HYBRIDGE

Distributed Control and Stochastic Analysis of Hybrid Systems
Supporting Safety Critical Real-Time Systems Design

WP1: Identification and modelling of uncertain hybrid systems

A Stochastic Hybrid System Modeling Framework

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May 29, 2003

Version: *1.1*
Task number: *1.2*
Deliverable number: *D1.2*
Contract: *IST-2001-32460 of European Commission*

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DOCUMENT CONTROL SHEET

Title of document: *A Stochastic Hybrid System Modeling Framework*

Authors of document: *M.L. Bujorianu, J. Lygeros, W. Glover and G. Pola*

Deliverable number: *D1.2*

Contract: *IST-2001-32460 of European Comission*

Project: *Distributed Control and Stochastic Analysis of Hybrid Systems Supporting Safety Critical Real-Time Systems Design (HYBRIDGE)*

DOCUMENT CHANGE LOG

Version #	Issue Date	Sections affected	Relevant information
0.1	1/12/2002	All	First draft
0.2	1/2/2003	All	Second draft
1.0	5/5/2003	All	Revised based on reviewer comments
1.1	29/5/2003	2, 3, References	Revised based on comments by H. Blom

Version 1.0		Organisation	Signature/Date
Authors	M.L. Bujorianu	UCAM	
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Internal reviewers	H. Blom	NLR	
	P. Lezaud	CENA	
	M. Prandini	UniBs	

HYBRIDGE, IST-2001-32460
Work Package WP1, Deliverable D1.2

A Stochastic Hybrid Process Modelling Framework

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Abstract

Deliverable D1.2 of WP1 of the HYBRIDGE project presents a study of a modelling framework for stochastic hybrid processes which allows one to capture the interaction of discrete dynamics, continuous dynamics and uncertainty in the continuous evolution, discrete transition times and discrete transition destinations. The report provides a comparative study of three classes of stochastic hybrid processes that have been proposed in the literature. This overview is followed by a discussion of how these classes of stochastic hybrid processes can be used to model the different safety critical air traffic management situations identified in deliverable D1.1 of WP1 of HYBRIDGE. The possibility of using system identification methods to tune the models developed for these safety critical situations is also discussed and the difficulties that this approach presents are highlighted.

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List of Acronyms

ATC	Air Traffic Control/Controller
ATM	Air Traffic Management
BADA	Base of Aircraft Data
CFMU	Central Flow Management Unit
CDR	Conflict Detection and Resolution
FMS	Flight Management System
GSHP	General Stochastic Hybrid Processes
ODE	Ordinary Differential Equation
PDMP	Piecewise Deterministic Markov Processes
SDE	Stochastic Differential Equation
SHP	Stochastic Hybrid Processes
SHS	Stochastic Hybrid Systems
SDP	Switching Diffusion Processes
TMA	Terminal Maneuvering Area

1 Objectives

Task 1.2 of WP1 of the HYBRIDGE project aims to establish a framework for developing formal mathematical models of the safety critical ATM situations identified in Task 1.1 of WP1 (documented in deliverable D1.1 [1]). We recall the statement of Task 1.2 from the Technical Annex of the HYBRIDGE contract.

> "Develop a modelling framework for stochastic hybrid systems. The framework will allow one to capture the interaction of discrete and continuous dynamics and uncertainty, in the continuous evolution, discrete transition times and discrete transition destinations. The model development will focus on the distributed nature of air traffic management."

Deliverable D1.2 aims to address these issues. The contribution of D1.2 to the HYBRIDGE project consists of two parts.

1. A comparative study of the classes of Stochastic Hybrid Processes (SHP) that have been studied in the literature: Piecewise Deterministic Markov Processes (PDMP), Switching Diffusion Processes (SDP), Stochastic Hybrid Systems (SHS) (Section 2).
2. A connection between these classes of SHP and the safety critical ATM situations identified in deliverable D1.1 (Section 3).

The report concludes that different classes of SHP are appropriate for the different safety critical situations. The desired properties of a more general class of SHP that would allow one to capture all safety critical situations in a single framework is presented in Section 2.6.

To ensure that the models developed for the different safety critical situations are realistic one needs to tune various model parameters to match data from the real system. Systematic methods for doing this have been developed in the control literature under the area of system identification. In Section 4 we investigate the possibility of using system identification methods to tune our models. We conclude that this approach may not be viable within the HYBRIDGE project due to

1. The unavailability of data.
2. The wide range of time scales involved.

An alternative approach for tuning our models based on existing studies of flight plan deviations and simulation is proposed.

2 Classification of Stochastic Hybrid Processes

We start by providing a review of three classes of SHP that have been studied in the literature. We also provide a comparison of their descriptive power. Two types of discrete transitions appear throughout the discussion. The first occurs at the boundaries of the state space, where continuous evolution becomes impossible. We refer to this class of discrete transitions as *forced transitions*. The second class are transitions that can take place in the interior of the state space and their occurrence is governed by a "rate" (as in continuous time Markov chains for example). We refer to this class of discrete transitions as *spontaneous transitions*.

Throughout this section $\mathcal{P}(Y)$ denotes the set of probability measures on a Polish space Y equipped with the topology of weak convergence. In other words, a sequence $\mu_n \in \mathcal{P}(Y)$ converges to $\mu \in \mathcal{P}(Y)$ in the weak topology if and only if $\mu_n(f) \to \mu(f)$ for all continuous and bounded $f : Y \to \mathbb{R}$, where

$$\mu(f) := \int_Y f(x)\mu(dx).$$

$\mathcal{B}(Y)$ denotes the Borel σ-algebra of Y.

2.1 Piecewise Deterministic Markov Processes

PDMP were introduced by M.H.A. Davis in [2, 3]. They are a class of non-linear continuous-time SHP which covers a wide range of non-diffusion phenomena. PDMP involve a hybrid state space, with both continuous and discrete states. The peculiarity of this model is that randomness appears only in the discrete transitions; between two consecutive transitions the continuous state evolves according to a nonlinear ordinary differential equation (hence the name "piecewise deterministic"). Transitions occur either when the state hits the state space boundary (forced transitions), or in the interior of the state space, according to a state dependent rate (spontaneous transitions). Whenever a transition occurs, the hybrid state is reset instantaneously according to a probability distribution which depends on the value taken by the hybrid state before the transition.

We introduce formally PDMP following the notation of [4, 5]. Let Q be a countable set of discrete states, and let $d : Q \to \mathbb{N}$ and $X : Q \to \mathbb{R}^{d(\cdot)}$ be two maps assigning to each discrete state $i \in Q$ an open subset of $\mathbb{R}^{d(i)}$. We denote by $\mathcal{D}(Q, d, X)$ the hybrid state space of the PDMP i.e.

$$\mathcal{D}(Q, d, X) = \cup_{i \in Q}\{i\} \times X(i)$$

and by $\alpha = (i, x) \in \mathcal{D}(Q, d, X)$ the hybrid state. The boundary of the hybrid state space is denoted by $\partial\mathcal{D}(Q, d, X)$.

A vector field f on the hybrid state space $\mathcal{D}(Q,d,X)$ is a function $f : \mathcal{D}(Q,d,X) \to \mathbb{R}^{d(\cdot)}$ assigning to each hybrid state $\alpha = (i,x) \in \mathcal{D}(Q,d,X)$ a direction $f(\alpha) \in \mathbb{R}^{d(i)}$. The flow of f is a function $\Phi : \mathcal{D}(Q,d,X) \times \mathbb{R} \to \mathcal{D}(Q,d,X)$ with

$$\Phi(\alpha,t) = \begin{bmatrix} \Phi_Q(\alpha,t) \\ \Phi_X(\alpha,t) \end{bmatrix},$$

$\Phi_Q(\alpha,t) \in Q$ and $\Phi_X(\alpha,t) \in X(i)$, such that for $\alpha = (i,x)$, $\Phi(\alpha,0) = \alpha$ and for all $t \in \mathbb{R}$, $\Phi_Q(\alpha,t) = i$ and $\frac{d}{dt}\Phi_X(\alpha,t) = f(\Phi(\alpha,t))$. Let $\Gamma((Q,d,X),f)$ be the set of boundary points which can be reached at some time $t > 0$ by evolving according to the flow Φ starting from a hybrid state $\alpha \in \mathcal{D}(Q,d,X)$,

$$\Gamma((Q,d,X),f) = \{(i,x) \in \partial\mathcal{D}(Q,d,X) | x = \Phi_X(\alpha,t) \text{ for some } \alpha \in \{i\}\times X(i) \text{ and some } t > 0\}$$

and $\overline{\mathcal{D}}(Q,d,X) = \mathcal{D}(Q,d,X) \cup \Gamma((Q,d,X),f)$. Consider also a Borel σ-algebra $\mathcal{B}(\overline{\mathcal{D}})$ on $\overline{\mathcal{D}}(Q,d,X)$ defined as in [4].

Definition 1 (Piecewise Deterministic Markov Process) *A Piecewise Deterministic Markov Process is a collection $H = ((Q,d,X), f, Init, \lambda, R)$ where*

- *Q is a countable set of discrete variables representing the discrete state space;*
- *$d : Q \to \mathbb{N}$ is a map giving the dimension of the continuous state space within each discrete state;*
- *$X : Q \to \mathbb{R}^{d(\cdot)}$ maps each $i \in Q$ into an open subset $X(i)$ of $\mathbb{R}^{d(i)}$;*
- *$f : \mathcal{D}(Q,d,X) \to \mathbb{R}^{d(\cdot)}$ is a vector field defined on the hybrid state space $\mathcal{D}(Q,d,X) = \cup_{i\in Q}\{(i,x) : x \in X(i)\}$;*
- *$Init : \mathcal{B}(\overline{\mathcal{D}}) \to [0,1]$ is an initial probability measure on $(\overline{\mathcal{D}}, \mathcal{B}(\overline{\mathcal{D}}))$, with $Init(\mathcal{D}^c) = 0$;*
- *$\lambda : \overline{\mathcal{D}}(Q,d,X) \to \mathbb{R}^+$ is a transition rate function;*
- *$R : \mathcal{B}(\overline{\mathcal{D}}) \times \overline{\mathcal{D}}(Q,d,X) \to [0,1]$ is a transition measure, with $R(\mathcal{D}^c,.) = 0$.*

To ensure that the execution of a PDMP is a well defined stochastic process, the following assumption is introduced.

Assumption 1 *For all $i \in Q$, $f(i,.)$ is globally Lipschitz continuous. $\lambda : \overline{\mathcal{D}}(Q,d,X) \to \mathbb{R}^+$ is measurable. For all $\alpha \in \mathcal{D}$ there exists $\varepsilon > 0$ such that the function $t \to \lambda(\Phi(\alpha,t))$ is integrable for all $t \in [0,\varepsilon)$. For all $A \in \mathcal{B}(\overline{\mathcal{D}})$, $R(A,\cdot)$ is measurable.*

Algorithm 1 (Generation a PDMP realization)

set $\tau = 0$

extract at random from $\overline{\mathcal{D}}$ a value $\hat{\alpha}$ for the random variable α_τ
according to *Init*

repeat

extract at random from $\mathbb{R}^+$ a value $\hat{T}$ for the random variable T
such that $P(T > t) = F(\hat{\alpha}, t)$

set $\alpha_t = \Phi(\hat{\alpha}, t - \tau)$ for all $t \in [\tau, \tau + \hat{T})$

extract at random from $\overline{\mathcal{D}}$ a value $\hat{\alpha}'$ for the random variable $\alpha_{\tau+\hat{T}}$
according to $R(., \Phi(\hat{\alpha}, \hat{T}))$

set $\tau = \tau + \hat{T}$ and $\hat{\alpha} = \hat{\alpha}'$

until true

Table 1: Generation of PDMP Executions

To define the realization of a PDMP execution we introduce the notion of exit time $t^* : \mathcal{D} \to \mathbb{R}^+ \cup \{\infty\}$

$$t^*(\alpha) = \inf\{t > 0 : \Phi(\alpha, t) \notin \mathcal{D}\}$$

($t^*(\alpha) = \infty$ if the set if empty) and of survivor function $F : \mathcal{D} \times \mathbb{R}^+ \to [0, 1]$

$$F(\alpha, t) = \begin{cases} \exp\left(-\int_0^t \lambda(\Phi(\alpha, \tau))d\tau\right) & \text{if } t < t^*(\alpha) \\ 0 & \text{if } t \geq t^*(\alpha). \end{cases}$$

A realization of the PDMP execution is described by the algorithm in Table 1. All random extractions in the algorithm are assumed to be independent of one another.

To ensure that the PDMP execution α_t is defined on the entire $\mathbb{R}^+$ it is necessary to exclude Zeno realizations (see [5]). The following assumption is introduced in [2, 3] to exclude Zeno behavior, at least in average.

Assumption 2 *Let N_t be the number of jumps in $[0, t]$. Then $\mathbb{E}[N_t] < \infty$ for all t.*

Under Assumptions 1 and 2, the following fact is established in [2, 3].

Theorem 1 *Algorithm 1 generates realizations of a strong Markov process.*

[2, 3] then proceed to characterize the extended generator of this process and based on it derive formulas for computing expectations, solving optimal control problems, etc. (see Section 2.5).

2.2 Switching Diffusion Processes

SDP are a class of non-linear continuous-time SHP that have been used to model a number of applications such as fault tolerant control systems, multiple target tracking, flexible manufacturing systems, etc. SDP involve a hybrid state space, with both continuous and discrete states. The continuous state evolves according to a Stochastic Differential Equation (SDE), while the discrete state is a Markov chain (spontaneous transitions). Both the dynamics of the SDE and the transition matrix of the Markov chain depend on the hybrid state. The characteristic feature of SDP is that the continuous hybrid state evolves without jumps, i.e. the evolution of the continuous state can be assumed to be a continuous function of time.

We formally introduce SDP following [6, 7, 8].

Definition 2 (Switching Diffusion Process) *A Switching Diffusion Process is a collection $H = (Q, X, f, Init, \sigma, \lambda)$ where*

- *Q is a finite set with cardinality $N \in \mathbb{N}$ representing the discrete state space;*
- *$X = \mathbb{R}^n$ is the continuous state space;*
- *$f : Q \times X \to \mathbb{R}^n$ is a vector field;*
- *$Init : \mathcal{B}(Q \times X) \to [0, 1]$ is an initial probability measure on $(Q \times X, \mathcal{B}(Q \times X))$;*
- *$\sigma : Q \times X \to \mathbb{R}^{n \times n}$ is a state dependent matrix;*
- *$\lambda_{ij} : X \to \mathbb{R}$, $i, j \in Q$ are a set of x-dependent transition rates, with $\lambda_{ij}(.) \geq 0$ if $i \neq j$ and $\sum_{j \in Q} \lambda_{ij}(x) = 0$ for all $i \in Q$, $x \in X$.*

As for PDMP we will use $\alpha = (q, x) \in Q \times X$ to denote the hybrid state of a SDP. To ensure the SDP model is well defined [6, 7, 8] introduce the following assumption.

Assumption 3 *For all $i \in Q$ the functions $f(i, \cdot)$, $\sigma_{kj}(i, \cdot)$ and $\lambda_{kj}(\cdot)$ are bounded and Lipschitz continuous.*

To determine a solution of an SDP one starts by extracting a random initial condition $(i, x_0) \in Q \times X$ according to $Init$. The evolution of the continuous state is then determined by the SDE corresponding to discrete state i

$$dx(t) = f(i, x(t))dt + \sigma(i, x(t))dW_t,$$

starting at $x(0) = x_0$, where W_t is a n-dimensional standard Wiener process. The discrete transitions are similar to those of a Markov chain and given by

$$P(q(t + \delta t) = j \mid q(t) = i, x(s), q(s), s \leq t) = \ lambda_{ij}(x(t))\delta t + o(\delta t), i \neq j.$$

after a discrete transition occurs continuous evolution resumes, according to the SDE corresponding to the new discrete state. Notice that only spontaneous transitions are allowed and the rates depend on the continuous state. Notice also that the continuous state remains constant during a discrete transition, therefore the evolution of the continuous state will be continuous in time.

[6, 7, 8] formalize the evolution of the discrete state by means of an auxiliary SDE driven by the Poisson random measure. Assumption 3 then ensures that for any initial condition, a strong solution of the SDP exists and is unique (see, for example, Theorem 6.2.2 in [9]). This allows the authors of [6, 7, 8] to determine the generator of the resulting Markov process and use it to solve optimal control problems (see Section 2.5).

2.3 Stochastic Hybrid Systems

SHS introduced in [10] are another class of non-linear, continuous-time SHP. SHS also involve a hybrid state space, with both continuous and discrete states. The continuous state evolves according to a SDE that depends on the discrete state. Transitions occur when the continuous state hits the boundary of the state space (forced transitions). Whenever a transition occurs the hybrid state is reset instantaneously to a new value. The value of the discrete state after the transition is deterministically given by the hybrid state before the transition. The new value of the continuous state, on the other hand, is governed by a probability law which depends on the last hybrid state.

Definition 3 (Stochastic Hybrid System) *A Stochastic Hybrid System is a collection* $H = (Q, X, Dom, f, g, Init, G, R)$ *where*

- Q *is a countable set representing the discrete state space;*
- $X = \mathbb{R}^n$ *is the continuous state space;*
- $Dom : Q \to 2^X$ *assigns to each* $i \in Q$ *an open subset of* X*;*
- $f, g : Q \times X \to \mathbb{R}^n$ *are vector fields;*
- $Init : \mathcal{B}(Q \times X) \to [0, 1]$ *is an initial probability measure on* $(Q \times X, \mathcal{B}(Q \times X))$ *concentrated on* $\cup_{i \in Q} \{i\} \times Dom(i)$*;*
- $G : Q \times Q \to 2^X$ *assigns to each* $(i, j) \in Q \times Q$ *a guard* $G(i, j) \subset X$ *such that*
  - *For each* $(i, j) \in Q \times Q$, $G(i, j)$ *is a measurable subset of* $\partial Dom(i)$ *(possibly empty);*
  - *For each* $i \in Q$*, the family* $\{G(i, j) \mid j \in Q\}$ *is a disjoint partition of* $\partial Dom(i)$*;*

- $R : Q \times Q \times X \to \mathcal{P}(X)$ *assigns to each* $(i, j) \in Q \times Q$ *and* $x \in G(i, j)$ *a reset probability kernel on* X *concentrated on* $Dom(j)$.

We again use $\alpha = (q, x)$ to denote the hybrid state of an SHS. To ensure that the model is well defined the following assumption is introduced in [10].

Assumption 4 *For all* $i \in Q$ *the functions* $f(i, \cdot)$ *and* $g(i, \cdot)$ *are bounded and Lipschitz continuous. For all* $i, j \in Q$ *and for any measurable set* $A \subset Dom(j)$, $R(i, j, x)(A)$ *is a measurable function in* x.

The first part of Assumption 4 ensures that for any $i \in Q$, the solution of the SDE

$$dx(t) = f(i, x(t))dt + g(i, x(t))dW_t,$$

where W_t is a 1-dimensional standard Wiener process, exists and is unique (see Theorem 6.2.2 in [9]). Moreover, the assumption on R ensures that "transition" events are measurable with respect to the underlying σ-field, hence their probabilities make sense.

Definition 4 (SHS Execution) *A stochastic process* $\alpha_t = (q(t), x(t))$ *is called a SHS execution if there exists a sequence of stopping times* $T_0 = 0 \le T_1 \le T_2 \le \ldots$ *such that for each* $j \in \mathbb{N}$,

- $\alpha(0) = (q(0), x(0))$ *is a* $Q \times X$*-valued random variable with distribution Init;*
- *For* $t \in [T_j, T_{j+1})$, $q(t) = q(T_j)$ *is constant and* $x(t)$ *is a (continuous) solution of the SDE:*
$$dx(t) = f(q(T_j), x(t))dt + g(q(T_j), x(t))dW_t$$
starting at $x(T_j)$, *where* W_t *is a* 1*-dimensional standard Wiener process;*
- $T_{j+1} = \inf \{t \ge T_j | x(t) \not\in Dom(q(T_j))\}$;
- $\lim_{t \uparrow T_{j+1}} x(t) \in G(q(T_j), q(T_{j+1})) \in \partial Dom(q(T_j))$;
- $x(T_{j+1})$ *is a random variable distributed according to* $R\left(q(T_j), q(T_{j+1}), \lim_{t \uparrow T_{j+1}} x(t)\right)$.

In [10] it is shown that if such an execution exists, then under Assumption 4, $\{\alpha_{T_n}\}$ defines a Markov process. Existence and uniqueness conditions for executions (in fact for a more general class of stochastic hybrid processes) are currently under investigation within WP2 of HYBRIDGE.

2.4 Comparison of Descriptive Power

Randomness enters in different places for the three classes of SHP discussed above. In this section we highlight the similarities and differences between these classes. We do this by developing subclasses of each class that can be reduced to one another (in a sense, establishing the common elements of the classes).

It is simple to check that the only stochastic processes that can be executions of all three models (PDMP, SDPand SHS) can be trivially represented by a family of nonlinear Ordinary Differential Equation (ODE) on $\mathbb{R}^n$, parametrized by a discrete state $q \in Q$ and random initial conditions (both for the discrete and the continuous state). Pairwise comparisons, however, provide some more insight into the differences in descriptive power between the three classes of model. To formalize the pairwise comparisons we introduce the concept of modification.

Definition 5 (Modification) *Given two stochastic processes α_t and $\hat{\alpha}_t$ defined on the same underlying probability space $(\Omega, \mathcal{F}, P)$, we say that α_t is a modification β_t if $P(\alpha_t = \beta_t) = 1$ for all t.*

Our aim will be to establish subclasses of PDMP, SDP and SHS that coincide in the sense of modification.

2.4.1 A comparison between PDMP and SDP

We first investigate the relationship between PDMP and SDP. We establish a subclass of PDMP that are equivalent to some SDP. To do this it is necessary to impose restrictions on the PDMP considered. First, it is necessary to assume that the number of discrete states of the PDMP is finite, since this is the case for all SDP. Second, it is necessary to assume for all discrete states i of the PDMP, $X(i) = \mathbb{R}^n$ (i.e. continuous evolution is unconstrained). This is because SDP only allow spontaneous transitions. Finally, since the evolution of the continuous state of an SDP can be assumed to be continuous as a function of time, it is necessary to impose some restrictions on the PDMP transition measure (roughly, that is leaves the continuous state unaffected). These qualitative remarks are formalized in the following lemma.

Lemma 1 (PDMP → SDP) *Consider a PDMP, $H^{PDMP} = ((Q, d, X), f, Init, \lambda, R)$, satisfying Assumptions 1 and 2. Assume further that*

1. $|Q| = N \in \mathbb{N}$;
2. $X(i) = \mathbb{R}^{d(i)}$, $\forall i \in Q$;

3. There exists finite n such that $d(i) \leq n$ for all $i \in Q$;

4. For any $i \in Q$, $x \in \mathbb{R}^n$, $R(\{(j,x), j \in Q\}, (i,x)) = 1$.

Then there exists a SDP, H^{SDP}, which is a modification of H^{PDMP}.

Condition 4 effectively implies that the evolution of the continuous state of H^{PDMP} is continuous as a function of time. Condition 2 implies that the PDMP exhibits no forced transitions; it can be weakened to $t^*(\alpha) = \infty$, $\forall \alpha \in \mathcal{D}$. It is easy to see that for a given H^{PDMP} there is in fact a whole family of SDP which are modifications of H^{PDMP} (for details see [4]). The proof of Lemma 1 is constructive; the H^{SDP} constructed in the proof can be considered as a canonical SDP representative of H^{PDMP}.

Conversely, to establish a class of SDP which are equivalent to some PDMP, we need to impose some restrictions on the SDP considered. First, we need to eliminate the diffusion element of the SDP, and assume that continuous motion is deterministic (since PDMP are deterministic processes between two consecutive jumps). Moreover, a relation between the SDP transition matrix $[\lambda_{ij}]$ and the PDMP transition rate and transition measure has to be established. The following lemma can be seen as a converse of Lemma 1.

Lemma 2 (SDP → PDMP) *Consider a SDP, $H^{SDP} = (Q, X, f, Init, \sigma, \lambda_{ij})$ satisfying Assumption 3. Suppose that $\sigma(i,x) = 0$ for all $(i,x) \in Q \times X$. Then, there exists a PDMP, H^{PDMP}, which is a modification of H^{SDP}.*

Lemmas 1 and 2 together indicate that the common model for SDP and PDMP is a SHP with deterministic continuous evolution between two consecutive jumps (as in PDMP) and with discrete state transitions governed by a transition rate depending on the hybrid state. Finally, the continuous state remains constant during discrete transitions (as in SDP).

2.4.2 A comparison between SHS and SDP

To find a class of SHS that are modifications of some SDP, it is necessary to assume that the number of discrete states of the SHS is finite, since this is the case for all SDP. It is also necessary to assume that $Dom(i) = \mathbb{R}^n$, since SDPdo not allow forced transitions. This in particular implies that the guards G of the SHS have to be empty and that the reset relation R of the SHS is never exercised. These remarks are formalized in the following lemma.

Lemma 3 (SHS → SDP) *Consider a SHS, $H^{SHS} = (Q, X, Inv, f, g, Init, G, R)$ satisfying Assumption 4. Assume that the cardinality of the set Q is finite, $Dom(i) = X$, for all $i \in Q$, and for all pairs $i, j \in Q$, $G(i,j) = \emptyset$. Then, there exists a SDP, H^{SDP} which is a modification of H^{SHS}.*

Again, H^{SDP} can be considered as a canonical SDP representative of H^{SHS} in the class of SDP.

To reduce a SDP to a SHS, we have to assume that the diffusion process for the SDP is governed by a standard 1-dimensional Wiener process (which is the case for all SHS). We also have to impose no jumps governed by transition rate λ_{ij} since this kind of jumps is absent in SHS. The following lemma can be seen as a converse of Lemma 3.

Lemma 4 (SDP→SHS) *Consider a SDP, $H^{SDP} = (Q, X, f, Init, \sigma, \lambda)$ satisfying Assumption 3. Assume that the rank of the matrix $[\sigma(i,x)]$ is at most 1 for all $(i,x) \in Q \times X$ and that $\lambda_{ij}(x) = 0$ for all $i, j \in Q$, $x \in X$. Then, there exists a SHS, H^{SHS}, which is a modification of H^{SDP}.*

Lemmas 3 and 4 indicate that the common model between SHS and SDP is a finite family of SDE (parametrized by q) driven by a 1-dimensional Wiener process. The particular SDE and its initial condition are determined according to a probability distribution and no discrete transitions are permitted from then on.

2.4.3 A comparison between SHS and PDMP

To establish equivalent subclasses of SHS and PDMP in the sense of modification, some assumptions are needed. First, we have to eliminate the diffusion component of the SHS, since PDMP are deterministic processes between two consecutive jumps. We have to assume that the dimension of the continuous state space (which for PDMP is allowed to depend on the discrete state) is bounded. We also have to eliminate jumps governed by the transition rate λ (since this kind of jumps are not allowed in the SHS) and establish a relation between the transition measures of SHS and PDMP. These qualitative remarks are formalized in Lemmas 5 and 6.

Lemma 5 (SHS → PDMP) *Consider a SHS, $H^{SHS} = (Q, X, Dom, f, g, Init, G, R)$, satisfying Assumption 4. Assume that $g(i,x) = 0$, for all $(i,x) \in Q \times X$. Then, there exists a PDMP H^{PDMP} which is a modification H^{SHS}.*

Lemma 6 *(PDMP→SHS] Consider a PDMP, $H^{PDMP} = ((Q, d, X), f, Init, \lambda, R)$ satisfying Assumptions 1 and 2. Assume further that*

1. *There exists $n \in \mathbb{N}$ such that $d(i) \leq n$ for all $i \in Q$;*
2. *$\lambda(i,x) = 0$ for all $(i,x) \in \mathcal{D}(Q, d, X)$;*
3. *For all $i, j \in Q$ the set $G(i,j) = \{x \in \partial X(i) : R(\{j\} \times X(j), (i,x)) \neq 0\}$ is measurable.*

Characteristics	PDMP	SHS	SDP
Stochastic continuous evolution		√	√
Forced transitions	√	√	
Spontaneous transitions	√		√
Continuous state reset	√	√	

Table 2: Overview of SHP models

Then, there exists a SHS, H^{SHS} which is a modification of H^{PDMP}.

Lemmas 5 and 6 suggest that the common model between SHS and PDMP is a SHP where continuous evolution between two consecutive jumps is deterministic (as in PDMP) and where switchings between two discrete states occur only when the continuous state hits the hybrid state space boundary. Whenever a switching occurs, the hybrid state jumps according to a probability law depending on the last hybrid state.

2.4.4 Summary

The comparison of the descriptive power of the three classes of SHP considered here is summarized in Table 2. The following differences in the technical details are omitted from the table:

- Countable vs. finite discrete states.
- Dependence of the dimension of the continuous state on the discrete state.
- One-dimensional vs. higher dimensional Weiner processes driving the diffusions..

The changes needed to extend models lacking some of these features are fairly minor.

2.5 Systems with Inputs

All the models discussed above are autonomous, i.e. have no inputs, or outputs and no explicit dependence of the dynamics on time. To address distributed nature of Air Traffic Management (ATM) one needs to be able to compose models of subsystems to form larger models. Hence, input/output models are needed. The discussion in Section 3 already points out some things that would be simpler to model if composition of subsystems is possible. Composition operations are expected to become even more important when dealing with distributed conflict detection and resolution (WP6) and with human error evolution control (WP7).

To date there does not appear to be a general theory that allows one to compose stochastic hybrid systems. The only results in this direction appear to come from theoretical computer science and deal primarily with discrete state (often also discrete time) stochastic systems [11, 12]. Composition operations and parallelism for stochastic systems appear to be better studied in the framework of Petri Nets; see for example the work on Dynamically Coloured Petri Nets of [13]. WP4 of HYBRIDGE will address these problems in an automaton framework for the class of PDMP.

For two of the classes discussed above the first step towards composition, i.e. the introduction of inputs has been taken. In both cases, the objective was not composition, but optimal control. Since optimal control is likely to play a prominent role in addressing reachability questions we will review some of the discussion here.

2.5.1 Controlled PDMP

The following overview will cover only the so called "continuous control" of PDMP, where control acts on f, λ and R. For a full treatment of the subject, as well as a discussion of control by intervention we refer to [3] (Chapters 4 and 5 respectively) [14, 15, 16].

The state space of a controlled PDMP is $\widehat{\mathcal{D}} = \mathcal{D}^* \cup \{\Delta\}$, where $\mathcal{D}^* = \{x \in \mathbb{R}^n : \Psi(x) < 0\}$ for some $\Psi \in \mathcal{C}^1(\mathbb{R}^n)$ such that $\|\nabla\Psi(x)\| \geq 1$ on $\partial\mathcal{D}^* = \{x \in \mathbb{R}^n : \Psi(x) = 0\}$ and Δ is the compactification point of $\mathbb{R}^n$. Note that $\mathcal{D}^*$ is not necessarily bounded or connected. We denote by $\mathbb{D}$ the set $Q \times \mathbb{R}^n$ and by U_0 and U_Γ the control sets.

A *controlled PDMP* is a collection $H = ((Q, d, X), U_0, U_\Gamma, f, Init, \lambda, R)$ where Q, d, X are defined as in Section 2.1, U_0 and U_Γ are the control sets and

- $Init : \mathcal{B}(\mathbb{D}) \to [0, 1]$ is an initial probability measure on $(\mathbb{D}, \mathcal{B}(\mathbb{D}))$, with $Init(\widehat{\mathcal{D}}^c) = 0$;
- $\lambda : \mathcal{D}^* \times U_0 \to \mathbb{R}^+$ is a transition rate function;
- $R : \mathcal{B}(\mathbb{D}) \times \mathcal{D}^* \times U_0 \to [0, 1]$ is a transition measure, with $R(\widehat{\mathcal{D}}^c, ., .) = 0$;
- $R_b : \mathcal{B}(\mathbb{D}) \times \partial\mathcal{D}^* \times U_\Gamma \to [0, 1]$ is a transition measure, with $R(\widehat{\mathcal{D}}^c, ., .) = 0$.

To ensure the model is well defined a number of assumptions are introduced in [3].

The class of controls considered in this context is the one of *feedback policies*, i.e., a pair of measurable functions $v_0 : \mathcal{D}^* \to U_0$ and $v_\Gamma : \partial\mathcal{D}^* \to U_\Gamma$. Given a control policy

(v_0, v_Γ), one can uniquely associate a PDMP of Definition 1 with the closed loop system (see [3]). In this context, a well-formulated optimal control can be defined by means of a cost function:

$$J_x(v) = \mathbb{E}_x^v \left\{ \int_0^\infty l(x_t, v_0(x_t))dt + \int_0^\infty c(x_{t^-}, v_\Gamma(x_{t^-}))dp^*(t) \right\}$$

where $l : \widehat{\mathcal{D}} \times U_0 \to \mathbb{R}^+$ and $c : \partial\mathcal{D}^* \times U_\Gamma \to \mathbb{R}^+$ are bounded nonnegative functions and dp^* is a counting process for the discrete transitions. The *optimal policy* $\hat{v}$ is the one that minimizes $J_x(v)$ for all $x \in \mathcal{D}^*$ over all admissible policies v.

Formulation above has to take into account that the trajectory x_t, between two consecutive jumps, is governed by the following nonlinear ODE

$$\frac{d}{dt}x_t = f(x_t, v_0(x_t))$$

In order to guarantee existence and uniqueness of the solution, the function $x \to f(x_t, v_0(x_t))$ must to be at least locally Lipschitz continuous and some further assumption have to be introduced on the control policies. To solve this problems Vermes [14] introduced the class of *piecewise open-loop controls*. For a complete treatment we refer to [3].

2.5.2 Controlled SDP

The following overview is based on [8]. The definition of controlled SDP can be given, by generalizing Definition 2. For completeness we give in the following the controlled SDP definition.

A *controlled SDP* is a collection $H = (Q, X, U, f, Init, \sigma, \lambda)$ where $Q, X, Init, \sigma$ are defined as in Section 2.2, U is the control set and

- $\bar{f} : Q \times X \times U \to \mathbf{T}X$ is a vector field;
- $\bar{\lambda}_{ij} : X \times U \to \mathbb{R}$, $\bar{\lambda}_{ij}(.) \geq 0$ if $i \neq j$ and $\sum_{j \in Q} \bar{\lambda}_{ij}(.) = 0$ for any $i \in Q$.

A number of assumptions are again needed to ensure that the model is well defined (see [6, 7]).

Let be $\mathcal{V} = \mathcal{P}(U)$ and let us define $f : Q \times X \times \mathcal{V} \to \mathbf{T}X$ and $\lambda_{ij} : X \times U \to \mathbb{R}$

$$\begin{aligned} f(.,.,v) &= \int_U \bar{f}(.,.,u)v(du) \\ \lambda_{ij}(.,v) &= \int_U \bar{\lambda}_{ij}(.,u)v(du) \end{aligned}$$

Let us also define a function $h : \mathbb{R}^n \times Q \times \mathcal{V} \times \mathbb{R} \to \mathbb{R}$

$$h(x,i,v,z) = \begin{cases} j-i & \text{if } z \in \Delta(i,j,x,v) \\ 0 & \text{otherwise.} \end{cases}$$

where, for any $i, j \in Q, x \in X, v \in \mathcal{V}$, $\Delta(i,j,x,v)$ denotes the consecutive, with respect to the lexicographic ordering on $Q \times Q$, left closed, right open intervals of the real line, each having length $\lambda_{ij}(x,v)$ (for details see [6]). We can now give the definition of controlled SDP execution.

Definition 6 (Controlled SDP execution) *A stochastic process* $(x(t), q(t)) \in X \times Q$ *is called a controlled SDP execution if it is the solution of the following stochastic differential equations:*

$$\begin{aligned} dx(t) &= f(q(t), x(t), v(t))dt + \sigma(q(t), x(t))dW_t, \\ dq(t) &= \int_{\mathbb{R}} h(x(t), q(t^-), v(t), z)\varphi(dt, dz) \end{aligned}$$

for $t \geq 0$ *with* $x(0) = x_0$, $q(0) = q_0$ *where*

- x_0 is a prescribed $\mathbb{R}^n$-valued random variable;
- q_0 is a prescribed Q-valued random variable;
- W_t is a n-dimensional standard Wiener process;
- $\varphi(dt, dz)$ is an $\mathcal{M}(\mathbb{R}^+ \times \mathbb{R})$-valued Poisson random measure with intensity $dt \times m(dz)$, m being the Lebesgue measure on $\mathbb{R}$ (see [17]);
- $\varphi(.,.)$, W_t, x_0 and q_0 are independent;
- $v(.)$ is a $\mathcal{V}$-valued process with measurable sample paths satisfying the nonanticipativity property (see [7]).

A process $v(.)$ satisfying last condition is said *admissible control policy*. An admissible control policy is said *feedback* if $v(.)$ is progressively measurable with respect to the natural filtration $\mathcal{F}_t = \{x(s), q(s), s \leq t\}$. A feedback control policy is said *homogeneous Markov policy* if $v(t) = \tilde{v}(x(t), q(t))$ for a given measurable map $\tilde{v} : \mathbb{R}^n \times Q \to \mathcal{V}$. It is shown in [7] that under a Markov policy v, the above SDE admits an almost sure unique strong solution.

In this context optimal control problems can be formulated by defining a cost function:

$$J(x,i,v) = \int_U c(x,i,u)v(du)$$

where $c : \mathbb{R}^n \times Q \times U \to \mathbb{R}^+$ such that $c(., i, .)$ is continuous for any $i \in Q$. Let $v(.)$ be an admissible policy and $(x(.), q(.))$ the corresponding process. The pathwise long-run average cost incurred under $v(.)$ is

$$\limsup_{T\to\infty} \frac{1}{T} \int_0^T c(x(t), q(t), v(t))dt.$$

[7] show how to minimize this cost over all admissible policies establishing the existence of a stable Markov policy which was a.s. optimal.

2.6 A General Class of Stochastic Hybrid Processes

From a theoretical point of view it is possible to define a general class of SHP which includes all the models considered so far as special cases. General Stochastic Hybrid Processes (GSHP) are a class of non-linear stochastic continuous-time hybrid dynamical systems. GSHPare characterized by an hybrid state defined by two components: a continuous state and the discrete state. The continuous state evolves in according to a SDE which depends on the hybrid state, both continuous and discrete. Switchings between two discrete states are either spontaneous or forced. Whenever a switching occurs, the hybrid state is instantaneously reset to a new value according to a probability measure which depends itself on the past hybrid state. In the following we use the notation developed for PDMP.

Definition 7 (General Stochastic Hybrid Processes) *A General Stochastic Hybrid Process is a collection* $H = ((Q, d, X), f, \sigma, Init, \lambda, R)$ *where*

- Q is a countable set of discrete variables;
- $d : Q \to \mathbb{N}$ is a map giving the dimensions of the continuous state spaces;
- $X : Q \to \mathbb{R}^{d(.)}$ maps each $q \in Q$ into an open subset $X(q)$ of $\mathbb{R}^{d(q)}$;
- $f : \mathcal{D}(Q, d, X) \to \mathbb{R}^{d(.)}$ is a vector field;
- $\sigma : \mathcal{D}(Q, d, X) \to \mathbb{R}^{d(.)\times m}$ is a $X(.)$-valued matrix with $m \in \mathbb{N}$;
- $Init : \mathcal{B}(\overline{\mathcal{D}}) \to [0, 1]$ is an initial probability measure on $(\overline{\mathcal{D}}, \mathcal{B}(\overline{\mathcal{D}}))$, with $Init(\mathcal{D}^c) = 0$;
- $\lambda : \mathcal{D}(Q, d, X) \to \mathbb{R}^+$ is a transition rate function;
- $R : \mathcal{B}(\overline{\mathcal{D}}) \times (\mathcal{D}(Q, d, X) \cup \Gamma((Q, d, X), f)) \to [0, 1]$ is a transition measure, with $R(\mathcal{D}^c, .) = 0$.

One can informally define solutions for this class of systems. Evolution starts by selecting an initial condition for the hybrid state, $q_0 \in Q$ and $x_0 \in X(q_0)$ according to the probability measure $Init$. Continuous evolution is then governed by the SDE

$$dx(t) = f(q_0, x(t))dt + \sigma(q_0, x(t))dW_t \tag{1}$$

starting at $x(0) = x_0$, where W_t is a m-dimensional standard Wiener process. $q(t) = q_0$ remains constant along continuous evolution.

Evolution along the solution of (1) can continue as long as $x(t) \in X(q(t))$. As soon as this condition is violated a forced transition will take place. During continuous evolution, spontaneous transitions can also take place; for small δt the probability that a spontaneous transition will take place in the interval $[t, t + \delta t]$ is

$$\lambda(q(t), x(t))\delta t + o(\delta t).$$

If either a spontaneous or a forced transition takes place at time T a new hybrid state, $(q(T), x(T))$ is selected according to the probability measure $R(\cdot, \lim_{t\uparrow T}(q(t), x(t))$ and the process is repeated.

This definition of a GSHPis indeed a natural generalization of PDMP, SDP and SHS. However a number of technical difficulties need to be resolved before it can be used in practice. For example, assumptions need to be imposed to ensure that solutions exist, are unique, have the Markov property, etc. An investigation along these lines is currently underway WP2 of HYBRIDGE.

3 Relevance to Safety Critical ATM Situations

In deliverable D1.1 of WP1 of HYBRIDGE we identified the following safety critical ATM situations, intended to be used as benchmark problems for the Conflict Detection and Resolution (CDR) studies:

1. Vertical crossings.
2. Overtake manoeuvres in unmanaged airspace.
3. ATC sector transitions, especially at Terminal Maneuvering Area (TMA) entry and exit points.
4. Missed approaches.

In D1.1 we also identified the most prominent types of discrete, continuous and stochastic dynamics for each one of these situations. In this section we juxtapose the discussion in D1.1 with the description of the hybrid stochastic processes in Section 2 in an attempt to determine which class of processes would be most suitable for each of the safety critical situations.

3.1 Vertical Crossings

Continuous Dynamics. Continuous dynamics are dominant in vertical crossing situations. In this case the continuous dynamics are primarily restricted to the vertical plane. They can be adequately modelled by a point mass model and energy balance equations [18, 19].

During climb an aircraft will keep its thrust constant at climb thrust and will control its speed by adjusting the flight path angle. In each phase of the climb the aircraft tries to maintain a certain speed, which depends on its altitude. This speed profile is piecewise constant as a function of altitude [18, 19]. Fixing thrust and speed (hence flight path angle) implicitly determines a certain rate of climb; the aircraft accept whatever rate of climb results from this process. Notice that in addition to the aircraft parameters (mass, climb thrust, etc.) the rate of climb also depends on altitude through the speed profile.

Discrete Dynamics. In addition to the discrete dynamics due to standard Air Traffic Control/Controller (ATC) practice (the use of way points, vectoring, altitude dependent speed profiles, etc.) vertical crossings involve other type of discrete dynamics if they are the result of a take-off, or an impending landing. Discrete changes occur at the time of the take off and landing themselves, when the aircraft joins or leaves the ATC system. Another type of discrete change relevant to landing aircraft occurs at the "top of descent", the point where the ATC commands the pilot to begin the final approach.

Another type of discrete dynamics relevant to vertical crossings involves the flight levels. The motion of an aircraft will change dramatically when it levels off at the desired flight level. Crossing of flight levels where other aircraft may be moving is another discrete phenomenon that may be important for CDR purposes.

Stochastic Phenomena. For climbing aircraft the primary source of uncertainty seems to be the continuous motion. Climbing rates depend on the aircraft weight, which may vary widely with aircraft type, route (and hence fuel), number of passengers, type of cargo, etc. Additional uncertainty comes from the settings of the Flight Management System (FMS)(e.g. the climb thrust). All this information is (at best) only partially known to ATC. Wind is another important source of uncertainty. The paths of climbing aircraft tend to be steeper in the presence of strong head winds.

Uncertainty in the continuous motion also induces uncertainty in the discrete transitions. The time at which the aircraft will reach a particular flight level and its horizontal position at that time may be difficult to predict with accuracy. This is important if conflicts are to be avoided with other aircraft moving at that flight level.

Uncertainty in the discrete transitions becomes more prominent in the case of descending aircraft. A major source of uncertainty in this case is the location of the top of descent. It appears that ATC has a lot of freedom in deciding when the descent maneuver should commence. This uncertainty can have important consequences for CDR and the efficiency of the system.

Connection with SHP. The above discussion suggests that SDP may be the most appropriate framework for modelling climbing aircraft. This is because of

1. the uncertainty in the continuous dynamics due to wind and
2. the spontaneous transitions that occur when, for example,
   - an aircraft levels off at its desired altitude,
   - moves from one flight level to the next,
   - changes its speed due to a change in altitude.

Uncertainty in parameter values (e.g. mass and climb thrust) can easily be modelled by introducing auxiliary variables (e.g. m and T) for these quantities governed by trivial differential equations (e.g. $\dot{m} = \dot{T} = 0$). Uncertainty in the parameter values will be reflected by uncertainty in the initial conditions. The continuous state will not change during discrete transitions.

Among the modelling frameworks in the literature, SHS appears to be the most appropriate for descending aircraft. The situation is complicated in this case, however, due to the uncertainty about the top of descend. This uncertainty seems to be naturally modelled by a spontaneous transition. Unfortunately, the modelling frameworks available for modelling spontaneous transitions seem to miss other important phenomena: PDMP makes it difficult to capture the effect of wind, while SDP makes it difficult to capture forced transitions due to levelling off, landing, etc. It appears that the ideal model for descending aircraft would take the form of a GSHP.

3.2 Overtake

Continuous Dynamics. Continuous dynamics are also important for overtake manoeuvres. Overtake manoeuvres in unmanaged airspace take place primarily at cruising altitudes.

Discrete Dynamics. The discrete dynamics in overtake situations are also limited. Initiation and termination of the manoeuvre are the two more prominent discrete phenomena. Depending on how the manoeuvre is executed, a distinction may also have to be drawn between the different phases of the manoeuvre. Figure 1 indicates how an overtake manoeuver may be executed using lateral motion or altitude changes.

Stochastic Phenomena. The main source of uncertainty here seems to be the weather and in particular wind. This again affects primarily the continuous motion of the aircraft.

Connection with SHP. Since the main source of uncertainty is the wind, a PDMP will not be sufficient for modelling overtakes because it only allows deterministic continuous motion. This leaves SDP and SHS. The overtake maneuver can be represented by a series of waypoints for the overtaking aircraft. These may, for example, take the overtaking aircraft to one side and past the slower aircraft and then return the aircraft to its original course. The flight plan can be naturally split into different segments, each representing the course from one way point to the next. The current segment is a discrete variable. The aircraft position and heading are continuous variables and do not change when the discrete state representing the current segment changes. The current segment will change when the aircraft reaches the next way point. There will be some uncertainty in when this occurs due to uncertainties in the tracking and navigation system. This suggests that SDP will be the most appropriate framework for modelling overtakes.

3.3 Sector Transitions

Continuous Dynamics. Accurate models of continuous dynamics seem to be less important for sector transitions. In certain cases it may also be possible to separate the vertical and horizontal components of the continuous model. For transitions at the TMA gates the horizontal and vertical may interact, since aircraft may be climbing when leaving the TMA, or descending when entering it.

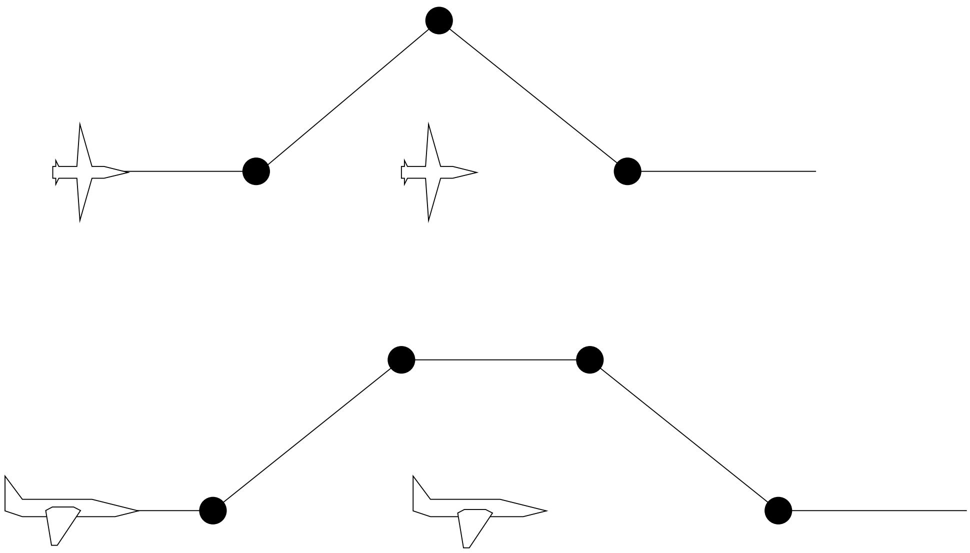

Figure 1: Phases of an overtake maneuver.

Discrete Dynamics. Sector transitions are intrinsically discrete phenomena. The state of the system undergoes a transition when an aircraft changes sector and is handed of from one ATC to another. One needs to ensure that the aircraft will find itself in a safe configuration in the new sector before the transition is allowed to go ahead. If not (for example, if the traffic density in the new sector is already too high) the transition should be delayed. In this case, one has to ensure that it is safe for the aircraft to remain in its current sector.

Currently all these tasks are performed by ATC based on established procedures. Some flexibility is allowed depending on traffic conditions; for example an air traffic controller may delay accepting a flight coming into a sector if there are more urgent problems to deal with. Introducing automation safely requires full understanding of these procedures. One would have to ensure, for example, that the system does not give rise to situations where an aircraft can neither be handed on nor remain in the current sector safely.

Stochastic Phenomena. There is some uncertainty in the time at which a hand-off from one sector to the other will take place. The exact timing of the transition may depend on the traffic in both sectors. Hand-off may take place earlier if traffic in the current sector is heavy, or may be delayed if traffic in the receiving sector is heavy. The effect of this uncertainty tends to be small in current conditions, but may become more pronounced as traffic densities increase.

Connection with SHP. Piecewise deterministic Markov processes can be used to capture the ATC sector transition process. The above characteristics suggest that it is possible to model sector transitions using a PDMP. Let $X(q)$ with $q \in Q$ denote the area covered by each ATC sector, where Q is the total number of the ATC sectors. Since precise information about continuous motion of the aircraft is not crucial, we can assume that the continuous motion of the aircraft in each sector is deterministic, governed by a given vector field f. The transition rate function λ will have support in a neighbourhood of the sector boundary. The magnitude of λ will depend on the traffic density in the current sector, ρ_c, and the traffic density in the new sector, ρ_n; more specifically λ increases as ρ_c increases or ρ_n decreases. Since the continuous state does not change during a sector transition and the next sector is known, R can be modelled by a simple function.

3.4 Missed Approaches

Continuous Dynamics. Missed approaches involve the closest coupling of the horizontal and vertical components. Whether an approach is aborted or not may depend on the combination of horizontal and vertical positions, in addition to external factors such as a runway incursion, or wind shear.

Discrete Dynamics. Missed approaches also contain an intrinsically discrete component: whether the approach is aborted or not. The "go around" manoeuvre may require additional discrete choices, such as vectoring commands and additional way points to slot the aircraft executing the go-around maneuver among the rest of the incoming traffic.

Stochastic Phenomena. The cause of the missed approach may be modelled as probabilistic. It may involve either the continuous motion (e.g. deviation from the approach pattern due to wind) or a discrete occurrence (e.g. a runway incursion). There is also uncertainty in whether the aircraft goes ahead with the current approach or whether it decides to goes around. In the latter case, the timing of the decision is another source of uncertainty.

Connection with SHP. Uncertainty in this case comes from both the continuous and the discrete dynamics. Therefore, PDMP and SHS are insufficient for modelling missed approaches. This leaves SDP.

SDPappear to be suitable for modelling situations when approaches are aborted as a result to deviations due to wind. The aircraft position and heading, which are the continuous variables of the model, do not change when the discrete state changes; this is required for an SDP model. SDP can also model changes in the flight plan segment when the aircraft reaches a way point (by introducing rate functions with support in a neighbourhood of the way point).

Missed approaches due to runway incursions are more difficult to capture in the SDP framework. While a runway incursion is often the result of human decision making in responce to some event, it may be easier to capture as an abrupt change in response to an external event, which would be more accurately modeled by a forced transition. This suggests that a GSHP model may be needed to accurately model this case.

3.5 Summary

The discussion in this section is summarized in Table 3. The conclusions seem to make a case for the need to develop further a more general class of stochastic hybrid processes than those found in the literature. This is because

1. Different types of models seem to be needed to capture the different situations. This implies that a number of different techniques and tools must be mastered to be able to deal with all the cases of interest. If a GSHP framework was available the process would be more efficient, since a single set of results, simulation procedures, etc. could be used in all cases.

2. Certain situations, such as vertical crossings during descent and missed approaches due to runway incursions, would be more accurately modelled by a GSHP.

In the examples considered npo explicit mention was made of situations that require resetting the continuous state of the system during discrete transitions. The main reason is that this often depends on the controlled variables and coordinate frames used in the definition of the continuous state soace. For example, if the positions of all aircraft are given in a global coordinate frame, then the continuous state will remain constant during discrete transitions. If on, the other hand, the positions of aircraft are given in coordinates relative to their flight plan or to one another, then the continuous state may experience discrete transitions whenever aircraft reach way points, execute turns, etc. Both these types of relative coordinates are common in CDR studies. Moreover, in both cases there are savings to be had by modelling the system in relative coordinates; typically the dimension of the continuous state space is smaller. Experience suggests that if resetting of continuous states is allowed, similar savings may also be possible in terms of the number of discrete states.

Situation	Continuous dynamics	Discrete dynamics	Stochastic phenomena	Hybrid Model
Vertical crossing	-primarily vertical	-take-off/landing -top-of-descend -flight levels	-wind -FMS settings -aircraft mass -top of descend	SDP (GSHP for descend)
Overtake manoeuvres	-horizontal or -vertical	-initiation -termination -maneuver phases	-wind	SDP
Sector transitions	-less important	-hand-off	-transition timing	PDMP
Missed approaches	-horizontal and -vertical -wind shear	-abort/complete -vectoring -way points -runway incursions	-wind -incursion	SDP (GSHP for incursions)

Table 3: Overview of modelling needs of safety critical ATM situations

4 Notes on System Identification

4.1 The need for identification

The stochastic models developed to capture the safety critical situations will invariably contain a number of “free” parameters whose values one has to select to ensure that the behaviour of the model is realistic. Examples of parameters include

- Aircraft masses, aerodynamic coefficients, etc.
- Details of flight plans (way points, speeds, altitudes).
- FMS gains.
- Wind variance and correlation structure.

Accurate values for some of these parameters are available. For example, many aircraft parameters are available through the Base of Aircraft Data (BADA) database (access to which has kindly been granted by Eurocontrol Experimental Center). Likewise, details of flight plans have been made available to us by Eurocontrol’s Central Flow Management Unit (CFMU).

Values for other parameters are less readily available, however. For example, the parameters of the simplified FMS models one needs to capture the safety critical situations may not correspond directly to parameters of a real FMS. Even if they did, the values of such parameters are typically proprietary. Likewise, setting of climb thrust, speed, etc. used by aircraft during vertical manoeuvres are typically not available to ATC. Finally, precise information about wind statistics (variances, spatio-temporal correlation, etc.) seems very difficult to obtain.

Educated guesses about bounds on the values of some these parameters can be made indirectly in some cases. For example, bounds on aircraft mass are available from BADA. One may also assume that the FMS will stabilize the aircraft trajectory about the flight plan. The requirement for stability will impose bounds on some of the FMS (and possibly wind) parameters. To make the system realistic, however, the parameter values should be chosen to match the observed behavior of real aircraft. This can be done through a process of *system identification*. System identification methods provide algorithms for systematically tuning the values of the parameters to get a better fit between the behavior predicted by the model and experimentally observed data.

4.2 System Identification Limitations

When we try to pose the problem of selecting values for the FMS gains in a system identification framework we immediately run against the problem of *time scales*. FMS control takes place at fairly high frequency (of the order of one sample every second, i.e. 1Hz). Ideally the data used for identification should be sampled at the same frequency. Unfortunately, because the available data is obtained through radar, it is likely to be much more infrequent: less than 0.1Hz if we assume a radar sampling time of 12 seconds. In the data collection experiments, the data may be even more sparse, e.g. 1 sample every minute, or 0.017Hz.

Is system identification possible when the data is so sparse? This question is difficult to answer before data is made available and experiments are conducted. Some obvious solutions immediately prove to be problematic. For example, slowing the FMS model down to the sampling rate of the data is unlikely to work. Basic stability arguments suggest that even though with high frequency FMS sampling (1Hz) the system is stable, low frequency sampling (e.g. 0.1Hz) requires very high gains and makes the system much harder to stabilize. Our initial simulations indicate that in the presence of noise models where the FMS operates at low frequency (say 0.1Hz) go unstable, even though the corresponding high frequency models (say 1Hz) work fine.

Another idea is to use system identification methods designed specifically to deal with missing data [20, 21, 22]. Roughly speaking, the idea behind these methods is to use Kalman filtering to fill in the missing data. This approach works when there is an occasional missing sample (e.g. every third sample is missing). It is unlikely to work in this case, however, since most of the samples (possibly 59 out of 60!) are missing.

We have tried to identify a number of solutions (listed below) that seem viable for this problem. The relative merits of each will have to be investigated in numerical experiments, once data becomes available.

Identify low frequency model Develop a low frequency (e.g. 0.017Hz) model of the entire system (*not just the FMS*). This approach has two advantages.

1. Identification can be carried our using standard methods.
2. The resulting model will be more efficient: simulations, detection algorithms, etc. will run much faster. For model sampling rates up to the radar sampling frequency (roughly 0.1Hz) the performance of conflict detection algorithms will be completely unaffected. Even for lower model sampling rates (e.g. low as 0.05Hz) the effect is likely to be small.

The main disadvantage of this approach is that much of the intuitive structure of the model will be lost. The parameters identified will no longer correspond to particular

FMS gains, or any other physically meaningful quantity. More importantly, the explicit dependence of the dynamics on things such as the nominal wind will be lost. If sets of data for constant values of the nominal wind are available, one can run a series of identification experiments and identify a different model for each value of the nominal wind. One can then use interpolation to capture other wind conditions. This approach is somewhat unsatisfactory. It is labor intensive, it requires a lot of very carefully selected data, and, in the end, the resulting model is still likely to be inaccurate.

Modify system identification methods Even though general purpose identification techniques would be difficult or impossible to derive for systems sampled so infrequently, we hope that we may be able to take advantage of the structure and physical intuition behind the model to develop custom made methods. One idea is to determine (by hand or using computer algebra packages such as Maple) how the parameters of a high frequency (e.g. 1Hz) model enter a low frequency (e.g. 0.1Hz) model. We can then identify the low frequency model and solve a set of nonlinear equations to obtain values for the high frequency system parameters. Even though this approach is likely to be intractable for data sampled every minute, because the model is low dimensional and the number of parameters small it may work for data sampled every 10-12 seconds.

Sample the data faster Any increase in the data sampling rate would make the identification problem simpler. Intuition suggests that shorter "chunks" (e.g. a few minutes) of frequently sampled data are likely to be more effective than longer chunks (e.g. an hour) at a low sampling rate. An other advantage of shorter chunks of data is that the nominal wind is more likely to be constant, which would also simplify the identification problem.

4.3 Data Availability

Formal system identification is impossible without access to data. To identify parameters for aircraft models from the point of view of ATC one needs access to the following pieces of data.

- The flight plans the aircraft were meant to follow.
- Their actual radar tracks.
- The weather conditions (ATC weather charts) when the flight took place.

Additional pieces of information (such as radar noise statistics) may improve the performance of the identification algorithms.

Besides the difficulties listed above, the main difficulty with the system identification approach seems to be that data is not available. Some is proprietary, some seems not to exist (e.g. wind correlation data). To circumvent this problem, we intend to make progress in two different directions. The first is through expert advice. In the HYBRIDGE project we are blessed with experts in a number of areas, including FMS design and air traffic control. Once models have been developed and simulated, we can get a rough idea of how realistic they are by talking to experts. To this end we intend to:

1. Interact with experts on flight management systems, to verify that the proposed structure of the model is reasonable from an FMS point of view. Among other things we would like to clarify:
   - the control objectives and control strategies of the FMS;
   - whether the FMS controller parameters are "gain scheduled" depending on the air speed and/or the wind speed;
   - when turns between flight plan segments are initiated;
   - how these turns are executed.
2. Interact with experts in meteorology and ATC to get a better idea of
   - the type of nominal wind information made available to the air traffic control system;
   - the properties of the stochastic deviation from this nominal data (primarily the covariance functions).
3. Interact with experts in ATC to get feedback on whether the trajectories produced by our models are reasonable.

The second approach to tuning the models is to use existing studies based on real data available in the literature, e.g. [23, 24, 25, 26, 27]. We can compare the output of simulations of our model with the statistics predicted by these studies and tune the parameters of our model until the macroscopic statistics agree. The point is of course that our model will not only be able to match the predictions of the models used to tune it, but also includes other phenomena (such as wind correlation) not present in earlier models. The disadvantage is that it is unlikely values for all parameters would be obtainable this way. For example, an implicit trade off exists between controller gains and wind variance; it is possible to get the same macroscopic statistics if we increase both the gains (making the system more stable) and the wind perturbation. A similar trade off makes it difficult to distinguish between the effect of the thrust setting during climb and that of the mass of the aircraft; similar climbing rates are possible if both parameters increase. Accurate knowledge of on be of the two parameters is needed to decouple the two phenomena in such cases.

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