General Stochastic Hybrid Systems*

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Abstract

We propose a model for General Stochastic Hybrid Systems (GSHS) which is a common generalization of Piecewise-Deterministic Markov Processes (PDMP), introduced by Davis and stochastic hybrid systems proposed by Hu, Lygeros and Sastry. We prove that this is a 'good model', i.e. it is a strong Markov process with the càdlàg property. Based on results available for PDMP, we provide a formula for the extended generator of the GSHS.

1 Introduction

One of the more general formal models for stochastic hybrid systems (SHS) was proposed by Hu et. al.[7], where the deterministic differential equations for the continuous flow are replaced by their stochastic counterparts, and the reset maps are generalized to (state-dependent) distributions that define the probability density of the state after a discrete transition. In this model transitions are always triggered by deterministic conditions (guards) on the state.

A class of stochastic processes, called piecewise-deterministic Markov processes (PDMP), introduced by Davis in [6], has been proposed as a model for studying stochastic hybrid systems in our paper [4].

We propose a new model for General Stochastic Hybrid Systems (GSHS) which is a generalization both of PDMP and SHS. The class of GSHS allows:
1) Diffusion processes in the continuous evolution.
2) Spontaneous discrete transitions (according to a transition rate).
3) Forced transitions (driven by a boundary hitting time).
4) Probabilistic reset of the discrete and continuous state as a result of discrete transitions.

The difference between GSHS and PDMP is that for GSHS between two consecutive jumps the process is a diffusion whilst for PDMP the inter-jumps motion is deterministic, according to a vector field. GSHS are, in fact, a kind of extended SHS for which the transitions between modes are triggered by some stochastic event (boundary hitting time and transition rate).

2 Description

General Stochastic Hybrid Systems (GSHS) are a class of non-linear stochastic continuous-time hybrid dynamical systems. GSHS are characterized by a hybrid state defined by two components: the continuous state and the discrete state. The continuous state evolves according to a SDE whose vector field and drift factor depend on the hybrid state, both continuous and discrete. Switching between two discrete states is governed by a probability law or occurs when the continuous state hits the boundary of its state space. Whenever a switching occurs, the hybrid state is reset instantly to a new state according to a prob-

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ability law which depends itself on the past hybrid state.

GSHS involve a hybrid state space, with both continuous and discrete states. The continuous and the discrete parts of the state variable have their own natural dynamics, but the main point is to capture the interaction between them.

The time t is measured continuously. The state of the system is represented by a continuous variable x and a discrete variable i. The continuous variable evolves in some "cells" X^i (open sets in the Euclidean space) and the discrete variable belongs to a countable set Q. The intrinsic difference between the discrete and continuous variables, consists of the way that they evolve through time. The continuous state is governed by an SDE that depends on the hybrid state. The discrete dynamics produces transitions in both (continuous and discrete) state variables x, i. Transitions occur when the continuous state hits the boundary of the state space (forced transitions) or according with a probability law. Whenever a transition occurs the hybrid state is reset instantly to a new value. The new value of the discrete state after the transition is determined by the hybrid state before the transition. On the other hand, the new value of the continuous state obeys a probability law which depends on the last hybrid state. Thus, a sample trajectory has the form $(q_t, x_t, t \geq 0)$, where $(x_t, t \geq 0)$ is piecewise continuous and $q_t \in Q$ is piecewise constant. Let $(0 \leq T_1 \leq T_2 \leq ... \leq T_i \leq T_{i+1} \leq ...)$ be the sequence of jump times at which the continuous and the discrete part of the system interact. This time sequence is generated when the state of the system hits the boundary or according with a transition rate.

3 The Mathematical Model

3.1 State space

Let Q be a countable set of discrete states, and let $d : Q \to \mathbb{N}$ and $\mathcal{X} : Q \to \mathbb{R}^{d(\cdot)}$ be two maps assigning to each discrete state $i \in Q$ an open subset X^i of $\mathbb{R}^{d(i)}$. We call the set

$$X(Q, d, \mathcal{X}) = \bigcup_{i \in Q} \{i\} \times X^i$$

the hybrid state space of the GSHS and $x = (i, x^i) \in X(Q, d, \mathcal{X})$ the hybrid state. The completion of the hybrid state space will be

$$\overline{X} = X \cup \partial X$$

where

$$\partial X = \bigcup_{i \in Q} \{i\} \times \partial X^i,$$

It is clear that, for each $i \in Q$, the state space X^i is a Borel space (homeomorphic to a Borel subset of a complete separable metric space). It is possible to define a metric ρ on X in such a way the restriction of ρ to any component X^i is equivalent to the usual Euclidean metric [6]. Then $(X, \mathcal{B}(X))$ is a Borel space. Moreover, X is a homeomorphic with a Borel subset of a compact metric space (Lusin space) because it is a locally compact Hausdorff space with countable base (see [6] and the references therein).

3.2 Construction

Assumption 1 *Suppose that* $b : Q \times X^{(\cdot)} \to \mathbb{R}^{d(\cdot)}$, $\sigma : Q \times X^{(\cdot)} \to \mathbb{R}^{d(\cdot) \times m}$, $m \in \mathbb{N}$, *are bounded and Lipschitz continuous in* x.

This assumption ensures, for any $i \in Q$, the existence and uniqueness (Theorem 6.2.2. in [1]) of the solution for the following stochastic differential equation (SDE)

$$dx(t) = b(i, x(t))dt + \sigma(i, x(t))dW_t, \tag{1}$$

where $(W_t, t \geq 0)$ is the m-dimensional standard Wiener process in a complete probability space.

In this way, when i runs in Q, the equation (1) defines a family of diffusion processes $\mathbb{M}^i = (\Omega^i, \mathcal{F}^i, \mathcal{F}^i_t, x^i_t, \theta^i_t, P^i)$, $i \in Q$ with the state spaces $\mathbb{R}^{d(i)}$, $i \in Q$. For each $i \in Q$, the elements $\mathcal{F}^i$, $\mathcal{F}^i_t$, θ^i_t, P^i, $P^i_{x^i}$ have the usual meaning as in the Markov process theory [3]:

The jump (switching) mechanism between the diffusions is governed by two functions: the jump rate λ and the transition measure R. The jump rate $\lambda : X \to \mathbb{R}_+$ is a measurable function and the transition measure R maps X into the set $\mathcal{P}(X)$ of probability measure on $(X, \mathcal{B}(X))$.

One can consider the transition measure $R : \overline{X} \times \mathcal{B}(X) \to [0,1]$ as a reset probability kernel such that: (i) for all $A \in \mathcal{B}(X)$, $R(\cdot, A)$ is measurable; (ii) for all $x \in \overline{X}$ the function $R(x, \cdot)$ is a probability measure.

Assumption 2 *(i)* $\lambda : X \to \mathbb{R}_+$ *is a measurable function such that* $t \to \lambda(x_t^i(\omega_i))$ *is integrable on* $[0, \varepsilon(x^i))$*, for some* $\varepsilon(x^i) > 0$*, for each* $x^i \in X^i$ *and each* ω_i *starting at* x^i*.*
(ii) $R(x, \{x\}) = 0$ *for* $x \in X$*.*

Since $\overline{X}$ is a Borel space, then $\overline{X}$ is homeomorphic to a subset of the Hilbert cube, $\mathcal{H}$[1] (Urysohn's theorem, Prop. 7.2 [2]). Therefore, its space of probabilities is homeomorphic to the space of probabilities of the corresponding subset of $\mathcal{H}$ (Lemma 7.10 [2]). There exists a measurable function $F : \mathcal{H} \times \overline{X} \to X$ such that $R(x, A) = \mathfrak{p}F^{-1}(A)$, $A \in \mathcal{B}(X)$, where $\mathfrak{p}$ is the probability measure on $\mathcal{H}$ associated to $R(x, \cdot)$ and $F^{-1}(A) = \{\boldsymbol{\omega} \in \mathcal{H} | F(\boldsymbol{\omega}, x) \in A\}$. The measurability of such a function is guaranteed by the measurability properties of the transition measure R.

We construct an GSHS as a *Markov 'sequence'* H which admits $(\mathbb{M}^i)$ as subprocesses. The sample path of the stochastic process $(x_t)_{t>0}$ with values in X, starting from a fixed initial point $x_0 = (i_0, x_0^{i_0}) \in X$ is defined in a similar manner as PDMP [6]. We have to precise, from the beginning, that the above recipe gives a sample path of GSHS starting with a initial diffusion path whose starting point is x_0. An arbitrary point x_0 does not define in a unique way a diffusion path!

Let ω_i a trajectory which starts in (i, x^i). Let $t_*(\omega_i)$ be the first hitting time of ∂X^i of the process (x_t^i). Let us define the function

$$F(t, \omega_i) = I_{(t < t_*(\omega_i))} \exp(-\int_0^t \lambda(i, x_s^i(\omega_i)))ds. \quad (2)$$

[1] $\mathcal{H}$ is the product of countable many copies of $[0,1]$.

This function will be the survivor function for the stopping time S^i associated to the diffusions (x_t^i), which will be employed in the construction of our model. This means that the stopping time S^i satisfies the condition

$$P^i[S^i > t] = P^i\{\omega_i | F(t, \omega_i) \geq e^{-c^i t}\}$$

where $c^i = \sup_{x^i \in X^i} \lambda(x^i)$. Obviously, the stopping time S^i is the minimum of two other stopping times:

1. first hitting time of boundary, i.e. $t_*|_{\Omega^i}$;
2. the stopping time $S^{i\prime}$ with the survivor function given by (2).

The first jump time of the process $T_1(\omega) = T_1(\omega_{i_0}) = S^{i_0}(\omega_{i_0})$. The sample path $x_t(\omega)$ up to the first jump time is now defined as follows:

if $T_1(\omega) = \infty :$ $x_t(\omega) = (i_0, x_t^{i_0}(\omega_{i_0}))$, $t \geq 0$
if $T_1(\omega) < \infty :$ $x_t(\omega) = (i_0, x_t^{i_0}(\omega_{i_0}))$, $0 \leq t < T_1(\omega)$
$x_{T_1}(\omega) = F(\boldsymbol{\omega}, (i_0, x_{T_1}^{i_0}(\omega_{i_0})))$.

The process restarts from $x_{T_1}(\omega) = (i_1, x_1^{i_1})$ according to the same recipe, using now the process $x_t^{i_1}$. Thus if $T_1(\omega) < \infty$ we define the next jump time

$$T_2(\omega) = T_2(\omega_{i_0}, \omega_{i_1}) = T_1(\omega_{i_0}) + S^{i_1}(\omega_{i_1})$$

The sample path $x_t(\omega)$ between the two jump times is now defined as follows:

if $T_2(\omega) = \infty :$ $x_t(\omega) = (i_1, x_{t-T_1}^{i_1}(\omega))$, $t \geq T_1(\omega)$
if $T_2(\omega) < \infty :$ $x_t(\omega) = (i_1, x_t^{i_1}(\omega))$, $0 \leq T_1(\omega) \leq t < T_2(\omega)$
$x_{T_2}(\omega) = F(\boldsymbol{\omega}, (i_1, x_{T_2}^{i_1}(\omega)))$.

and so on.

Let $T_1 < T_2 < ... < T_n < ...$ be the sequence of stopping times obtained by the above method. Let $T_\infty = \lim_{n \to \infty} T_n$.

We denote

$$N_t(\omega) = \sum I_{(t \geq T_k)}$$

Assumption 3 *For every starting point* $x \in X$*,* $EN_t < \infty$*, for all* $t \in \mathbb{R}_+$*.*

We suppose that the assumption 3 is in force.

3.3 Formal Definitions

We can introduce the following definition.

Definition 1 *A General Stochastic Hybrid System (GSHS) is a collection* $H = ((Q, d, \mathcal{X}), b, \sigma, Init, \lambda, R)$ *where*

- Q *is a countable set of discrete variables;*
- $d : Q \to \mathbb{N}$ *is a map giving the dimensions of the continuous state spaces;*
- $\mathcal{X} : Q \to \mathbb{R}^{d(\cdot)}$ *maps each* $q \in Q$ *into an open subset* X^q *of* $\mathbb{R}^{d(q)}$*;*
- $b : X(Q, d, \mathcal{X}) \to \mathbb{R}^{d(\cdot)}$ *is a vector field;*
- $\sigma : X(Q, d, \mathcal{X}) \to \mathbb{R}^{d(\cdot)\times m}$ *is a* $X^{(\cdot)}$*-valued matrix,* $m \in \mathbb{N}$*;*
- $Init : \mathcal{B}(X) \to [0, 1]$ *is an initial probability measure on* $(X, \mathcal{B}(S))$*;*
- $\lambda : \overline{X}(Q, d, \mathcal{X}) \to \mathbb{R}^{+}$ *is a transition rate function;*
- $R : \overline{X} \times \mathcal{B}(\overline{X}) \to [0, 1]$ *is a transition measure.*

We can introduce the GSHS execution.

Definition 2 (GSHS Execution) *A stochastic process* $x_t = (q(t), x(t))$ *is called a GSHS execution if there exists a sequence of stopping times* $T_0 = 0 \leq T_1 \leq T_2 \leq \ldots$ *such that for each* $k \in \mathbb{N}$*,*

- $x_0 = (q_0, x_0^{q_0})$ *is a* $Q \times X$*-valued random variable extracted according to the probability measure* $Init$*;*
- *For* $t \in [T_k, T_{k+1})$*,* $q_t = q_{T_k}$ *is constant and* $x(t)$ *is a (continuous) solution of the SDE:*

$$dx(t) = b(q_{T_k}, x(t))dt + \sigma(q_{T_k}, x(t))dW_t \quad (3)$$

  where W_t *is a the* m*-dimensional standard Wiener;*
- $T_{k+1} = T_k + S^{i_k}$ *where* S^{i_k} *is chosen according with the survivor function (2).*
- *The probability distribution of* $x(T_{k+1})$ *is governed by the law* $R\left((q_{T_k}, x(T_{k+1}^{-})), \cdot\right)$*.*

4 Properties

Notations. Given a function $f \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ and a vector field $b : \mathbb{R}^n \to \mathbb{R}^n$, we use $\mathcal{L}_b f$ to denote the Lie derivative of f along b, i.e. $\mathcal{L}_b f(x) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x) f_i(x)$. Given a function $f \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$, we use $\mathbb{H}^f$ to denote the Hamiltonian operator applied to f, i.e. $\mathbb{H}^f(x) = (h_{ij}(x))_{i,j=1\ldots n} \in \mathbb{R}^{n\times n}$, where $h_{ij}(x) = \frac{\partial^2 f}{\partial x_i \partial x_j}(x)$. Given a matrix $A = (a_{ij})_{i,j=1\ldots n} \in \mathbb{R}^{n\times m}$, A^T denotes the transpose matrix of A and $Tr(A)$ denotes its trace, i.e. $Tr(A) = \sum_{i=1}^{n} a_{ii}$.

4.1 Strong Markov Property

Proposition 3 *Any General Stochastic Hybrid System* H*, under the standard assumptions of section 3, is a Borel right process.*

Proof. In order to prove that H is a right Markov process, we want to apply the results, from [8], concerning the "melange" operation of Markov processes. We can suppose without loss of generality that $\Omega^i \cap \Omega^j = \emptyset$. Then, the renewal kernel Ψ, used in Th.1, [8], can be defined as follows

$$\Psi : \{\bigcup_{i\in Q} \Omega^i\} \times \mathcal{B}(X) \to [0, 1]$$

such that

$$\Psi(\omega_i, A) = R(x^i_{S^i(\omega_i)}, A)$$

We need to check that: If $0 < t < S^i(\omega_i)$ then $\Psi(\theta^i_t \omega_i, \cdot) = \Psi(\omega_i, \cdot)$, i.e. the 'memoryless' of the stopping times (S^i)

$$R(x^i_{S^i(\theta^i_t\omega_i)}, \cdot) = R(x^i_{S^i(\omega_i)}, \cdot).$$

In fact, we have to prove that, if $0 < t < t + s < S^i(\omega_i)$ then

$$P^{x^i}(S^i > t + s | S^i > t) = P^{x^i_t}(S^i > s) \quad (4)$$

Using the survivor function defined by (2), the left hand side of (4) becomes

$$P^{x^i}(S^i > t + s | S^i > t) = \frac{F(t + s, x^i)}{F(t, x^i)} =$$

$$= \frac{I_{\{t+s<t_*(\omega_i)\}} \exp(-\int_0^{t+s} \lambda(x^i_\tau(\omega_i))d\tau)}{I_{\{t<t_*(\omega_i)\}} \exp(-\int_0^{t} \lambda(x^i_\tau(\omega_i))d\tau)} =$$
$$= I_{\{t+s<t_*(\omega_i)\}} \exp(-\int_t^{t+s} \lambda(x^i_\tau(\omega_i))d\tau) =$$
$$= I_{\{t+s<t_*(\omega_i)\}} \exp(-\int_0^{s} \lambda(x^i_{\tau+t}(\omega_i))d\tau) =$$
$$= I_{\{t+s<t_*(\omega_i)\}} \exp(-\int_0^{s} \lambda(x^i_\tau \circ \theta^i_t(\omega_i))d\tau)$$

The right hand side of (4) is

$$P^{x^i_t}(S^i > s) = I_{\{s<t_*(\theta^i_t\omega_i)\}} \exp(-\int_0^{s} \lambda(x^i_\tau \circ \theta^i_t(\omega_i))d\tau)$$

Since $t^*(\theta^i_t\omega_i) = t_*(\omega_i) - t$ we get $t+s < t_*(\omega_i) \Leftrightarrow s < t_*(\theta^i_t\omega_i)$ and (4) is proved.

Therefore, H is a Markov string obtained by mixing some diffusion processes. Moreover, since the state space is a Lusin space, H is a Borel right process (i.e. a right Markov process whose semigroup maps $\mathcal{B}(\overline{\mathbb{S}})^2$ into itself).

4.2 Cadlag Property

Proposition 4 *Any General Stochastic Hybrid System H, under the standard assumptions of section 3, enjoys the càdlàg property, i.e.*
for all $\omega \in \Omega$ the trajectories $t \mapsto x_t(\omega)$ are right continuous on $[0, \infty)$ with left limits on $(0, \infty)$.

Proof. The result is a direct consequence of two facts:

1. the sample paths of (x_t) are obtained by concatenation of sample paths of component process;
2. the component processes are continuous (being diffusions).

4.3 The Process Generator

We denote by $\mathcal{B}_b(X)$ the set of all bounded measurable functions $f : X \to \mathbb{R}$. This is a Banach space under the norm

$$\|f\| = \sup_{x\in X} |f(x)|$$

Let (P_t) be the semigroup of the whole Markov process (x_t),

$$P_t f(x) = E_x f(x_t) = E_x\{f(x_t)|t < \zeta\}$$

where g is bounded $\mathcal{B}$-measurable function and ζ is the lifetime when the process retires to Δ, i.e.

$$\zeta := \inf\{t|x_t = \Delta\}.$$

Associated with the semigroup (P_t) is its *strong generator* which, loosely speaking, is the derivative of P_t at $t = 0$. Let $D(L) \subset \mathcal{B}_b(X)$ be the set of functions f for which the following limit exists

$$\lim_{t\searrow 0} \frac{1}{t}(P_t f - f) \tag{5}$$

and denote this limit Lf. The limit refers to convergence in the norm $\|\cdot\|$, i.e. for $f \in D(L)$ we have

$$\lim_{t\searrow 0} ||\frac{1}{t}(P_t f - f) - Lf|| = 0.$$

Specifying the domain $D(L)$ is an essential part of specifying the operator L.

Let $\mathcal{B}_0$ be the subset of $\mathcal{B}_b(X)$ consisting of those functions f for which $\lim_{t\searrow 0} ||P_t f - f|| = 0$. The semigroup is said to be *strongly continuous* on $\mathcal{B}_0$. $\mathcal{B}_0$ is a closed linear subspace of $\mathcal{B}_b(X)$.

Proposition 5 (martingale property) *[6] For $f \in D(L)$ we define the real-valued process $(C^f_t)_{t\geq 0}$ by*

$$C^f_t = f(x_t) - f(x_0) - \int_0^t Lf(x_s)ds. \tag{6}$$

Then for any $x \in X$, the process $(C^f_t)_{t\geq 0}$ is a martingale on $(\Omega, \mathcal{F}, \mathcal{F}_t, P_x)$.

There may be other functions f, not in $D(L)$, for which something akin to (6) is still true. In this way we get the notion of *extended generator* of the process.

[2] here $\mathcal{B}(\overline{\mathbb{S}})$ is understood as the set of all real Borel functions defined on $\mathbb{S}$.

Let $D(\widehat{L})$ denote the set of measurable functions $f : X \rightarrow \mathbb{R}$ with the following property: there exists a measurable function $h : X \rightarrow \mathbb{R}$ such that the function $t \rightarrow h(x_t)$ is integrable $P_x - a.s.$ for each $x \in X$ and the process

$$C_t^f = f(x_t) - f(x_0) - \int_0^t h(x_s)ds$$

is a local martingale. Then we write $h = \widehat{L}f$ and call $(\widehat{L}, D(\widehat{L}))$ the extended generator of the process (x_t).

Following [6], for $A \in \mathcal{B}(\overline{X})$ define processes p, p^* and $\widetilde{p}$ as follows:

$$\begin{aligned} p(t, A) &= \sum_{k=1}^{\infty} I_{(t \geq T_k)} I_{(x_{T_k} \in A)}; \\ p^*(t) &= \sum_{k=1}^{\infty} I_{(t \geq T_k)} I_{(x_{T_k^-} \in \partial X)}; \\ \widetilde{p}(t, A) &= \textstyle\int_0^t R(x_s, A)\lambda(x_s)ds + \int_0^t R(A, x_{s-})dp^*(s) \\ &= \sum_{T_k \leq t} R(x_{T_k -}, A). \end{aligned}$$

Note that p, p^* are counting processes, $p^*(t)$ counting the number of jumps from the boundary of the process (x_t). $\widetilde{p}(t, A)$ is the compensator of $p(t, A)$ (see [6] for more explanations). The process $q(t, A) = p(t, A) - \widetilde{p}(t, A)$ is a local martingale.

Theorem 6 *Let H be an GSHM as in definition 1. Then the domain $D(L)$ of the extended generator L of H, as a Markov process, consists of those measurable functions f on $X \cup \partial X$ satisfying:*

1. *$f : \overline{X} \rightarrow \mathbb{R}$, $\mathcal{B}-$measurable; $t \rightarrow f(x_t^i(\omega_i))$ should have second order derivatives on $[0, S^i(\omega_i))$, for all $\omega_i \in \Omega^i$;*

2. *Boundary condition*

$$f(x) = \int_{\mathbb{X}} f(y)R(x, dy)), \; x \in \partial X;$$

3. *$Bf \in L_1^{loc}(p)$, where*

$$Bf(x, s, \omega) := f(x) - f(x_{s-}(\omega));$$

For $f \in D(L)$, Lf is given by

$$Lf \; (x) = L_{cont} f(x) + \lambda(x) \int_{\overline{X}} (f(y) - f(x))R(x, dy)$$

where: $L_{cont} \; f(x) = \mathcal{L}_b f(x) + \frac{1}{2} Tr(\sigma(x)\sigma(x)^T \mathbb{H}^f(x))$.

A detailed proof of this theorem can be found in [5].

5 Conclusions

In this paper we propose a very general model for stochastic hybrid systems. The model answers important practical challenges and thus needs to be explored. The contributions of this paper are twofold:

•to define the model;

•to prove correctness of this model (existence of the solution process which is a "Borel right process").

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