Can linear stabilizability analysis be generalized to switching systems? *

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Abstract

Stabilizability and asymptotic stabilizability of switching systems is characterized in terms of the existence of controlled invariant sets and domains of attraction. Then the classical Kalman decomposition for linear dynamical system is extended to switching systems. This shows that the core issue for solving the stabilizability problem for switching systems is the stability of an autonomous (i.e. without continuous input) switching subsystem.

keywords: Switching systems, stabilizability, Kalman decomposition

1 Introduction

Hybrid systems have captured over the past few years great attention in the scientific community because of their generality and expressive power. In particular, important theoretical results have been achieved for safety problems where the control specifications require the evolution of the controlled system to stay out of sets of states called the "bad" states. A systematic procedure for solving these problems has been proposed in [12]. On the other hand, stability issues of hybrid systems have been investigated e.g. in [4], [17], [11] and references therein. A very recent paper [9] discusses the issue of uniform stability of switched linear systems.

In this paper, we focus on the subclass of hybrid systems where the continuous dynamics and the reset functions are linear and the transitions depend only on a disturbance event (switching systems). We first study the relations between stabilizability and safety properties for switching systems (a preliminary result on this equivalence can be found in [6] in the discrete time domain).

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We introduce the notion of strong safety and we show that strong safety implies the existence of a controlled safe set with non-empty interior. Then, we fully characterize controlled invariant sets and domains of attraction for switching systems. This characterization is useful for assessing stabilizability and asymptotic stabilizability of switching systems in terms of the existence of controlled invariant sets or domains of attraction. We then extend the classical Kalman decomposition for linear dynamical system to switching systems by also showing that asymptotic stabilizability of a switching system can be reduced to the stability properties of a particular autonomous switching system extracted from the original one. Some recent results on this issue can be found in [15].

2 Definitions

We consider the subclass of hybrid systems where the continuous dynamics and the reset functions are linear and the transitions depend only on a disturbance event (switching transitions). This sub-class can be viewed as a particular case of general hybrid systems as defined in [16] The continuous state space associated with each discrete state is characterized by its own dimension that is not necessarily the same for all the discrete states.

Definition 1 *A linear continuous time switching system* $\mathcal{S}$ *is a tuple* $(\Xi, \mathbf{Q}, \mathbf{V}, \mathbb{R}^m, \mathbf{S}_C, S, E, R)$ *where:*

- $\Xi = \bigcup_{i \in J} \{q_i\} \times \mathbb{R}^{n_i}$ *is the hybrid state space;* $J = \{1, 2, \ldots, N\}$.
- $\mathbf{Q} = \{q_i, i \in J\}$ *is the set of discrete states;*
- $\mathbf{V}$ *is the finite set of discrete disturbances;*
- $\mathbb{R}^m$ *is the continuous input space; we denote by* $\mathcal{U}$ *the class of piecewise continuous control functions* $u : \mathbb{R} \to \mathbb{R}^m$*;*
- $\mathbf{S}_C$ *is a subclass of linear, continuous time dynamical systems . The system* $S_h \in \mathbf{S}_C$ *is defined by the equation:*

  $$\dot{x}(t) = A_h x(t) + B_h u(t)$$

  with $h \in J$, $A_h \in \mathbb{R}^{n_h \times n_h}$, $B_h \in \mathbb{R}^{n_h \times m}$*;*
- $S : \mathbf{Q} \to \mathbf{S}_C$ *is a mapping that associates a continuous time (resp. a discrete time) dynamical system to every discrete state; for simplicity* $S(q_i) = S_i$.
- $E \subset \mathbf{Q} \times \mathbf{V} \times \mathbf{Q}$ *is a collection of discrete transitions;*
- $R : E \times \Xi \to \Xi$ *is the linear reset function, i.e. given* $e = (q_i, \sigma, q_j) \in E$ *and* $\xi = (q_i, x) \in \Xi$, $R(e, \xi) = (q_j, M_{ij} x)$, $M_{ij} \in \mathbb{R}^{n_j \times n_i}$.

The triple $(\mathbf{Q}, \mathbf{V}, E)$ can be viewed as a Finite State Machine (FSM) having state set $\mathbf{Q}$ and transitions defined by E. This FSM characterizes the structure of the discrete transitions and w.l.o.g. is supposed to be connected.

Following [12], we recall that a hybrid time basis τ is an infinite or finite sequence of sets I_j satisfying the following conditions: $I_j = \left\{t \in \mathbb{R} : t_j \leq t \leq t'_j\right\}$; if $card(\tau) = L + 1 < \infty$, then I_L may be of the form $I_L = \{t \in \mathbb{R} : t_L \leq t < \infty\}$ and $t'_L = \infty$; for all j, $t_j \leq t'_j$ and for $j > 0$, $t_j = t'_{j-1}$. Denote by $\mathcal{T}$ the set of all hybrid time bases. The switching system temporal evolution is then defined as follows:

Definition 2 *(Switching System Execution) An execution χ of a switching system $\mathcal{S}$ is a collection $\chi = (\xi_0, \tau, \sigma, u, \xi)$ with $\xi_0 = (\widehat{q}, x_0) \in \Xi$, $\tau \in \mathcal{T}$, $\sigma : \mathbb{N} \to \mathbf{V}$, $u \in \mathcal{U}$, $\xi : \mathbb{R} \times \mathbb{N} \to \Xi$. Setting $\xi(t, j) = (q(j), x(t, j))$, $\forall t \in I_j$ the function ξ is defined as follows:*

$$\begin{array}{l} \xi(t_0, 0) = \xi_0 = (\widehat{q}, x_0); \\ \xi(t_{j+1}, j+1) = R(e_j, \xi(t'_j, j)); \\ e_j = (q(j), \sigma(j), q(j+1)) \in E; \\ x(t, j) = x_h(t, x(t_j, j), u) \end{array}$$

where $x_h(t, x(t_j, j), u)$ is the (unique) solution at time t of the dynamical system $S_h = S(q(j))$, with initial time t_j, initial condition $x(t_j, j)$ and input $u|_{[t_j, t)}$.

We will say that $\chi = (\xi_0, \tau, \sigma, u, \xi)$ is an execution of $\mathcal{S}$ with initial state $\xi_0 \in \Xi$.

Throughout the paper we make the following assumptions:

Assumption 1 (Minimum dwell time) Given the switching system $\mathcal{S}$, $t'_j - t_j \geq \delta_m > 0$, $\forall j = 0, 1, ..., L$, for any hybrid execution.

Assumption 1 implies that all executions are non-Zeno. Then, all executions may be assumed w.l.o.g. to be infinite.

We can now define the class of state feedback functions for a switching system.

Definition 3 *(Hybrid state feedback) Given $\mathcal{S}$, a state feedback function is a function $\varphi : \Xi \to \mathbb{R}^m$. A closed loop execution of $\mathcal{S}$ is a hybrid execution where $u(t) = \varphi(\xi(t, j))$, $t \in [t_j, t'_j)$, $j \in \{0, 1, ..., L\}$.*

The following definition generalizes some well-known properties of subsets of finite dimensional vector spaces to subsets of the hybrid state space. Recall that a $C-$set in $\mathbb{R}^n$ is a convex and compact set with the origin as an interior point.

Definition 4 *Consider a set $\Pi = \bigcup_{i \in J' \subset J} \{q_i\} \times \Sigma_i \subset \Xi$. Given some $\alpha \in \mathbb{R}$, we write $\alpha\Pi := \bigcup_{i \in J' \subset J} (\{q_i\} \times \alpha\Sigma_i)$. The set Π is*

- symmetric, (convex, polyhedral) if Σ_i is symmetric (convex, polyhedral resp.) $\forall i \in J'$;
- a $C-$set if $J' = J$ and Σ_i is a $C-$set in $\mathbb{R}^{n_i}$, for any $i \in J$;
- a C_s-set if $J' = J$ and Σ_i is a convex symmetric bounded set with the origin as an interior point in $\mathbb{R}^{n_i}$, for any $i \in J$.

We are now ready to define stabilizability and asymptotic stabilizability for linear switching systems. Let $\mathcal{B} = \bigcup_{i \in J} \{q_i\} \times \mathcal{B}_i$, $\mathcal{B}_i = \{x \in \mathbb{R}^{n_i} : \|x\| \leq 1\}$.

Definition 5 *Given* $\mathbf{Q}_0 \subset \mathbf{Q}$*, let* $\mathcal{B}_0 = \bigcup_{i \in J_0} \{q_i\} \times \mathcal{B}_i \subset \Xi$*,* $J_0 = \{i \in J : q_i \in \mathbf{Q}_0\}$*. A system* $\mathcal{S}$ *is* $\mathbf{Q}_0-$ *stabilizable by state feedback if for all* $\varepsilon > 0$ *there exist* $\rho > 0$ *and a state feedback function* φ *such that for any* $\xi_0 \in \rho\mathcal{B}_0$*,* $\xi(t,j) \in \varepsilon\mathcal{B}$*,* $\forall t \geq 0$*,* $\forall j \geq 0$*, for all closed loop executions of the switching system, with initial state* ξ_0*. A system* $\mathcal{S}$ *is* $\mathbf{Q}_0-$ *asymptotically stabilizable by state feedback if it is* $\mathbf{Q}_0-$*stabilizable by state feedback and the state feedback function* φ *is such that, for any arbitrarily small* $\varepsilon > 0$ *and for any* $\xi_0 \in \alpha\mathcal{B}_0$*, with arbitrarily large* α*, there exists a finite time* $\mathbf{t}$ *such that* $\xi(t,j) \in \varepsilon\mathcal{B}$*,* $\forall t \in I_j \cap [\mathbf{t}, \infty)$*,* $\forall j \geq \mathbf{j}$*, for all closed loop executions of the switching system with initial state* ξ_0*, where* $\mathbf{j} = \inf j : \mathbf{t} \in I_j$*.*

If the above definition holds with $\mathbf{Q}_0 = \mathbf{Q}$, the system will be called stabilizable (asymptotically stabilizable) by state feedback . For shortness, in the sequel we'll omit the specification "by state feedback" for the stabilizability.

3 Lyapunov-like analysis

In this section, stabilizability properties of a switching system will be analyzed and their precise relationship with safety properties will be established. Therefore we need now some more preliminaries.

Hybrid control problems with safety specifications are described by giving a set of good states within which the controlled switching system should evolve.

A switching system with constraints

$$\xi(t,j) \in \Omega = \bigcup_{i \in J} \{q_i\} \times \Omega_i \subset \Xi \qquad \forall t \in I_j, \forall I_j \in \tau, \forall \tau \in \mathcal{T} \tag{1}$$

will be called **state constrained switching system**.

The definitions below generalize to hybrid systems the concepts of controlled invariant set, domain of attraction (see [3]) and controlled safe set (see [2]).

Definition 6 *A set* $\Pi = \bigcup_{i \in J} (\{q_i\} \times \Sigma_i) \subset \Xi$ *is* ***controlled invariant*** *for a switching system* $\mathcal{S}$ *if there exists a state feedback function* φ *such that for any closed loop execution of* $\mathcal{S}$ *with initial state in* Π*,* $\xi(t,j) \in \Pi$*,* $\forall t \in I_j$*,* $\forall j \in$

$\{0,1,...,L\}$. *We say that* φ *makes the set* Π *invariant for system* $\mathcal{S}$. *Given a real* $\beta > 0$, *a set* $\Pi \subset \Xi$ *is a* $\beta-$***domain of attraction*** *for* $\mathcal{S}$ *if there exists a state feedback function* φ *such that for any closed loop execution of* $\mathcal{S}$ *with initial state in* Π, $\xi(t,j) \in e^{-\beta t}\Pi$, $\forall t \in I_j$, $\forall j \in \{0,1,...,L\}$.

Definition 7 *Consider a state constrained switching system* $\mathcal{S}$. *A set* $\Pi \subset \Omega \subset \Xi$ *is* ***controlled safe*** *for* $\mathcal{S}$ *if a state feedback function* φ *exists such that the constraints* (1) *are satisfied for any closed loop execution of* $\mathcal{S}$ *with initial state in* Π. *We say that* φ *makes the set* Π *safe for the system* $\mathcal{S}$.

Given a state constrained switching system $\mathcal{S}$, the class of controlled safe sets has a maximal element, the so-called **maximal safe set** of $\mathcal{S}$, denoted by $X = \bigcup_{i \in J' \subset J} \{q_i\} \times X_i$, where each set X_i is controlled!!! safe for the dynamical system S_i, with state constraining set $\Omega_i \subset \mathbb{R}^{n_i}$.

The constrained switching system can be classified on the basis of the maximal safe set properties according to the following:

Definition 8 *A system* $\mathcal{S}$ *is* ***safe*** *(resp.* ***strongly safe****) with respect to constraints* (1) *if* $X \neq \emptyset$ *(resp. if* X *has nonempty interior).*

Remark 9 *By Definitions 6 and 7, a controlled invariant subset of* Ω *for a switching system* $\mathcal{S}$ *is also controlled safe for* $\mathcal{S}$, *under constraints* (1) *and the maximal safe set coincides with the maximal controlled invariant subset of the state constraining set.*

Throughout this section, the following assumption holds:

Assumption 2 The state constraining set $\Omega \subset \Xi$ is a $C-$set .

Under Assumption 2, any switching system is safe, because the origin is safe with respect to any $q \in \mathbf{Q}$. Therefore we'll be interested in characterizing the strong safety property.

In what follows $J_i = \{j : (q_i, \sigma, q_j) \in E$, for some $\sigma \in \mathbf{V}\}$. Given some set $\Sigma \subset \mathbb{R}^{n_i}$, $\mathcal{I}_i(\Sigma)$ and $\mathcal{I}_{i\beta}(\Sigma)$ denote the maximal controlled invariant subset of Σ and the maximal $\beta-$domain of attraction in Σ for system S_i, respectively. Obviously $\mathcal{I}_{i0}(\Sigma) = \mathcal{I}_i(\Sigma)$.

The following theorem establishes the equivalence between stabilizability and safety.

Theorem 10 *A switching system* $\mathcal{S}$ *is stabilizable if and only if it is strongly safe with respect to constraints* (1), *for any given* $C-$*set* Ω.

Proof. The sufficiency is obvious. The necessity follows from the fact that, by definition of stabilizability, $\forall \rho > 0\ \exists \varepsilon > 0$ such that $\varepsilon \mathcal{B}_i$ is a safe set with respect to $q_i \in \mathbf{Q}$, $\forall i \in J$, and with respect to any $C-$set Ω, containing the set $\rho\mathcal{B}$. ■

The following result fully characterizes controlled invariant sets and domains of attraction for switching systems. Such characterization will be useful for assessing stabilizability of a switching system in terms of the existence of controlled invariant sets or domains of attraction. The symbol $M_{ij}^{-1}\Sigma_j$ denotes the inverse image of the set Σ_j through the operator M_{ij}.

Lemma 11 *A set* $\Pi = \bigcup_{i \in J} (\{q_i\} \times \Sigma_i) \subset \Xi$ *is a* $\beta-$*domain of attraction (resp. controlled invariant set) for the system* $\mathcal{S}$ *if and only if for any* $i \in J$ *a state feedback function exists such that for any* $x_0 \in \Sigma_i$ *the following condition holds (resp. the following condition holds with* $\beta = 0$*):*

$$x_i(t, x_0) \in e^{-\beta t}\Sigma_i, \forall t \in [0, \delta_m) \text{ and}$$
$$x_i(t, x_0) \in e^{-\beta t}\mathcal{I}_{i\beta}\left(\bigcap_{j \in J_i} \left(M_{ij}^{-1}\Sigma_j\right) \bigcap \Sigma_i\right), \forall t \geq \delta_m.$$

where $x_i(t, x_0)$ *is the closed loop evolution at time* t *of the system* S_i*, starting at* $t = 0$ *from the initial state* x_0*.*

Proof. For the characterization of controlled invariance, see [7]. As for the property of being a $\beta-$domain of attraction, a set is a $\beta-$domain of attraction for the system $\mathcal{S}$ if and only if it is controlled invariant for the system $\mathcal{S}'$, obtained from the given system $\mathcal{S}$ by the perturbation of the continuous dynamics: $A_i' = A_i + \beta I$, where A_i', $i \in J$, denotes the state matrix of the perturbed dynamical system. Therefore by linearity of $\mathcal{S}$ and $\mathcal{S}'$ the result follows. ■

Given a set $\Pi = \bigcup_{i \in J} (\{q_i\} \times \Sigma_i) \subset \Xi$, let Π_s be the symmetric set

$$\{\xi \in \Xi : \xi = \alpha_1\xi_1 + \alpha_2\xi_2,\ \xi_1, \xi_2 \in \Pi,\ \alpha_1, \alpha_2 \in \mathbb{R} : |\alpha_1| + |\alpha_2| \leq 1\}$$

The following result holds:

Lemma 12 *If a set* $\Pi = \bigcup_{i \in J} (\{q_i\} \times \Sigma_i) \subset \Xi$ *is a* $\beta-$*domain of attraction (resp. controlled invariant set) for the system* $\mathcal{S}$*, then the set* Π_s *is a* $\beta-$*domain of attraction (resp. controlled invariant set) for* $\mathcal{S}$*.*

Proof. Straightforward from linearity and from Lemma 11. ■

The following theorem establishes an equivalence between asymptotic stabilizability and the existence of a domain of attraction for the class of linear switching systems. The "if" implication is obvious, and it was already stated in the literature for a broader class of systems: in fact, the existence of a domain of attraction implies the existence of "multiple Lyapunov-like functions" (see [5], [9] and the references therein), concatenated together to form a Lyapunov function, suitable for assessing stability of switching systems. The necessity is a consequence of the linearity of the switching system and of the results stated in this section.

Theorem 13 *A system $\mathcal{S}$ is stabilizable (resp. asymptotically stabilizable) if and only if there exists a C_s-set $\Pi \subset \Xi$, which is controlled invariant (resp. $\beta-$domain of attraction for some $\beta > 0$) for $\mathcal{S}$.*

Proof. The sufficiency is obvious. Necessity: if $\mathcal{S}$ is stabilizable, then Theorem 10 implies that the maximal safe set has nonempty interior. From Remark 9 the maximal safe set is controlled invariant, and finally Lemma 12 implies that there exists a controlled invariant C_s-set. If $\mathcal{S}$ is asymptotically stabilizable, then the system $\mathcal{S}'$ defined in the proof of Lemma 11 is stabilizable. Therefore there exists a controlled invariant C_s-set for $\mathcal{S}'$, which, as already pointed out in the same proof, is a domain of attraction for $\mathcal{S}$. ■

We now define the multiple Lyapunov-like function induced by the controlled invariant (resp. $\beta-$domain of attraction) C_s-set $\Pi \subset \Xi$. Consider the hybrid generalization of the Minkowski functional $\Psi_\Pi : \Xi \to \mathbb{R}$ (see [10]) :

$$\Psi_\Pi(\xi) = \inf \{\mu \in \mathbb{R}, \mu \geq 0 : \xi \in \mu\Pi\}$$

Since Π is controlled invariant (resp. $\beta-$domain of attraction, for some $\beta > 0$) for $\mathcal{S}$, then, setting

$$V(\xi(t,j)) = \Psi_\Pi(\xi(t,j))$$

there exists a closed loop execution such that for any hybrid initial state and for any $j \in \{0, \dots, L\}$,

- $\dot{V}(\xi(t,j)) \leq 0$ (resp. $\dot{V}(\xi(t,j)) \leq \eta < 0$), for any time $t \in I_j$;
- $V(\xi(t_k,k)) \leq V(\xi(t_j,j))$ (resp. $V(\xi(t_k,k)) \leq \gamma V(\xi(t_j,j))$, for some $\gamma \in (0,1)$), $\forall k > j : q(k) = q(j)$.

Therefore $V : \Xi \to \mathbb{R}$ is a Lyapunov-like function [4].

4 Kalman-like continuous state space decomposition

In [8] we presented a number of techniques for simplifying the structure of the finite state machine associated to a hybrid system, which allow a significant reduction of the effort required to check stabilizability and detectability. A consequence of those results is that stabilizability of a given switching system depends on a suitable subsystem, where none of the continuous dynamics is controllable. Moreover, a switching system is (asymptotically) stabilizable if and only if each strongly connected component is (asymptotically) stabilizable. Hence, without loss of generality, we consider a strongly connected switching system $\mathcal{S}$ where all the dynamical systems are not controllable.

We start with a decomposition with respect to the continuous component of the state space.

We will show that, as in the case of a linear dynamical system, a switching system $\mathcal{S}$ can be decomposed into two subsystems.

Given $\mathcal{S} = (\Xi, \mathbf{Q}, \mathbf{W}, \mathbb{R}^m, \mathbf{S}_C, S, E, R)$, we can assume without loss of generality that the dynamical systems are in controllability canonical form

$$A_i = \begin{pmatrix} A_i^{(11)} & A_i^{(12)} \\ 0 & A_i^{(22)} \end{pmatrix}, B_i = \begin{pmatrix} B_i^{(1)} \\ 0 \end{pmatrix} \tag{2}$$

where $A_i^{(22)} \in \mathbb{R}^{d_i \times d_i}$, $d_i > 0$, $\forall i \in J$. In fact, if $\mathcal{S}$ is not in this form, it is possible to define the following hybrid state space transformation: given the hybrid state $\xi = (q_i, x)$ and the reset matrices M_{ij}, i, j such that $(q_i, \sigma, q_j) \in E$, the transformed hybrid state and reset matrices are

$$\widehat{\xi} = (q_i, T_i x),\ \widehat{M}_{ij} = T_j M_{ij} T_i^{-1}$$

where T_i is a nonsingular matrix such that $T_i A_i T_i^{-1}$ and $T_i B_i$ are in the desired controllability canonical form. The matrix M_{ij} can be partitioned as

$$M_{ij} = \begin{pmatrix} M_{ij}^{(11)} & M_{ij}^{(12)} \\ M_{ij}^{(21)} & M_{ij}^{(22)} \end{pmatrix}$$

with $M_{ij}^{(22)} \in \mathbb{R}^{d_j \times d_i}$; the continuous component of the hybrid state (q_i, x) can be partitioned as $x = \begin{pmatrix} x^{(1)} \\ x^{(2)} \end{pmatrix}$ where $x^{(1)} \in \mathbb{R}^{n_i - d_i}$, $x^{(2)} \in \mathbb{R}^{d_i}$.

Given the system $\mathcal{S}$, define the switching system

$$\widetilde{\mathcal{S}} = \left(\widetilde{\Xi}, \mathbf{Q}, \mathbf{W}, \mathbb{R}^m, \widetilde{\mathbf{S}}_C, \widetilde{S}, E, \widetilde{R}\right)$$

with $\widetilde{\Xi} = \bigcup\limits_{i \in J} \{q_i\} \times \mathbb{R}^{d_i}$, $\widetilde{S}_i \in \widetilde{\mathbf{S}}_C$ described by the equation

$$\dot{z}(t) = A_i^{(22)} z(t)$$

$\widetilde{S}(q_i) = \widetilde{S}_i$, $\widetilde{R}(e, \widetilde{\xi}) = (q_j, M_{ij}^{(22)} z)$, $e = (q_i, \sigma, q_j) \in E$, $\widetilde{\xi} = (q_i, z) \in \widetilde{\Xi}$.

The following Theorem 15 characterizes asymptotic stabilizability of the system $\mathcal{S}$ in terms of stabilizability of $\widetilde{\mathcal{S}}$ and can be seen as an extension to hybrid systems of the classical Kalman decomposition for linear dynamical systems. The proof of Theorem 15 requires the following lemma, which generalizes the "Squashing Lemma" proved previously in [14].

Lemma 14 *Let (A, B) be a controllable matrix pair with $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and consider the system described by the equation*

$$\dot{x}(t) = Ax(t) + Bu(t) + d(t)$$

with $\|d(t)\| \leq De^{-\beta t}$, $\forall t \in [0, T]$, for some $D \geq 0$, $\beta > 0$, where the symbol $\|.\|$ denotes the Euclidean norm in the space $\mathbb{R}^n$. Then $\forall \Delta \in (0, T)$ and $\forall \varepsilon > 0$ there exists $\gamma > 0$ and a state feedback gain F, such that the solution $x(t)$ of the closed loop system satisfies the condition

$$\|x(t)\| \leq \varepsilon \left(e^{-\gamma(t-\Delta)} \|x(0)\| + e^{-\beta(t-\Delta)} D\right), \quad \forall t \in [0, T].$$

Proof. We use the same arguments as in the proof of the Squashing Lemma [14]. Given any set of n distinct real numbers $\lambda_1, ..., \lambda_1$, since the pair (A, B) is controllable, there exists a polynomial matrix $H(\lambda)$ that assigns to $A + BH(\lambda)$ the characteristic polynomial $p(s, \lambda) = \prod_{i=1}^{n} (s + \lambda (1 + \lambda_i))$ [13], and therefore the eigenvalues of $\widehat{A} = A + BH(\lambda)$ are real, distinct and bounded above by $-\lambda$. This implies that the matrices T and T^{-1} such that $T^{-1}\widehat{A}T = \Lambda$, being Λ a diagonal matrix with the eigenvalues of $\widehat{A}$ on the diagonal, are rational and continuous functions of the parameter $\lambda > 0$. Then

$$\begin{aligned} \|x(t)\| &\leq \left\| e^{\widehat{A}t} \right\| \|x(0)\| + \int_0^t \left\| e^{\widehat{A}(t-\tau)} \right\| \|d(\tau)\| \, d\tau \\ &\leq \|T\| \left\|T^{-1}\right\| \left(e^{-\lambda t} \|x(0)\| + D \int_0^t e^{-\lambda(t-\tau)} e^{-\beta\tau} d\tau \right) \\ &= \|T\| \left\|T^{-1}\right\| \left(e^{-\lambda t} \|x(0)\| + \frac{D}{\lambda - \beta} \left(e^{-\beta t} - e^{-\lambda t} \right) \right). \end{aligned}$$

Since the matrices T and T^{-1} are rational and are continuous functions of the parameter $\lambda > 0$, then it is possible to select $\lambda > \beta + 1$ such that $\|T\| \left\|T^{-1}\right\| e^{-\lambda\Delta} \leq \varepsilon$. Therefore

$$\begin{aligned} \|x(t)\| &\leq \|T\| \left\|T^{-1}\right\| \left(e^{-\lambda t} \|x(0)\| + D \left(e^{-\beta t} - e^{-\lambda t} \right) \right) \\ &\leq \varepsilon \left(e^{-\lambda(t-\Delta)} \|x(0)\| + D \left(e^{-\beta(t-\Delta)} - e^{-\lambda(t-\Delta)} \right) \right) \\ &\leq \varepsilon \left(e^{-\lambda(t-\Delta)} \|x(0)\| + e^{-\beta(t-\Delta)} D \right), \quad t \in [0, T]. \end{aligned}$$

Setting $\gamma = \lambda$ and $F = H(\lambda)$, the result follows. ■

The last result of this paper shows that in the stabilizability problem for switching systems, the core problem is the stability of the autonomous (i.e. without continuous input) switching subsystem $\widetilde{\mathcal{S}}$.

Theorem 15 *The system $\mathcal{S}$ is asymptotically stabilizable if and only if the system $\widetilde{\mathcal{S}}$ is asymptotically stable.*

Proof. Consider the interval $I_j$of the time basis and let $q(j) = q_i$.

(Sufficiency) Since the switching system $\widetilde{\mathcal{S}}$ is asymptotically stable, there exists a C_s-set $\widetilde{\Pi}$ which is a $\beta-$domain of attraction for it. Set

$$\alpha = \max_{i \in J} \left\| A_i^{(12)} \right\|, \qquad \mu_{11} = \max_{i \in J, j \in J_i} \left\| M_{ij}^{(11)} \right\|,$$

$$\mu_{12} = \max_{i \in J, j \in J_i} \left\| M_{ij}^{(12)} \right\|, \quad \mu_{21} = \max_{i \in J, j \in J_i} \left\| M_{ij}^{(21)} \right\|.$$

Without loss of generality, let $\widetilde{\Pi}$ be such that $\sup_{i\in J,(q_i,x))\in\widetilde{\Pi}} \|x\| = 1$, and

$$\left\|x^{(1)}(t_j,j)\right\| \leq 1+\mu_{12}, \quad x^{(2)}(t_j,j)\in\widetilde{\Pi}. \tag{3}$$

By Lemma 14 and setting $\Delta = \delta_m$, $\forall\varepsilon > 0$ there exists $\gamma > 0$ and a state feedback gain $H(q_i)$, such that the state evolution of the closed loop system $S(q_i)$ satisfies the condition,

$$\begin{aligned} \|x(t)\| &\leq \varepsilon\left(e^{-\gamma(t-\delta_m-t_j)}\left\|x^{(1)}(t_j,j)\right\| + \alpha e^{-\beta(t-\delta_m-t_j)}\right) \\ &\leq \varepsilon\left(\left\|x^{(1)}(t_j,j)\right\|+\alpha\right), \forall t\in\left[t_j,t_j'\right], \end{aligned} \tag{4}$$

Moreover, the stabilizability of the decoupled system $\widetilde{\mathcal{S}}$ ensures that $x^{(2)}(t,j)\in e^{-\beta\delta_m}\widetilde{\Pi},\ \forall t\in\left[t_j+\delta_m,t_j'\right]$. After the first switching from the discrete state q_i to some discrete state q_h,

$$\begin{aligned} x^{(1)}(t_{j+1},j+1) &= M_{ih}^{(11)}x^{(1)}(t_j',j)+M_{ih}^{(12)}x^{(2)}(t_j',j), \\ x^{(2)}(t_{j+1},j+1) &= M_{ih}^{(21)}x^{(1)}(t_j',j)+M_{ih}^{(22)}x^{(2)}(t_j',j). \end{aligned}$$

Therefore,

$$\begin{aligned} &\left\|x^{(1)}(t_{j+1},j+1)\right\| \leq \mu_{11}\varepsilon\left(\left\|x^{(1)}(t_j,j)\right\|+\alpha\right)+\mu_{12}e^{-\beta\delta_m}, \\ &x^{(2)}(t_{j+1},j+1)\in\mu_{21}\varepsilon\left(\left\|x^{(1)}(t_j,j)\right\|+\alpha\right)\mathcal{B}+e^{-\beta\delta_m}\widetilde{\Pi}, \end{aligned}$$

and hence, by condition (3),

$$\begin{aligned} &\left\|x^{(1)}(t_{j+1},j+1)\right\| \leq \mu_{11}\varepsilon\left(1+\alpha+\mu_{12}\right)+\mu_{12}e^{-\beta\delta_m}, \\ &x^{(2)}(t_{j+1},j+1)\in\mu_{21}\varepsilon\left(1+\alpha+\mu_{12}\right)\mathcal{B}+e^{-\beta\delta_m}\widetilde{\Pi}. \end{aligned}$$

By selecting $\lambda\in\left(e^{-\beta\delta_m},1\right)$ and $\varepsilon>0$ such that

$$\begin{aligned} &\mu_{11}\varepsilon\left(1+\alpha+\mu_{12}\right)\leq\lambda, \\ &\mu_{21}\varepsilon\left(1+\alpha+\mu_{12}\right)\mathcal{B}+e^{-\beta\delta_m}\widetilde{\Pi}\subset\lambda\widetilde{\Pi}, \end{aligned}$$

the following conditions hold:

$$\begin{aligned} &\left\|x^{(1)}(t_{j+1},j+1)\right\| \leq \lambda+\mu_{12}e^{-\beta\delta_m} \leq \lambda\left(1+\mu_{12}\right), \\ &x^{(2)}(t_{j+1},j+1)\in\lambda\widetilde{\Pi}. \end{aligned}$$

By iterating this step for k discrete transitions, it is easily seen that

$$\begin{aligned} &\left\|x^{(1)}(t_{j+k},j+k)\right\| \leq \lambda^k\left(1+\mu_{12}\right), \\ &x^{(2)}(t_{j+k},j+k)\in\lambda^k\widetilde{\Pi}. \end{aligned}$$

If $L=\infty$, the conditions above guarantee that $\lim_{k\to\infty}x^{(1)}(t_{j+k},k+1)=0$ and $\lim_{k\to\infty}x^{(2)}(t_{j+k},k+1)=0$. Hence by condition (4) the result follows.

If $L < \infty$, since any $S(q_i), i \in J$ is asymptotically stable with the state feedback gain $H(q_i)$ the hybrid state evolution converges to the origin for $t \to \infty$.

(Necessity) Let $L = \infty$. Given $\mathcal{S}$, consider the augmented system $\mathcal{S}^{(a)}$ with the dynamical systems defined by the matrices

$$A_i^{(a)} = \begin{pmatrix} A_i^{(11)} & A_i^{(12)} & 0 \\ 0 & A_i^{(22)} & 0 \\ 0 & 0 & A_i^{(22)} \end{pmatrix}, \quad M_{ij}^{(a)} = \begin{pmatrix} M_{ij}^{(11)} & M_{ij}^{(12)} & 0 \\ M_{ij}^{(21)} & M_{ij}^{(22)} & 0 \\ M_{ij}^{(21)} & 0 & 0 \end{pmatrix},$$

and $B_i^{(a)} = \begin{pmatrix} B_i^{(1)} \\ 0 \\ 0 \end{pmatrix}$. Let the continuous state of the system $\mathcal{S}^{(a)}$ be denoted by w, partitioned as $w = \begin{pmatrix} w^{(1)} \\ w^{(2)} \\ w^{(3)} \end{pmatrix}$ with appropriate dimensions. Consider the infinite closed loop executions of systems $\mathcal{S}$, $\mathcal{S}^{(a)}$ and $\widetilde{\mathcal{S}}$ with initial states $\left(\widehat{q}, \begin{pmatrix} x_0^{(1)} \\ x_0^{(2)} \end{pmatrix}\right)$, $\left(\widehat{q}, \begin{pmatrix} x_0^{(1)} \\ x_0^{(2)} \\ 0 \end{pmatrix}\right)$ and $(\widehat{q}, x_0^{(2)})$, with time basis τ, discrete state sequence $\{q\,(i)\,, i = 0, 1, ...\}$ and feedback control law φ, where φ is asymptotically stabilizing for system $\mathcal{S}$. By construction for any I_j

$$\begin{aligned} x^{(1)}(t,j) &= w^{(1)}(t,j) \\ x^{(2)}(t,j) &= w^{(3)}(t,j) + z(t,j). \end{aligned} \tag{5}$$

Since φ is asymptotically stabilizing for $\mathcal{S}$ then

$$\lim_{j\to\infty} x^{(1)}(t,j) = \lim_{j\to\infty} w^{(1)}(t,j) = 0, \tag{6}$$

$$\lim_{j\to\infty} x^{(2)}(t,j) = 0. \tag{7}$$

Moreover when a switching occurs from q_i to some q_h, $w^{(3)}(t_{j+1}, j+1) = M_{ih}^{(21)} x^{(1)}(t'_j, j)$. Hence by (6)

$$\lim_{j\to\infty} w^{(3)}(t,j) = 0. \tag{8}$$

Finally by combining conditions (5), (7) and (8), we obtain that $\lim_{j\to\infty} z(t,j) = 0$ and the result follows. ■

5 Conclusions

In this paper, we addressed the relationships between safety and stabilizability for a special class of hybrid systems, switching systems. We defined strong safety, as well as stabilizability and asymptotic stabilizability. The equivalence between these properties and safety ones was established. We proposed a Kalman decomposition for switching systems that is important in showing that the core problem for solving the stabilizability problem is the stability of the autonomous (i.e. without continuous input) switching subsystem.

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