Borel subset of a compact metric space (Lusin space) because it is a locally compact Hausdorff space with countable base (see [7] and the references therein).

Construction

Consider a function $m : Q \to \mathbb{N}$ and two functions $b : Q \times X^{(\cdot)} \to \mathbb{R}^{d(\cdot)}$, $\sigma : Q \times X^{(\cdot)} \to \mathbb{R}^{d(\cdot)\times m(\cdot)}$.

Assumption 1: For all i the functions $b(i, \cdot) : X^i \to \mathbb{R}^{d(i)}$ and $\sigma(i, \cdot) : X^i \to \mathbb{R}^{d(i)\times m(i)}$ are bounded and Lipschitz continuous.

This assumption ensures the existence and uniqueness of the solution of the SDE

$$dx(t) = b(i, x(t))dt + \sigma(i, x(t))dW_t, \qquad (1)$$

for any $i \in Q$ (Th.6.2.2. in [1]). $(W_t, t \geq 0)$ is an $m(i)$-dimensional standard Wiener process in a complete probability space. Equation (1) defines a family of diffusion processes $\mathbb{M}^i = (\Omega^i, \mathcal{F}^i, \mathcal{F}^i_t, x^i_t, \theta^i_t, P^i, P^i_{x^i})$, $i \in Q$ with state spaces $\mathbb{R}^{d(i)}$, $i \in Q$. For each $i \in Q$ we have: $(\Omega^i, \mathcal{F}^i, P^i)$ -the underlying probability space, $\mathcal{F}^i_t$ -the natural filtrations, θ^i_t the shift operator, $P^i_{x^i}$ probabilities on the trajectories, with the usual meaning, as in the theory of Markov processes [7].

The switching mechanism between the diffusions is governed by two functions: a jump rate $\lambda : X \to \mathbb{R}_+$ and a transition measure $R : \overline{X} \times \mathcal{B}(X) \to [0, 1]$.

Assumption 2: (i) $\lambda : X \to \mathbb{R}_+$ is a measurable function such that $t \to \lambda(x^i_t(\omega_i))$ is integrable on $[0, \varepsilon(x^i))$, for some $\varepsilon(x^i) > 0$ for each $x^i \in X^i$ and each ω_i starting at x^i. (ii) For each $i \in Q$ the restriction of λ to X^i is bounded. Let $c^i = \sup_{x^i\in X^i} \lambda(x^i)$. (iii) For all $A \in \mathcal{B}(X)$, $R(\cdot, A)$ is measurable and for all $x \in \overline{X}$ the function $R(x, \cdot)$ is a probability measure.

Since $\overline{X}$ is a Borel space, it is homeomorphic to a subset of the Hilbert cube[1], $\mathcal{H}$ (Urysohn's theorem, Prop. 7.2 [2]). Therefore, its space of probabilities is homeomorphic to the space of probabilities of the corresponding subset of $\mathcal{H}$ (Lemma 7.10 [2]). There exists a measurable function $\digamma : \mathcal{H} \times \overline{X} \to X$ such that $R(x, A) = \mathfrak{p}\digamma^{-1}(A)$, $A \in \mathcal{B}(X)$, where $\mathfrak{p}$ is the probability measure on $\mathcal{H}$ associated to $R(x, \cdot)$ and $\digamma^{-1}(A) = \{\boldsymbol{\omega} \in \mathcal{H} | \digamma(\boldsymbol{\omega}, x) \in A\}$. The measurability of such a function is guaranteed by the measurability properties of the transition measure R.

The sample path of the stochastic process $(x_t)_{t>0}$ with values in X, starting from a fixed initial point $x_0 = (i_0, x^{i_0}) \in X$ is defined in a similar manner as PDMP [7]. We have to precise, from the beginning, that the above recipe gives a sample path of GSHS starting with a initial diffusion path whose starting point is x_0. An arbitrary point x_0 does not define in a unique way a diffusion path!

Let x^i_t be a sample path of the diffusion process (1) which starts at (i, x^i) and let ω_i be the associated event in the underlying probability space $(\Omega^i, \mathcal{F}^i, P^i)$. Let $t_*(\omega_i)$ be the first exit time of x^i_t from the set X^i. Define a function $F : \mathbb{R}_+ \times \Omega^i \to [0, 1]$ by

$$F(t, \omega_i) = I_{(t<t_*(\omega_i))} \exp(-\int_0^t \lambda(i, x^i_s(\omega_i)))ds. \qquad (2)$$

Using this function we define a stopping time S^i associated to the diffusions (x^i_t). In other words, F can be thought of as the *survivor function* for the stopping time. Obviously, the stopping time S^i is the minimum of two stopping times: the first exit time from X^i and the stopping time with the exponential survivor function equal to $\exp(-\int_0^t \lambda(i, x^i_s(\omega_i)))ds$,

[1] $\mathcal{H}$ is the product of countably many copies of $[0, 1]$.

i.e.

$$\begin{aligned} S^i(\omega_i) &= \inf\{t>0|F(t,\omega_i)\le e^{-c^i t}\}, \text{ or} \\ P^i[S^i &> t] = P^i\{\omega_i|F(t,\omega_i)\ge e^{-c^i t}\}. \end{aligned}$$

To construct the stochastic process associated with the GSHS we define the event ω and the associated sample path inductively. Select an initial hybrid state $(i_0, x_0^{i_0})$. Take a solution of (1) starting at $x_0^{i_0}$ and let ω_{i_0} be the associated event in the underlying probability space. Define the first jump time of the process $T_1(\omega) = S^{i_0}(\omega_{i_0})$. Define the sample path $x_t(\omega)$ up to the first jump time by: (i) If $T_1(\omega)=\infty$ then $x_t(\omega) = (i_0, x_t^{i_0}(\omega_{i_0}))$, for all $t\ge 0$. (ii) If $T_1(\omega)<\infty$, then $x_t(\omega) = (i_0, x_t^{i_0}(\omega_{i_0}))$ for all $0\le t<T_1(\omega)$ and $x_{T_1}(\omega) = F(\boldsymbol{\omega}, (i_0, x_{T_1}^{i_0}(\omega_{i_0})))$.

The process restarts from $x_{T_1}(\omega) = (i_1, x_1^{i_1})$ according to the same recipe, using now the process $(x_t^{i_1})$. Thus if $T_1(\omega)<\infty$ we define the next jump time $T_2(\omega) = T_2(\omega_{i_0}, \omega_{i_1}) = T_1(\omega_{i_0}) + S^{i_1}(\omega_{i_1})$. The sample path $x_t(\omega)$ between the two jump times is defined by: (i) If $T_2(\omega)=\infty$ then $x_t(\omega) = (i_1, x_{t-T_1}^{i_1}(\omega))$ for all $t\ge T_1(\omega)$. (ii) If $T_2(\omega)<\infty$ then $x_t(\omega) = (i_1, x_t^{i_1}(\omega))$ for all $0\le T_1(\omega)\le t<T_2(\omega)$ and $x_{T_2}(\omega) = F(\boldsymbol{\omega}, (i_1, x_{T_2}^{i_1}(\omega)))$. And so on.

We denote by $N_t(\omega) = \sum_k I_{(t\ge T_k)}$ the number of jump times in the interval $[0,t]$. To eliminate pathological solutions that take an infinite number of discrete transitions in a finite amount of time (known as Zeno solutions) we impose the following assumption.

Assumption 3: For every starting point $x\in X$, $EN_t<\infty$, for all $t\in\mathbb{R}_+$.

Formal Definitions

Definition 1: A GSHS is a collection $H = ((Q,d,m,\mathcal{X}), b, \sigma, Init, \lambda, R)$ where
- Q is a countable set of discrete variables;
- $d: Q\to\mathbb{N}$ gives the dimensions of the modes;
- $m: Q\to\mathbb{N}$ gives the dimension of the Weiner processes that govern the continuous state evolution;
- $\mathcal{X}: Q\to\mathbb{R}^{d(\cdot)}$ maps each $q\in Q$ into an open subset X^q of $\mathbb{R}^{d(q)}$;
- $b: X(Q,d,\mathcal{X})\to\mathbb{R}^{d(\cdot)}$ is a vector field;
- $\sigma: X(Q,d,\mathcal{X})\to\mathbb{R}^{d(\cdot)\times m(\cdot)}$ is a $X^{(\cdot)}$-valued matrix;
- $Init: \mathcal{B}(X)\to[0,1]$ is an initial probability on X;
- $\lambda: \overline{X}(Q,d,\mathcal{X})\to\mathbb{R}^+$ is a transition rate function;
- $R: \overline{X}\times\mathcal{B}(\overline{X})\to[0,1]$ is a transition measure.

Now we can define the GSHS execution:

Definition 2: A stochastic process $x_t = (q(t), x(t))$ is called a GSHS execution if there exists a sequence of stopping times $T_0 = 0 < T_1 < \ldots$ such that $\forall k\in\mathbb{N}$,
- $x_0 = (q_0, x_0^{q_0})$ is a $Q\times X$-valued random variable extracted according to the probability measure $Init$;
- For $t\in[T_k, T_{k+1})$, $q_t = q_{T_k}$ is constant and $x(t)$ is a (continuous) solution of the SDE:

$$dx(t) = b(q_{T_k}, x(t))dt + \sigma(q_{T_k}, x(t))dW_t \tag{3}$$

where W_t is the m-dimensional standard Wiener;
- $T_{k+1} = T_k + S^{i_k}$ where S^{i_k} is chosen according with the survivor function (2).
- The probability distribution of $x(T_{k+1})$ is governed by the law $R\left((q_{T_k}, x(T_{k+1}^-)), \cdot\right)$.

The executions of the GSHS can be thought of as being generated by the following algorithm.

Algorithm 1 (GSHS Executions): **set** $T=0$
select X-valued random variable $\widehat{x}$ according to $Init$
repeat
  set $i = \mathcal{X}^{-1}(\widehat{x})$
  select $\mathbb{R}^+$-valued random variable $\widehat{S}$ such that
    $\widehat{S} = \inf\{t>0|F(t,\cdot)\le e^{-c^i t}\}$
  set x_t as solution of (1) with initial condition
    equal to $\widehat{x}$, for all $t\in[T, T+\widehat{S})$
  select X-valued random variable $\widehat{x}$
    according to $R(., x_{\widehat{S}})$
  set $T = T+\widehat{S}$
until true

All random extractions in Algorithm 1 are assumed to be independent.

III. Model Properties

In [5] we proved that, under Assumptions 1, 2 and 3, any GSHS defines: 1. a Borel right process; 2. a càdlàg process, i.e. for all ω the trajectories $t\mapsto x_t(\omega)$ are right continuous on $[0,\infty)$ with left limits on $(0,\infty)$.

The Process Generator

We denote by $\mathcal{B}_b(X)$ the set of all bounded measurable functions $f: X\to\mathbb{R}$. This is a Banach space under the norm $\|f\| = \sup_{x\in X}|f(x)|$. Let (P_t) be the semigroup of the whole Markov process (x_t), i.e. $P_t f(x) = E_x f(x_t) = E_x\{f(x_t)|t<\zeta\}$, where f is bounded $\mathcal{B}$-measurable function and ζ is the lifetime when the process retires to Δ (where Δ is the cemetery point for X, i.e.an adjoined point to X, whose existence is assumed in order to have a probabilistic interpretation of $P_x(x_t\in X)<1$), i.e. $\zeta = \inf\{t|x_t=\Delta\}$. Associated with the semigroup (P_t) is its *strong generator* which is the 'derivative' of P_t at $t=0$. Let $D(L)\subset\mathcal{B}_b(X)$ be the set of functions f for which the following limit exists $\lim_{t\searrow 0}\frac{1}{t}(P_t f - f)$ and denote this limit Lf. This refers to convergence in the norm $\|\cdot\|$, i.e. for $f\in D(L)$ we have $\lim_{t\searrow 0}||\frac{1}{t}(P_t f - f) - Lf|| = 0$. Specifying the domain $D(L)$ is an essential part of specifying L.

Proposition 1 (Martingale property): [7] For $f\in D(L)$ we define the real-valued process $(C_t^f)_{t\ge 0}$ by

$$C_t^f = f(x_t) - f(x_0) - \int_0^t Lf(x_s)ds. \tag{4}$$

Then for any $x\in X$, the process $(C_t^f)_{t\ge 0}$ is a martingale on $(\Omega,\mathcal{F},\mathcal{F}_t,P_x)$.

There may be other functions f, not in $D(L)$, for which something akin to (4) is still true. In this way we get the notion of *extended generator* of the process.

Let $D(\widehat{L})$ be the set of measurable functions $f: X\to\mathbb{R}$ with the following property: there exists a measurable function $h : X \to \mathbb{R}$ such that $t \to h(x_t)$ is integrable $P_x - a.s.$ for each $x \in X$ and the process

$$C_t^f = f(x_t) - f(x_0) - \int_0^t h(x_s)ds$$

is a local martingale. Then we write $h = \widehat{L}f$ and call $(\widehat{L}, D(\widehat{L}))$ the extended generator of the process (x_t).
Following [7], for $A \in \mathcal{B}(\overline{X})$ define p, p^* and $\widetilde{p}$ as follows:

$$p(t,A) = \sum_{k=1}^{\infty} I_{(t \geq T_k)} I_{(x_{T_k} \in A)};\ p^*(t) = \sum_{t \geq T_k} I_{(x_{T_k^-} \in \partial X)};$$
$$\widetilde{p}(t,A) = \int_0^t R(x_s, A)\lambda(x_s)ds + \int_0^t R(A, x_{s-})dp^*(s) = \sum_{T_k \leq t} R(x_{T_k-}, A).$$

Note that p, p^* are counting processes, $p^*(t)$ is counting the number of jumps from the boundary of the process (x_t). $\widetilde{p}(t,A)$ is the compensator of $p(t,A)$ (see [7] for more explanations). The process $q(t,A) = p(t,A) - \widetilde{p}(t,A)$ is a local martingale.

Given a function $f \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ and a vector field $b : \mathbb{R}^n \to \mathbb{R}^n$, we use $\mathcal{L}_b f$ to denote the Lie derivative of f along b given by $\mathcal{L}_b f(x) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x) f_i(x)$. Given a function $f \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$, we use $\mathbb{H}^f$ to denote the Hamiltonian operator applied to f, i.e. $\mathbb{H}^f(x) = (h_{ij}(x))_{i,j=1\ldots n} \in \mathbb{R}^{n \times n}$, where $h_{ij}(x) = \frac{\partial^2 f}{\partial x_i \partial x_j}(x)$. A^T denotes the transpose matrix of a matrix $A = (a_{ij})_{i,j=1\ldots n} \in \mathbb{R}^{n \times m}$ and $Tr(A)$ denotes its trace.

Theorem 2: Let H be an GSHS as in definition 1. Then the domain $D(L)$ of the extended generator L of H, as a Markov process, consists of those measurable functions f on $X \cup \partial X$ satisfying:
1. $f : \overline{X} \to \mathbb{R}$, $\mathcal{B}-$measurable; $t \to f(x_t^i(\omega_i))$ have second order derivatives on $[0, S^i(\omega_i))$, for all $\omega_i \in \Omega^i$;
2. the boundary condition

$$f(x) = \int_{\mathbb{X}} f(y)R(x,dy)),\ x \in \partial X;$$

3. $Bf \in L_1^{loc}(p)$ (see the definition in [7])[2] where

$$Bf(x,s,\omega) := f(x) - f(x_{s-}(\omega)).$$

For $f \in D(L)$, Lf is given by

$$Lf(x) = L_{cont}f(x) + \lambda(x)\int_{\overline{X}}(f(y) - f(x))R(x,dy) \quad (5)$$

where:

$$L_{cont}f(x) = \mathcal{L}_b f(x) + \frac{1}{2}Tr(\sigma(x)\sigma(x)^T \mathbb{H}^f(x)). \quad (6)$$

Proof: Let $(\widetilde{L}, D(\widetilde{L}))$ be the extended generator of (x_t). We want to show that $(\widetilde{L}, D(\widetilde{L})) = (L, D(L))$.

[2] f is in $L_1^{loc}(p)$ if for some sequence of stopping times $\sigma_n \uparrow \infty$

$$E_x \sum_i |f(x_{T_i \wedge \sigma_n}) - f(x_{T_i \wedge \sigma_n -})| < \infty$$

Suppose first that f satisfies 1-3. Then $Bf \in L_1^{loc}(\widetilde{p})$ and $\int_{[0,t]\times\overline{X}} Bf d\widetilde{p} = I_1 + I_2$, where

$$I_1 = \int_{[0,t]}\int_{\overline{X}}(f(y) - f(x_s))R(x_s,dy)\lambda(x_s)ds$$
$$I_2 = \int_{[0,t]}\int_{\overline{X}}(f(y) - f(x_{s-}))R(x_{s-},dy)dp^*(s).$$

Now the support of p^* is contained in the countable set $\{s : x_{s-} \in \partial X\}$ and because of the boundary condition 2. the second integral I_2 vanishes. Thus

$$\int_{[0,t]\times\overline{X}} Bf dq = \sum_{T_k \leq t} (f(x_{T_k}) - f(x_{T_k-})) - \int_{[0,t]}\int_{\overline{X}}(f(y) - f(x_s))R(x_s,dy)\lambda(x_s)ds.$$

This is a local martingale because of condition 3. Let T_m denote the last jump time prior or equal to t. Then

$$\sum_{T_k \leq t} (f(x_{T_k}) - f(x_{T_k-})) = \{f(x_t) - f(x_{Tm})\} + S_m$$

where

$$S_m = \sum_{k=1}^{m} (f(x_{T_k}) - f(x_{T_{k-1}}))\} - \{f(x_t) - f(x_{Tm}) + \sum_{k=1}^{m} (f(x_{T_k-}) - f(x_{T_{k-1}}))\}.$$

The first bracketed term on the right is equal to $f(x_t) - f(x)$. Note that $x_{T_k-} = x^{i_{k-1}}_{T_k - T_{k-1}}$, if $x_{T_{k-1}} = (i_{k-1}, x^{i_{k-1}}_{k-1})$. Then Itô-formula gives the second term

$$f(x_{T_k-}) - f(x_{T_{k-1}}) = \int_{T_{k-1}}^{T_k} L_{cont}f(x_s)ds + \int_{T_{k-1}}^{T_k} < \sigma(x_s), \nabla f(x_s) > dW(s).$$

The second term is therefore equal to $\int_0^t L_{cont}f(x_s)ds + \int_0^t < \sigma(x_s), \nabla f(x_s) > dW(s)$ and we obtain

$$C_t^f := f(x_t) - f(x) - \int_0^t Lf(x_s)ds = \int_0^t < \sigma(x_s), \nabla f(x_s) > dW(s) + \int_{[0,t]\times\overline{X}} Bf dq$$

is a local martingale (the sum between a continuous martingale and a discrete martingale), where L is given by (5). Thus $f \in D(\widehat{L})$ and $\widehat{L}f = Lf$.
Conversely, suppose that $f \in D(\widehat{L})$. Then the process $M_t := f(x_t) - f(x) - \int_0^t h(x_s)ds$ is a local martingale, where $h = \widehat{L}f$. Then M_t must be the sum between a continuous martingale M_t^c and a discrete martingale M_t^d. From Th.(26.12), p.69 [7], we have $M_t^d = M_t^\rho$ for some predictable integrand $\rho \in L_1^{loc}(p)$, where
$M_t^\rho = \int_{\overline{X}\times\mathbb{R}_+} \rho I_{(s \leq t)} dq = \sum_{T_k \leq t} \rho(x_{T_k}, T_k, \omega) - \int_0^t \int_{\overline{X}} \rho(y,s,\omega)\{R(x_s,dy)\lambda(x_s)ds - R(x_{s-},dy)dp^*(s)\}$.
Since M_t^d and M_t^ρ agree, their jumps ΔM_t^d and ΔM_t^ρ must agree; these only occur when $t = T_k$ for some k and are given by: $\Delta M_t^d = f(x_t) - f(x_{t-})$; $\Delta M_t^\rho = \rho(x_t, t, \omega) - \int_{\overline{X}} \rho(y,t,\omega)R(x_{t-},dy)I_{(x_{t-} \in \partial X)}$. Thus $\rho(x_t,t,\omega) = f(x_t) - f(x_{t-})$ on the set $(x_{t-} \notin \partial X)$, which implies that $\rho(x,t,\omega) = f(x) - f(x_{t-})$ for all (x,t)

except perhaps a set to which the process 'never jumps', i.e. $G \subset \mathbb{R}_+ \times X$ such that $E_z \int_G p(dt, dx) = 0$, $\forall z \in X$. Suppose that $z = x_{t-} \in \partial X$. Then equating ΔM_t^d and ΔM_t^ρ gives $f(x_t) - f(z) = \rho(x_t, t, \omega) - \int_X \rho(y, t, \omega) R(z, dy)$ and hence $f(x) - f(z) = \rho(x, t, \omega) - \int_X \rho(y, t, \omega) R(z, dy)$, except on a set $A \in \mathcal{B}(X)$ such that $R(z, A) = 0$. Integrating both sides of the previous equality with respect to $R(z, dx)$, we obtain $\int_X f(x) R(z, dx) - f(z) = \int_X \rho(x, t, \omega) R(z, dx) - \int_X \rho(y, t, \omega) R(z, dy) = 0$. Thus f satisfies the boundary condition. For fixed z, define $\widetilde{\rho}(x, t, \omega) = \rho(x, t, \omega) - (f(x) - f(z))$.
Using the boundary condition we get $\int_X \widetilde{\rho}(y, t, \omega) R(z, dy) = \int_X \rho(y, t, \omega) R(z, dy) = \widetilde{\rho}(x, t, \omega)$. Then $\widetilde{\rho}(x, t, \omega) = \int_X \widetilde{\rho}(y, t, \omega) R(z, dy)$.
However, the right-hand side does not depend on x, and hence $\widetilde{\rho}(x, t, \omega) = u(t, \omega)$ for some predictable process u. The general expression for ρ is thus

$$\rho(x, t, \omega) = f(x) - f(x_{t-}) + u(t, \omega) I_{(x_{t-} \in \partial X)}.$$

Inserting this in the expression of M_t^ρ we find that M_t^ρ does not depend on u, then we can take $u \equiv 0$, obtaining $\rho = Bf$; hence the part 3 of theorem is satisfied.

Finally, consider the sample paths of M_t, $M_t^{Bf} + M_t^c$, for $t < T_1(\omega)$, starting at $x \in X$. We have

$$M_t = f(x_t(\omega_{i_0})) - f(x) + \int_0^t h(x_s(\omega_{i_0})) ds$$

while, because $p = p^* = 0$ on $[0, T_1)$,

$$M_t^{Bf} = -\int_{[0,t]} \int_X (f(y) - f(x_s(\omega_{i_0}))) R(x_s(\omega_{i_0}), dy) \lambda(x_s(\omega_{i_0})) ds.$$

So, since $M_t = M_t^{Bf} + M_t^c$ for all t a.s., it must be the case that $M_t = M_t^c$ for $t \in [0, T_1)$ and the generator coincides with the generator L_{cont} associated to the stochastic equation, the function $f(x_t(\omega_{i_0}))$ should have second order derivatives on $[0, T_1)$. The general case follows by concatenation. Similar calculations show that

$$M_t^{Bf} + M_t^c = f(x_t) - f(x) - \int_0^t Lf(x_s) ds, \; \forall t \geq 0$$

with L given by (5). Hence $f \in D(L)$ and $Lf = \widehat{L}f$. This completes the proof.□

The Differential Formula for GSHS

We need a further operator C, defined as follows

$$Cf(z) := \int_X f(y) R(z, dy) - f(z), z \in \partial X$$

We can state the following result, which is a simple corollary of Th.2, but which plays a fundamental role in the GSHS control theory.

Theorem 3 (GSHS Differential Formula): If f satisfies the conditions 1 and 3 of the Th.2. Then $\forall t \geq 0$

$$\begin{aligned} f_{(x_t)} - f_{(x)} &= \int_0^t Lf_{(x_s)} ds + \int_0^t \sigma_{(x_s)} \cdot \nabla f_{(x_s)} dW^+_{(s)} \\ & \int Bf \; q_{(s,du)} + \int_0^t Cf_{(x_{s-})} dp^*_{(s)} \end{aligned}$$

IV. Control of GSHS

In this section, we suppose that the state space of the GSHS is a subset X of $\mathbb{R}^d$, i.e. all the components which appear in the general definition 1 can be embedded in a possibly higher-dimensional space Euclidean space. Control arises when b, σ from (1) or other local characteristics λ or R depend on an additional control parameters. It is important to distinguish at the outset between control in the interior of the state space and control on the boundary ∂X. We suppose that the parameters associated with them take values in possibly different sets U_0, U_Γ, respectively.

Assumption 4: (i) The state space is $X_\Delta = X \cup \{\Delta\}$.
(ii) U_0, U_Γ are compact metrizable spaces.
(iii) The functions $b : X_\Delta \times U_0 \to \mathbb{R}^d$, $\sigma : X_\Delta \times U_0 \to \mathbb{R}^{d \times d}$ and $\lambda : X_\Delta \times U_0 \to \mathbb{R}_+$ are bounded and Lipschitz continuous on X, uniformly in U_0; $\lambda(\Delta, u) = b(\Delta, u) = 0$ and $\sigma(\Delta, u) = \mathbf{0}$ for all $u \in U_0$.
(iv) $R : X \times U_0 \times \mathcal{B}(X_\Delta) \to [0, 1]$ and $Q : \partial X \times U_\Gamma \times \mathcal{B}(X) \to [0, 1]$ are continuous functions such that for all $\theta \in C_b(X_\Delta)$ the maps $(x, u_0) \mapsto \int_{X_\Delta} \theta(y) R(x, u_0, dy)$ ($x \in X, u_0 \in U_0$) and $(x, u_\Gamma) \mapsto \int_X \theta(y) Q(x, u_\Gamma, dy)$ ($x \in \partial X$, $u_\Gamma \in U_\Gamma$) are Lipschitz continuous in x, uniformly in u_0 and u_Γ, respectively.

Feedback policies

The natural class of controls in Markovian optimization problems is that of feedback policies, a feedback policy v being in the present context a pair of measurable functions $v_0 : X \to U_0$ and $v_\Gamma : \partial X \to U_\Gamma$. These are described as a policy because they describe a rule of action: if the state is x, apply control $v_0(x)$; if the boundary is hit at z, apply boundary control $v_\Gamma(z)$. We now wish to construct a GSHS corresponding to a control policy (v_0, v_Γ). This policy defines a set of local characteristics $(b^v, \sigma^v, \lambda^v, R^v, Q^v)$ by the recipe: $b^v(x) = b(x, v_0(x))$, $\sigma^v(x) = \sigma(x, v_0(x))$, $\lambda^v(x) = \lambda(x, v_0(x))$, $R^v(x, dy) = R(x, dy, v_0(x))$, $Q^v(z, dy) = Q(z, dy, v_\Gamma(z))$. We construct a GSHS having these local characteristics as in section II. The problem is to choose a policy v to minimize a given *cost function*, which is assumed to be of the form $J_x(v) = E_x^v \{ \int_0^\infty l(x_t, v_0(x_t)) dt + \int_0^\infty c(x_{t-}, v_\Gamma(x_{t-})) dp^*(t) \}$, where $l : X_\Delta \times U_0 \to \mathbb{R}_+$ and $c : \partial X \times U_\Gamma \to \mathbb{R}_+$ are bounded non-negative functions. A policy $\widehat{v}$ minimizing $J_x(v)$ for all $x \in X$ over all admissible policies v is *optimal*. But, also, the admissible policies should be chosen such that between jump times, the trajectory x_t must satisfy the SDE (3). To guarantee existence and uniqueness of a solution the functions $x \mapsto b^v(x)$ and $x \mapsto \sigma^v(x)$ must satisfy the assumption 1.

We denote the set of all measurable functions $v_0 : X \to U_0$ by $\mathcal{U}_0'$ and the set of all measurable functions $v_\Gamma : \partial X \to U_\Gamma$ by $\mathcal{U}_\Gamma$. Let $\mathcal{U}_0$ be the subset of $\mathcal{U}_0'$ such that for $u_0 \in \mathcal{U}_0$ the equation (1), with b, σ defined as in assumption 4, has a unique solution. Thus $\mathcal{U}_0$ consists of those control functions for which the controlled GSHS can be constructed in the direct way as in section II. We denote $\mathcal{U}_F := \mathcal{U}_0 \times \mathcal{U}_\Gamma$. The aim is to chose a control $v \in \mathcal{U}_F$, which minimizes the cost function J_x.

Assumption 5: (i) $l : X \times U_0 \to \mathbb{R}_+$ and $c : \partial X \times U_\Gamma \to \mathbb{R}_+$ are non-negative, bounded measurable functions. (ii) For any $x \in X$, $v \in \mathcal{U}_\Gamma$ and $t > 0$, $E_x^v N_t < \infty$, where $N_t = \sum_i I_{(t \geq T_i)}$ and T_i are the jump times of the process with control v. In particular, $T_i \to \infty$ (P_x^v a.s.).

For $u_0 \in U_0$, $u_\Gamma \in U_\Gamma$ and $f \in C^2(\overline{X})$ we denote

$$\begin{aligned} L^{u_0} f(x) &= L^{u_0}_{cont} f(x) + \\ &\quad + \lambda(x, u_0) \int_{\overline{X}} (f(y) - f(x)) R(x, u_0, dy), \end{aligned} \tag{7}$$

where $x \in X$ and $L^{u_0}_{cont} f(x)$ is given by (6) with the function b, σ depending on u_0. As well, we denote

$$C^{u_\Gamma} f(x) = \int_X (f(y) - f(x)) Q(x, u_\Gamma, dy),\ x \in \partial X. \tag{8}$$

In the same manner, L^v, C^v denote (7) and (8) with $v_0(x)$ and $v_\Gamma(x)$ replacing u_0 and u_Γ, respectively, for $v = (v_0, v_\Gamma)$. Thus L^v is the generator of the controlled process with control v.

Hamilton-Jacobi-Bellman (HJB) equations

Because of boundary conditions, the HJB equation for GSHS is a pair of equations:

$$\min_{u_0 \in U_0} \{ L^{u_0} V(x) + l(x, u_0) \} = 0,\ x \in X; \tag{9}$$

$$\min_{u_\Gamma \in U_\Gamma} \{ C^{u_\Gamma} V(x) + c(x, u_\Gamma) \} = 0,\ x \in \partial X. \tag{10}$$

Theorem 4: Suppose assumptions 4 and 5 are satisfied and that

1. V is a piecewise C^2 solution of (9) and (10).
2. $\forall v \in \mathcal{U}_F$, $E_x^v \sum\limits_i I_{(t \geq T_i)} |V(x_{T_i}) - V(x_{T_i -})| < \infty$.
3. $\forall v \in \mathcal{U}_F$, $E_x^v V(x_t) \to 0$ as $t \to \infty$.
4. There exists $\widehat{v} \in \mathcal{U}_F$ such that at each $x \in X$ (resp. $x \in \partial X$) the value $\widehat{v}_0(x)$ (resp. $\widehat{v}_\Gamma(x)$) achieves the minimum in (9) (resp. in (10)).

Then $\widehat{v}$ is optimal in $\mathcal{U}_F$ and $V(x) = J_x(\widehat{v})$.

Proof: Suppose V is a solution to (9) and (10) satisfying the hypotheses of the theorem. Let $v \in \mathcal{U}_F$ be an arbitrary feedback control and let be (x_t) the corresponding controlled process. Then by differential formula

$$\begin{aligned} V_{(x_t)} - V_{(x)} &= \int_0^t L^v V_{(x_s)} ds + \int_0^t \sigma_{(x_s)} \cdot V_{(x_s)} dW_{(s)} \\ &\quad + \int_{[0,t] \times \overline{X}} B^v V dq + \int_0^t C^v V_{(x_{s_-})} dp^*_{(s)} \end{aligned}$$

Now from (9) and (10) we get $L^v V(x) \geq -l(x, u_0)$ and $C^v V(x) \geq -c(x, u_\Gamma)$. Using the condition 2 from the hypothesis of Th.2, the sum $\int_0^t \langle \sigma(x_s), V(x_s) \rangle dW(s) + \int_{[0,t] \times \overline{X}} B^v V dq$ is a martingale. Thus, taking expectations in in the previous differential formula we obtain $V(x) \leq E_x^v \{ \int_0^s l(x_s, u_0(x_s)) ds + \int_0^s c(x_{s-}, u_\Gamma(x_{s-})) dp^*(s) \} + E_x^v V(x_t)$.

Now let $t \to \infty$ and invoke condition 2 of the to conclude that $V(x) \leq J_x(v)$. Finally, let $\widehat{v}$ be the control policy referred to in condition 4 of the theorem. Then, in an analogous way as above, but with the inequality replaced by equality we get that $V(x) = J_x(\widehat{v})$. Therefore, $\widehat{v}$ is optimal.□

V. Conclusions

In this paper we review and develop a very general model for stochastic hybrid systems, proposed in [4], [5]. The model answers important practical challenges and thus needs to be explored. The generality of the model is an essential asset, as it can be instantiated with almost all stochastic hybrid system proposed in the literature. The main technical contributions of this paper are: 1. Define the model and establish some basic properties of the model (existence of solution process, Borel “right” property); 2. Define an algorithm to derive the GSHS executions; 3. Prove the expression of the process generator; 4. Give the differential formula for GSHS; 5. Define the dynamic programming for GSHS.

Further developments of our model will include two main tracks. First it is necessary a study of the reachability problem for GSHS. Second it is natural to generalize the results on relaxed controls, control via discrete-time dynamic programming, non-smooth analysis, from PDMP to GSHS.

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