Distributed Control and Stochastic Analysis of Hybrid Systems
Supporting Safety Critical Real-Time Systems Design

WP2: Stochastic hybrid systems based modelling of accident risk

Generalised stochastic hybrid processes as strong solutions of stochastic differential equations

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1. Introduction

Many modelling and control studies for complex dynamical multi-agent systems have in common that they make use of continuous-time strong Markov processes the state of which is hybrid, i.e. one state component evolves in an Euclidean space, the other state component evolves in a discrete set, and each component may influence the evolution of the other component. Recently, Hu et al. 2000 [13] noticed that there is a need to formally characterize hybrid state processes of which an Euclidean valued jump may depend of the simultaneous switching. For short, we refer to such simultaneous jumps with switching dependency as hybrid jumps.

There are two types of hybrid jumps possible: those that happen at instants of hitting some boundary, and those that happen at a sudden instant (i.e. Poisson type). A well-known class of semimartingale Markov processes are the Piecewise Deterministic Markov processes (Davis 1984, 1993 [6, 7]; Vermes 1985 [17]). They incorporate both types of hybrid jumps, however they do not include diffusion. Moreover, their specific formulation does not allow a straightforward inclusion of diffusion. At the other side of the spectrum there is the class of switching diffusion as solutions of stochastic differential equations (Ghosh et al. 1993, 1997 [8, 9]). These processes incorporate diffusions, but are lacking many interesting phenomenon of interaction between the Euclidean and discrete valued process components. There is a clear gap in the spectrum of stochastic hybrid processes, with PDPs on one side and switching diffusions at the other side. Because the stochastic analysis hurdles to be overcome are significant, few authors only have tried to help closing this gap. Borkar et al. 1991 [5] have studied switching diffusion processes the control strategy of which included the possibility of an instantaneous jump at some boundary. The solution of this problem was characterized in the setting of solutions to the Hamilton Jacobi Bellman equation. Inherent to the limitation in scope, this study did not characterize semimartingale and strong Markov properties of the resulting stochastic process. Bensoussan & Menaldi 2000 [1] have significantly extended this stochastic hybrid control framework, with a similar restriction of the scope. In order to contribute to the filling of this gap, Blom 1990 [2] (chapter IV) started the development of an alterative approach: study a stochastic differential equation (SDE) on a hybrid space that is driven by Brownian motion and Poisson random measure, and characterize the class of stochastic processes that are defined by pathwise unique solutions of such an SDE. One of the results was the explicit characterization of hybrid jumps as a discontinuity in the Euclidean valued process component that happens synchronous with a discontinuity in the discrete valued process component and the size of which depends of the discrete valued process component prior and after the jump. The paper by Hu 2001 [13] stimulated the development of an improved version and a publication of this approach for the first time at a conference (Blom 2003 [4]). Basically this paper places the SDE of Lepeltier & Marchal 1976 [16] on a hybrid state space, identifies the resulting class of stochastic hybrid processes and shows that the semimartingale and strong Markov properties identified by Lepeltier & Marchal 1976 [16] carry over to the resulting stochastic hybrid processes. The main limitation of this approach was that the understanding of the stochastic technicalities involved with the pathwise uniqueness, semimartingale and strong Markov properties remained hidden in the technical proofs by Lepeltier & Marchal 1976 [16]. This created difficulties in the further extension of the approach while allowing instantaneous jumps at boundaries.

In order to improve this situation, Ghosh & Bagchi 2004 [10] have performed a study on some of these technicalities for a more restricted class of SDE's and relating these to the hybrid stochastic control processes of Borkar et al. 1991 [5] and Bensoussan & Menaldi 2000 [1]. Although this did not solve all outstanding issues, it did lead to a better understanding of the remaining technical challenges.

The aim of the current report is to significantly further the study of SDE's on a hybrid space, including characterizations of its solutions in terms of pathwise uniqueness, semimartingale and strong Markov process properties. We use Jacod & Shyriayev 1987 [14] and Gihman & Skorohod 1982 [12] as a starting point for characterizing jump-diffusion process solutions of SDE's. This yields a valuable improvement over the Lepeltier & Marchal 1976 [16] regarding the understanding of semimartingale property and pathwise uniqueness of jump-diffusions. From this point on we follow a similar path as taken by Blom 1990, 2003 [2, 4] in transferring this pathwise uniqueness and semimartingale understanding to the class of stochastic hybrid processes. This subsequently allows to incorporate instantaneous jumps at a boundary within the same framework including pathwise uniqueness and semimartingale property. Finally we introduce a completely novel approach in showing strong Markov property.

The report is organized as follows. Section 2 provides a brief introduction to semimartingales. Section 3 presents the existence and uniqueness results for $\mathbb{R}^n$-valued jump-diffusions. Section 4 extends these results to hybrid state processes with Poisson and hybrid Poisson jumps. In section 5 we characterize a general stochastic hybrid process which includes jumps at the boundaries. Section 6 presents a brief comparison of different stochastic models. Finally, the Markov and the Strong Markov properties for a general stochastic hybrid process are shown in sections 7 and 8.

2. Semimartingales and characteristics

We are interested in general semimartingale processes which are solutions of SDEs. Semimartingales form the most general class of stochastic processes for which a full stochastic calculus, including Itô's lemma exists. Very large classes of diffusions and jump-diffusions can be studied as semimartingale solutions of a wide class of SDEs. Following Jacod & Shiryaev 1987 [14] we provide basic results concerning semimartingales, their canonical representation and their relation with the large class of SDEs to be studied in this report.

Throughout this report we assume that a probability space $(\Omega, \mathcal{F}, P)$ is equipped with a right-continuous filtration $(\mathcal{F}_t)_{t\geq 0}$. The stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ is called complete if the σ-algebra $\mathcal{F}$ is P-complete and if every $\mathcal{F}_t$ contains all P-null sets of $\mathcal{F}$. Note that it is always possible to "complete" a given stochastic basis, if it is not complete, by adding all subsets of P-null sets to $\mathcal{F}$ and $\mathcal{F}_t$. We will always assume that a given stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ is "completed".

The *predictable σ-algebra* is the σ-algebra $\mathcal{P}$ on $\Omega \times \mathbb{R}_+$ that is generated by all cág (left-continuous) adapted process (considered as mappings on $\Omega \times \mathbb{R}_+$). A process or random set that is $\mathcal{P}$-measurable is called *predictable*.

The $\mathbb{R}^n$-valued càdlàg stochastic process $\{X_t\}$ defined on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ is a *semimartingale* if X_t admits a decomposition of the form

$$X_t = X_0 + A_t + M_t,\ t \geq 0, \tag{2.1}$$

where X_0 is a finite-valued and $\mathcal{F}_0$-measurable, where $\{A_t\} \in \mathcal{V}^n$ (a process of *bounded variation*), $\{M_t\} \in \mathcal{M}^n_{loc}$ (an n-dimensional *local martingale* starting at 0). Furthermore, we have that for each $t \geq 0$, A_t and M_t are $\mathcal{F}_t$-measurable. $\{M_t\} \in \mathcal{M}^n_{loc}$ if and only if there exists a sequence of $(\mathcal{F}_t)_{t\geq 0}$-stopping times $(\tau_k)_{k\geq 1}$ such that $\tau_k \uparrow \infty$ (P-a.s.) for $k \longrightarrow \infty$ and for each $k \geq 1$, the *stopped process*

$$\{M_t^{\tau_k}\} \text{ with } M_t^{\tau_k} = M_{t\wedge\tau_k},\ k \geq 1, \tag{2.2}$$

is a *martingale*:

$$\mathbb{E}|M_t^{\tau_k}| < \infty,\ \mathbb{E}[M_t^{\tau_k}|\mathcal{F}_s] = M_s^{\tau_k}\ (P - \text{a.s.}),\ s \leq t. \tag{2.3}$$

Denote by $\mu = \mu(\omega; ds, dx)$ the measure describing the jump structure of $\{X_t\}$:

$$\mu(\omega; (0,t] \times B) = \sum_{0<s\leq t} I(\Delta X_s(\omega) \in B),\ t > 0, \tag{2.4}$$

where $B \in \mathcal{B}(\mathbb{R}^n \setminus \{0\})$, $\Delta X_s = X_s - X_{s-}$ and $I(\cdot)$ stands for the indicator function. By $\nu = \nu(\omega; ds, dx)$ we denote a compensator of μ, i.e. a predictable measure with the property that $\mu - \nu$ is a local martingale measure. This means that for each $B \in \mathcal{B}(\mathbb{R}^n \setminus \{0\})$:

$$(\mu(\omega; (0,t] \times B) - \nu(\omega; (0,t] \times B))_{t>0} \tag{2.5}$$

is a local martingale with value 0 for $t = 0$.

A semimartingale $\{X_t\}$ is called *special* if there exists a decomposition (2.1) with a *predictable* process $\{A_t\}$. Every semimartingale with *bounded jumps* ($|\Delta X_t(\omega)| \leq b < \infty, \omega \in \Omega, t > 0$) is special (see Jacod & Shiryaev 1987 [14], Chapter I, 4.24).

Let h be a truncation function, i.e. $\Delta X_s - h(\Delta X_s) \neq 0$ if and only if $|\Delta X_s| > b$ for some $b > 0$. Hence

$$\widetilde{X}_t = \sum_{0<s\leq t} (\Delta X_s - h(\Delta X_s)) \tag{2.6}$$

denotes the jump part of $\{X_t\}$ corresponding to *large jumps*. The number of the large jumps still is finite on $[0,t]$, for all $t > 0$, because for all semimartingales (Jacod & Shiryaev 1987 [14], Chapter I, 4.47)

$$\sum_{0<s\leq t} (\Delta X_s)^2 < \infty,\ P - a.s. \tag{2.7}$$

The process $\{X_t - \widetilde{X}_t\}$ is a semimartingale with *bounded jumps* and hence it is special:

$$X_t - \widetilde{X}_t = X_0 + \widetilde{B}_t + \widetilde{M}_t \tag{2.8}$$

where $\{\widetilde{B}_t\}$ is a predictable process and $\{\widetilde{M}_t\}$ is a local martingale. The "tilde" above process denotes the dependence on truncation function h.

Every local martingale $\widetilde{M}_t$ can be decomposed as:

$$\widetilde{M}_t = \widetilde{M}_t^c + \widetilde{M}_t^d \tag{2.9}$$

where $\widetilde{M}_t^c$ is a *continuous* (martingale) part and $\widetilde{M}_t^d$ is a *purely discontinuous* (martingale) part which satisfies:

$$\widetilde{M}_t^d = \int_0^t \int h(x)(\mu(ds,dx) - \nu(ds,dx)). \tag{2.10}$$

However the continuous martingale part $\widetilde{M}_t^c$ does not depend on h and will be denoted by M_t^c (the *continuous martingale* part of X_t). $\widetilde{M}_t = M_t^c + \widetilde{M}_t^d$
By definition of μ and $\{\widetilde{X}_t\}$ we have

$$\widetilde{X}_t = \int_0^t \int (x - h(x))\mu(ds,dx). \tag{2.11}$$

Consequently, substitution of (2.9 - 2.11) into (2.8) yields the following canonical representation of semimartingale $\{X_t\}$:
(2.12)

$$X_t = X_0 + \widetilde{B}_t + M_t^c + \int_0^t \int h(x)(\mu(ds,dx) - \nu(ds,dx)) + \int_0^t \int (x - h(x))\mu(ds,dx).$$

Next we may assume $h(x) = xI(|x| < 1)$ and replace $\widetilde{B}_t$ by B_t. Then (2.12) takes on the form:
(2.13)

$$X_t = X_0 + B_t + M_t^c + \int_0^t \int_{|x|<1} x(\mu(ds,dx) - \nu(ds,dx)) + \int_0^t \int_{|x|\geq 1} x\mu(ds,dx).$$

We denote by $\langle M_t^c \rangle$ the predictable quadratic variation of $\{M_t^c\}$, hence $(M_t^c)^2 - \langle M_t^c \rangle$ is a local martingale.

We call *characteristics* associated with h of the semimartingale $\{X_t\}$ (if there may be an ambiguity on h) the triplet (B_t, C_t, ν) consisting in :

1) $B_t = (B_t^i)_{i\leq n}$ is a predictable process in $\mathcal{V}^n$, namely the process $B_t = \widetilde{B}_t$ appearing in (2.8).

2) $C_t = (C_t^{ij})_{i,j\leq n}$ is a continuous process in $\mathcal{V}^{n\times n}$ (a process of bounded variation), namely $C_t = \langle M_t^c \rangle$.

3) ν is a predictable random measure on $\mathbb{R}_+ \times \mathbb{R}^n$, namely the compensator of random measure μ associated to the jumps of X by (2.4).

3. Semimartingale strong solution of SDE

3.1. **Semimartingale solution of an SDE.**

Definition 3.1. *The canonical setting.* Ω is the "*canonical space*" (also denoted by $\mathbb{D}(\mathbb{R}^n)$) of all càdlàg functions $\omega : \mathbb{R}_+ \to \mathbb{R}^n$; X is the "canonical process" defined by $X_t(\omega) = \omega(t)$; $\mathcal{H} = \sigma(X_0)$; finally $(\mathcal{F}_t)_{t\geq 0}$ is generated by X and $\mathcal{H}$, by which we mean:

(i) $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s^0$ and $\mathcal{F}_s^0 = \mathcal{H} \vee \sigma(X_r : r \leq s)$ (in other words, $(\mathcal{F}_t)_{t\geq 0}$ is the smallest filtration such that X is adapted and $\mathcal{H} \subset \mathcal{F}_0$);

(ii) $\mathcal{F} = \mathcal{F}_{\infty-} (= \bigvee_t \mathcal{F}_t)$.

Throughout this section, the *canonical setting* 3.1 is in force.

Definition 3.2. Let P be a probability measure on $(\Omega, \mathcal{F})$. Then $\{X_t\}$ is called a *jump diffusion* on $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, P)$ if it is a semimartingale with the following characteristics:

$$(3.1) \quad \begin{cases} B_t^i(\omega) = \int_0^t \alpha^i(s, X_s(\omega))ds & (= +\infty \text{ if the integral diverges}) \\ C_t^{ij}(\omega) = \int_0^t \beta^{ij}(s, X_s(\omega))ds & (= +\infty \text{ if the integral diverges}) \\ \nu(\omega; dt \times dx) = dt \times K_t(\omega, X_t(\omega), dx) \end{cases}$$

where:

$$\begin{cases} \alpha : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^n & \text{is Borel} \\ \beta : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}^n \times \mathbb{R}^n & \text{is Borel, } c(s,x) \text{ is symmetric nonnegative} \\ K_t(\omega, x, dy) & \text{is a Borel transition kernel from } \Omega \times \mathbb{R}_+ \times \mathbb{R}^n \text{ into } \mathbb{R}^n, \end{cases}$$

with $K_t(\omega, x, \{0\}) = 0$.

Next, we relate the above with stochastic differential equations, partially following [14].

Let $\mathcal{B} = (\Omega, \mathcal{F}, \mathbf{F}, P)$ be a stochastic basis endowed with:

1. $W = (W^i)_{i\leq m}$, an m-dimensional standard Wiener process (i.e., each W^i is a standard Wiener process, and the W^i's are independent);
2. p_i are Poisson random measures on $\mathbb{R}_+ \times U$ with intensity measure $dt \cdot m_i(du)$, $i = 1, 2$; here, $(U, \mathcal{U})$ is an arbitrary Blackwell space (one may take $U = \mathbb{R}^d$ for practical applications), and m_i, $i = 1, 2$, is a positive σ-finite measure on $U, \mathcal{U}$; We denote the compensated Poisson random measure by $q_i(dt, du) = p_i(dt, du) - dt \cdot m_i(du)$, $i = 1, 2$.

Let us also be given the coefficients:

$$(3.2) \qquad \begin{cases} a = (a^i)_{i\leq n}, & \text{a Borel function: } \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^n \\ b = (b^{ij})_{i\leq n, j\leq m}, & \text{a Borel function: } \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^n \times \mathbb{R}^m \\ f_1 = (f_1^i)_{i\leq n} & \text{a Borel function: } \mathbb{R}_+ \times \mathbb{R}^n \times U \longrightarrow \mathbb{R}^n, \\ f_2 = (f_2^i)_{i\leq n} & \text{a Borel function: } \mathbb{R}_+ \times \mathbb{R}^n \times U \longrightarrow \mathbb{R}^n. \end{cases}$$

Let the initial variable be an $\mathcal{F}_0$-measurable $\mathbb{R}^n$-valued random variable X_0. The stochastic differential equation is as follows:

$$(3.3) \quad dX_t = a(t, X_t)dt + b(t, X_t)dW_t + \int_U f_1(t, X_{t-}, u)q_1(dt, du) + \int_U f_2(t, X_{t-}, u)p_2(dt, du),$$

Define two stochastic sets:

$$D_1 = \{(\omega, t) : p_1(\omega; \{t\} \times U) = 1\},$$
$$D_2 = \{(\omega, t) : p_2(\omega; \{t\} \times U) = 1\}.$$

If at least one of the Poisson random measures, p_1 or p_2, has a "jump" at point (t, u), then $\Delta X_t(\omega) = \mathbf{1}_{D_1}(\omega, t) \cdot f_1(t, X_{t-}(\omega), u) + \mathbf{1}_{D_2}(\omega, t) \cdot f_2(t, X_{t-}(\omega), u)$.

Next, let us assume that the following integrals make sense.

$$\int_0^t |a(s, X_s)|ds < \infty, \ P\text{-a.s.} \tag{3.4}$$

$$\int_0^t \int_U |f_1(s, X_{s-}, u)|^2 ds\, m_1(du) < \infty, \ P\text{-a.s.}, \tag{3.5}$$

$$\int_0^t \int_U |f_2(s, X_{s-}, u)| p_2(ds, du) < \infty, \ P\text{-a.s.}, \tag{3.6}$$

$$\int_0^t |b^{ij}(s, X_s)|^2 ds < \infty, \ P\text{-a.s. for any } i, j \in \{1, \dots, n\} \tag{3.7}$$

for every $t \in \mathbb{R}_+$. By a solution to the SDE (3.3) we mean a càdlàg $\mathcal{F}_t$-adapted process $\{X_t\}$ such that the following equation is satisfied with probability one for every $t \in \mathbb{R}_+$

$$X_t = X_0 + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dW_s + \int_0^t \int_U f_1(s, X_{s-}, u)q_1(ds, du) + \int_0^t \int_U f_2(s, X_{s-}, u)p_2(ds, du). \tag{3.8}$$

If such process $\{X_t\}$ exists and conditions (3.4)-(3.7) are satisfied then it is a semimartingale with the characteristics (associated with truncation function $h = xI(|x| < 1)$) given by (3.1), where

$$\alpha(t, X_t(\omega)) = \Big[a(t, X_t(\omega)) - \int_{|f_1| \geq 1} f_1(t, X_{t-}(\omega), u)m_1(du) + \int_{|f_2|<1} f_2(t, X_{t-}(\omega), u)m_2(du)\Big],$$

$$\beta(t, X_t(\omega)) = b(t, X_t(\omega))b^T(t, X_t(\omega)),$$

$$K_t(\omega, X_t(\omega), A) = \mathbf{1}_{D_1}(\omega, t) \cdot \int_U \mathbf{1}_{A\setminus\{0\}}(f_1(t, X_{t-}(\omega), u))m_1(du) + \mathbf{1}_{D_2}(\omega, t) \cdot \int_U \mathbf{1}_{A\setminus\{0\}}(f_2(t, X_{t-}(\omega), u))m_2(du).$$

3.2. **Existence and uniqueness concepts.** There are two important notions of the sense in which a solution to stochastic differential equation can be said to *exist* and also two senses in which *uniqueness* is said to hold.

Definition 3.3. Strong Existence. We say that strong existence holds if given a probability space $(\Omega, \mathcal{F}, P)$, a filtration $\mathcal{F}_t$, an $\mathcal{F}_t$-Wiener process W, an $\mathcal{F}_t$-Poisson random measures p_1, p_2, and an $\mathcal{F}_0$-measurable initial condition X_0, then an $\mathcal{F}_t$-adapted process $\{X_t\}$ exists satisfying (3.8) for all $t \geq 0$.

Definition 3.4. Weak Existence. We say that weak existence holds if given any probability measure η on $\mathbb{R}^n$ there exists a probability space $(\Omega, \mathcal{F}, P)$, a filtration $\mathcal{F}_t$, an $\mathcal{F}_t$-Wiener process W, an $\mathcal{F}_t$-Poisson random measures p_1, p_2, and an $\mathcal{F}_t$-adapted process $\{X_t\}$ satisfying (3.8) for all $t \geq 0$ as well as $P(X_0 \in B) = \eta(B)$.

Strong existence of a solution requires that the probability space, filtration, and driving terms (W, p_1, p_2) be given first and that the solution $\{X_t\}$ then be found for the given data. Weak sense existence allows these objects to be constructed together with the process $\{X_t\}$. Clearly, strong existence implies weak existence.

Definition 3.5. Strong Uniqueness. Suppose that a fixed probability $(\Omega, \mathcal{F}, P)$, a filtration $(\mathcal{F}_t)_{t\geq 0}$, an $\mathcal{F}_t$-Wiener process W, and an $\mathcal{F}_t$-Poisson random measures p_1 and p_2 are given. Let $\{X_t\}$ and $\{X'_t\}$ be two solutions of (3.3) for the given driving terms (W, p_1, p_2). We say that strong uniqueness holds true if

$$P(X_0 = X'_0) = 1 \Longrightarrow P(X_t = X'_t \text{ for all } t \geq 0) = 1, \tag{3.9}$$

i.e. $\{X_t\}$ and $\{X'_t\}$ are indistinguishable.

Remark 3.6. Since solutions of (3.3) are càdlàg processes the requirement (3.9) can be relaxed to:

$$P(X_0 = X'_0) = 1 \Longrightarrow P(X_t = X'_t) = 1, \text{ for every } t \geq 0. \tag{3.10}$$

Definition 3.7. Weak Uniqueness. Suppose we are given weak sense solutions

$$\{(\Omega_i, \mathcal{F}_i, P_i), (\mathcal{F}_{i,t})_{t\geq 0}, \{X_{i,t}\}\},\ i = 1, 2,$$

to (3.3). We say that weak uniqueness holds if equality of the distributions induced on $\mathbb{R}^n$ by $X_{i,0}$ under P_i, $i = 1, 2$, implies the equality of the distributions induced on $\mathbb{D}(\mathbb{R}^n)$ by $\{X_{i,t}\}$ under P_i, $i = 1, 2$.

Strong uniqueness is also referred to as *pathwise uniqueness*, whereas weak uniqueness is often called *uniqueness in (the sense of probability) law.* Strong uniqueness implies weak uniqueness.

3.3. **Strong Uniqueness.** In what follows we will state and prove strong existence and strong uniqueness theorems for SDE (3.3), following [12], pp.223-245.

We assume that Wiener process W and Poisson random measures p_1 and p_2 are mutually independent. Suppose $\{W_t\}$, p_1 and p_2 are adapted to the given filtration $(\mathcal{F}_t)_{t\geq 0}$. If τ is a stopping time relative to $\mathcal{F}_t$ and X_τ is an $\mathcal{F}_\tau$ measurable random variable, then we will be looking for an $\{\mathcal{F}_t\}$-adapted process $\{X_t\}$, defined for $t > \tau$, for which the following equation holds with probability 1

$$\begin{aligned} X_t = X_\tau + \int_\tau^t a(s, X_s)ds + \int_\tau^t b(s, X_s)dW_s + \int_\tau^t \int_U f_1(s, X_{s-}, u)q_1(ds, du) \\ + \int_\tau^t \int_U f_2(s, X_{s-}, u)p_2(ds, du). \end{aligned} \tag{3.11}$$

If equality (3.11) holds for all $t \in (\tau, \zeta)$, with ζ another stopping time, $\zeta > \tau$, then we will say that $\{X_t\}$ is the solution of SDE (3.3) on interval (τ, ζ), if started at X_τ.

Theorem 3.8. *Assume that the coefficients of equation (3.3) satisfy the following condition:*

(i) for each $r > 0$ there exist a constant l_r, for which

$$|a(s,x) - a(s,y)|^2 + |b(s,x) - b(s,y)|^2 + \int_U |f_1(s,x,u) - f_1(s,y,u)|^2 m_1(du) \le l_r |x-y|^2,$$

for all $|x| \le r$, $|y| \le r$, $s \le r$.

(ii) Assume that condition (3.6) is satisfied

(iii) Let S be the support of $f_2(\cdot,\cdot,\cdot)$ and let S_u be the projection of S on U, then assume that $m_2(S_u) < \infty$.

Then a solution of equation (3.3) for any given X_0 is strongly unique.

Proof. We fix some admissible filtration $\{\mathcal{F}_t\}$ and consider only $\mathcal{F}_t$-measurable solutions. Suppose $\tau_1 < \tau_2 < \dots$ are all jump moments of the Poisson process $p_2(S_u, [0,t])$. Since it is a homogeneous process with parameter $m_2(S_u) < \infty$, there will be only finite number of jumps on every finite interval. Let $\tau_0 = 0$. Note, that it suffices to establish the uniqueness of solution of equation (3.3) on interval $[\tau_k, \tau_{k+1}]$, with assumption that X_{τ_k} is given. Then we establish by induction that a solution of (3.3) is unique on any interval $[0, \tau_k]$, and $\bigcup [0,\tau_k] = \mathbb{R}_+$. Suppose $\{X_t\}$ and $\{\overline{X}_t\}$ are two solutions of (3.3) on $[\tau_k, \tau_{k+1})$, for which $X_{\tau_k} = \overline{X}_{\tau_k}$. For $\tau_k \le t < \tau_{k+1}$

$$X_t = X_{\tau_k} + \int_{\tau_k}^t a(s,X_s)ds + \int_{\tau_k}^t b(s,X_s)dW_s + \int_{\tau_k}^t \int_U f_1(s,X_s,u) q_1(ds,du), \tag{3.12}$$

since the last integral with respect to measure p in (3.11) will be equal to zero. Similar equality holds for solution $\overline{X}_t$. Let $\zeta_r = \inf\{t > \tau_k, |X_t| + |\overline{X}_t| \ge r\} \wedge r$. Since

$$X_{t\wedge\zeta_r} - \overline{X}_{t\wedge\zeta_r} = \int_{\tau_k}^{t\wedge\zeta_r} [a(s,X_s) - a(s,\overline{X}_s)]ds + \int_{\tau_k}^{t\wedge\zeta_r} [b(s,X_s) - b(s,\overline{X}_s)]dW_s + \int_{\tau_k}^{t\wedge\zeta_r} [f_1(s,X_s,u) - f_1(s,\overline{X}_s,u)]q_1(ds,du),$$

$$\mathbb{E}\Big(\Big| \int_{\tau_k}^{t\wedge\zeta_r} [a(s,X_s) - a(s,\overline{X}_s)]ds \Big|^2 |\mathcal{F}_{\tau_k}\Big) \le l_r(t-\tau_k)\mathbb{E}\Big(\int_{\tau_k}^{t\wedge\zeta_r} |X_s - \overline{X}_s|^2 ds |\mathcal{F}_{\tau_k}\Big),$$

$$\mathbb{E}\Big(\Big| \int_{\tau_k}^{t\wedge\zeta_r} [b(s,X_s) - b(s,\overline{X}_s)]dW_s \Big|^2 |\mathcal{F}_{\tau_k}\Big) \le l_r n^2 \mathbb{E}\Big(\int_{\tau_k}^{t\wedge\zeta_r} |X_s - \overline{X}_s|^2 ds |\mathcal{F}_{\tau_k}\Big),$$

where n - dimensionality of X,

$$\mathbb{E}\Big(\Big| \int_{\tau_k}^{t\wedge\zeta_r} \int_U (f_1(s,X_s,u) - f_1(s,\overline{X}_s,u))q_1(ds,du) \Big|^2 |\mathcal{F}_{\tau_k}\Big) \le l_r \mathbb{E}\Big(\int_{\tau_k}^{t\wedge\zeta_r} |X_s - \overline{X}_s|^2 ds |\mathcal{F}_{\tau_k}\Big),$$

(we have made use of the properties of stochastic integrals and theorem conditions), then for some L (it is a $\mathcal{F}_{\tau_k}$-measurable quantity)

$$\mathbb{E}(|X_{t\wedge\zeta_r} - \overline{X}_{t\wedge\zeta_r}|^2|\mathcal{F}_{\tau_k}) \leq L\mathbb{E}\Big(\int_{\tau_k}^{t\wedge\zeta_r} |X_s - \overline{X}_s|^2 ds|\mathcal{F}_{\tau_k}\Big).$$

But then the following holds

$$\mathbb{E}(|X_t - \overline{X}_t|^2 I_{\{\zeta_r>t\}}|\mathcal{F}_{\tau_k}) \leq L\int_{\tau_k}^{t} \mathbb{E}(|X_s - \overline{X}_s|^2 I_{\{\zeta_r>s\}}|\mathcal{F}_{\tau_k})ds. \tag{3.13}$$

Hence, because of Gronwall's lemma :

$$\mathbb{E}|X_s - \overline{X}_s|^2 I_{\{\zeta_r>t\}} = 0.$$

Since $I_{\{\zeta_r>t\}} \longrightarrow 1$ as $r \longrightarrow \infty$, thus $X_t = \overline{X}_t$ for $\tau_k \leq t < \tau_k$. It remains to show that $X_{\tau_{k+1}} = \overline{X}_{\tau_{k+1}}$. Suppose X_t^* is a solution of (3.12). It was already shown that it is unique, does not have discontinuities of second kind, and thus is continuous with probability 1 at the point $t = \tau_{k+1}$, because τ_{k+1} is independent of X_t^* and its distribution is continuous, and number of discontinuity points of X_t is at most countable. Now note, that solution of equation (3.11) at point τ_{k+1} can be expressed in terms of X_t^* on $[\tau_k, \tau_{k+1}]$ in the following way:

$$\begin{aligned} X_{\tau_{k+1}} &= X^*_{\tau_{k+1}} + f_2(\tau_{k+1}, X^*_{\tau_{k+1}}, \hat{u}_{k+1}) \\ &= X_{\tau_{k+1}-0} + f_2(\tau_{k+1}, X_{\tau_{k+1}-0}, \hat{u}_{k+1}), \end{aligned}$$

where $\hat{u}_{k+1}$ - such a point from U, that $p_2(\{\hat{u}_{k+1}\}\times\{\tau_k\}) = 1$. From the coincidence of $X_{\tau_{k+1}-0}$ and $\overline{X}_{\tau_{k+1}-0}$ follows the coincidence $X_{\tau_{k+1}} = \overline{X}_{\tau_{k+1}}$. □

It is easy to see from the proof of theorem 3.8 that not only two solutions of one equation coincide, but also solutions with equal initial conditions of two different equations coincide as long as their coefficients coincide. We formulate this statement precisely, known as the theorem of local uniqueness.

Theorem 3.9. *Suppose $\{X_t\}$ is a solution of equation (3.8), and $\{\tilde{X}_t\}$ is a solution of equation*

$$\begin{aligned} \tilde{X}_t = \tilde{X}_0 &+ \int_0^t \tilde{a}(s, \tilde{X}_s)ds + \int_0^t \tilde{b}(s, \tilde{X}_s)dW_s \\ &+ \int_0^t\int_U \tilde{f}_1(s, \tilde{X}_s, u)q_1(ds, du) + \int_0^t\int_U \tilde{f}_2(s, \tilde{X}_s, u)p_2(ds, du). \end{aligned}$$

If conditions of theorem 3.8 are satisfied and $a(s,x) = \tilde{a}(s,x)$, $b(s,x) = \tilde{b}(s,x)$, $f_k(s,x,u) = \tilde{f}_k(s,x,u)$ given $|x| \leq N$, then $X_s = \tilde{X}_s$ for $s \leq \tau$, where $\tau = \inf\{s : |X_s| \geq N\}$.

3.4. **Strong Existence.** First, we state the classical existence results for the following equation:

$$X_t = X_0 + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dW_s + \int_0^t\int_U f_1(s, X_s, u)q_1(ds, du). \tag{3.14}$$

Theorem 3.10. *Assume that coefficients of equation (3.14) satisfy the following conditions:*

(1) *$a(s,0)$, $b(s,0)$, $\int |f_1(s,0,u)|^2 m_1(du)$ are locally bounded with respect to s,*

(2) *there exists increasing function $l(s)$ such that*

$$|a(s,x)-a(s,y)|^2+|b(s,x)-b(s,y)|^2 + \int_U |f_1(s,x,u)-f_1(s,y,u)|^2 m_1(du) \le l(s)|x-y|^2.$$

Let us denote by $\mathcal{F}_t$ the σ-algebra generated by X_0, $q_1(ds,du)$, W_s with $s \le t$. If X_0 is independent of W_s, $q_1(ds,du)$ and $\mathbb{E}|X_0|^2 < \infty$, then equation (3.14) has $\mathcal{F}_t$-measurable solution, moreover $\mathbb{E}|X_s|^2 < \infty$.

Theorem 3.11. *Assume that for coefficients of equation (3.14) the following condition holds:*

$$|a(t,x)|^2+|b(t,x)|^2+\int_U |f_1(t,x,u)|^2 m_1(du) \le l(1+|x|^2),$$

and for any $r>0$ one can specify constant l_r such that

$$|a(s,x)-a(s,y)|^2+|b(s,x)-b(s,y)|^2 + \int_U |f_1(s,x,u)-f_1(s,y,u)|^2 m_1(du) \le l_r|x-y|^2$$

for $s \le r$, $|x| \le r$, $|y| \le r$. If X_0 is independent of $\{W_s, q_1(ds,du)\}$, and σ-algebras $\mathcal{F}_t$ are constructed as in theorem 3.10, then there exists an $\mathcal{F}_t$-measurable solution of (3.14) for every $t \in \mathbb{R}_+$.

Remark 3.12. Suppose $\{\hat{\mathcal{F}}_t\}$ is some admissible filtration, τ is a stopping time relative to this filtration. Let us consider the SDE for $t > \tau$:

(3.15)
$$X_t = X_\tau + \int_\tau^t a(s,X_s)ds + \int_\tau^t b(s,X_s)dW_s + \int_\tau^t \int_U f_1(s,X_s,u)q_1(ds,du).$$

Under conditions of theorem 3.11 equation (3.15) has $\hat{\mathcal{F}}_t$-measurable solution, no matter what is the $\hat{\mathcal{F}}_\tau$-measurable variable X_τ. To prove this, it suffices to consider the process $\hat{X}_t$ which is a solution of the following equation

(3.16)
$$\hat{X}_t = \hat{X}_0+\int_0^t a(s+\tau,\hat{X}_s)ds+\int_0^t b(s+\tau,\hat{X}_s)d\hat{W}_s+\int_0^t\int_U f_1(s+\tau,\hat{X}_s,u)\hat{q}_1(ds,du),$$

where

(3.17) $$\hat{W}_s = W(s+\tau)-W_\tau; \quad \hat{q}_1([s_1,s_2]\times du) = q_1([s_1+\tau,s_2+\tau]\times du).$$

Obviously, that $\hat{W}$ and $\hat{q}_1$ possess the same properties as W, q_1 and are independent of $\mathcal{F}_\tau$. Thus, for equation (3.16), all derivations which were verified for equation (3.14), hold as well, if mathematical expectations and conditional mathematical expectations with given X_0 are substituted by conditional mathematical expectation with respect to σ-algebra $\hat{\mathcal{F}}_\tau$. Obviously, then $X_t = \hat{X}_{t-\tau}$ will be the solution of equation (3.15).

Now we prove the existence theorem for general SDE (3.3).

Theorem 3.13. *Assume that for equation (3.3) the following conditions are satisfied:*

(1) *coefficients a, b, f_1 satisfy the conditions of theorem 3.11;*
(2) *X_0 is independent of $\{W_s, q_1(ds,du), p_2(ds,du)\}$.*
(3) *Conditions (ii) and (iii) of theorem (3.8) are satisfied.*

Let $\mathcal{F}_t$ denote the σ-algebra generated by X_0 and $\{W_s,\ q_1([0,s],du),\ p_2([0,s],du),\ s \le t\}$. Then there exists $\mathcal{F}_t$-measurable solution of equation (3.3).

Proof. Let $\tau_1 < \tau_2 < \dots < \tau_n < \dots$ denote all stopping times that are the "growth" points of the last integral term in (3.8). The number of "growth" points on every finite time interval will be finite due to condition (3). It suffices to construct the solution of equation (3.3) on each interval $[0,\tau_1)$, $[\tau_1,\ \tau_2)$, $\dots$, $[\tau_n,\tau_{n+1})$, $\dots$ Since $\int_{\tau_n}^t \int f_2(s,X_s,u)p_2(ds,du) = 0$ when $t \in [\tau_n,\tau_{n+1})$, then on each of the specified intervals equation (3.3) turns into equation of type (3.15), where τ equals $0,\tau_1,\tau_2,\dots$ and so on. As it was pointed out in remark 3.12, there exists a solution of this equation if X_τ is $\mathcal{F}_\tau$-measurable. Let us prove that this is indeed the case. If $\tau = 0$, then X_0 is $\mathcal{F}_0$-measurable by definition of σ-algebra $\mathcal{F}_t$. Suppose, that X_{τ_n} is $\mathcal{F}_{\tau_n}$-measurable. We will show that then $X_{\tau_{n+1}}$ will be $\mathcal{F}_{\tau_{n+1}}$-measurable. Let X_t^n be the solution of the following equation

$$X_t^n = X_{\tau_n}^n + \int_{\tau_n}^t a(s,X_s^n)ds + \int_{\tau_n}^t b(s,X_s^n)dW_s + \int_{\tau_n}^t \int_U f_1(s,X_s^n,u)q_1(ds,du)$$

for $t \ge \tau_n$. In consequence of remark 3.12 such solution exists. Set $X_t = X_t^n$ for $t < \tau_{n+1}$. Let u_{n+1} be such a point in U that $p_2(\{\tau_{n+1}\} \times \{u_{n+1}\}) = 1$. Now let us define $X_{\tau_{n+1}}$ by the equality

$$X_{\tau_{n+1}} = X_{\tau_{n+1}}^n + f_2(\tau_{n+1}, X_{\tau_{n+1}}^n, u_{n+1}). \tag{3.18}$$

Since X_t^n is $\mathcal{F}_t$-measurable, has no discontinuities of the second kind and is continuous with probability 1 at the point τ_{n+1}, then $X_{\tau_{n+1}}^n$ is $\mathcal{F}_{\tau_{n+1}}$-measurable. Similarly u_{n+1} is also $\mathcal{F}_{\tau_{n+1}}$-measurable. Therefore both summands in the right hand side of (3.18) are $\mathcal{F}_{\tau_{n+1}}$-measurable, i.e. $X_{\tau_{n+1}}$ is $\mathcal{F}_{\tau_{n+1}}$-measurable. Thus, we can successively construct $\mathcal{F}_t$-measurable process X_t. In order to make certain that it is indeed a solution of (3.3), it suffices to see that

$$f_2(\tau_{n+1}, X_{\tau_{n+1}}^n, u_{n+1}) = \int_{\tau_n}^{\tau_{n+1}} \int_U f_2(t,X_t,u)p_2(dt,du).$$

□

Remark 3.14. Solution, existence of which was established in theorem 3.13, is unique. Indeed, by theorem 3.8 we have that for any enlargement of initial probability space and any admissible filtration of σ-algebras $\tilde{\mathcal{F}}_t$, and $\mathcal{F}_0$-measurable initial variable X_0, $\tilde{\mathcal{F}}_t$-measurable solution of equation (3.3) is unique. Since $\mathcal{F}_t \subset \tilde{\mathcal{F}}_t$, thus solution X_t constructed in theorem 3.13 will be also $\tilde{\mathcal{F}}_t$-measurable, and therefore, there will be no other $\tilde{\mathcal{F}}_t$-measurable solutions of equation (3.3).

Remark 3.15. The solution constructed in theorem 3.13 is fully determined by initial condition, Wiener process W and Poisson random measures p_1 and p_2, i.e. it is a "strong" solution (solution-process). Thus, theorem 3.13 states that there exists a strong solution of equation (3.3) (strong existence), and from the remark 3.14 it follows that under conditions of theorem 3.13 any solution of (3.3) is unique (strong uniqueness).

Remark 3.16. Under condition of theorem 3.13 the solution of SDE (3.3) admits the decomposition (2.1) with

$$A_t = \int_0^t a(s, X_s)ds + \int_0^t \int_U f_2(s, X_{s-}, u)p_2(ds, du) \in \mathcal{V}^n,$$

$$M_t = \int_0^t b(s, X_s)dW_s + \int_0^t \int_U f_1(s, X_{s-}, u)q_1(ds, du) \in \mathcal{M}_{loc}^n,$$

hence it is a semimartingale.

4. Stochastic Hybrid Processes as Solutions of SDEs

4.1. **SDE on hybrid state space.** In this section we construct a switching jump diffusion $\{X_t, \theta_t\}$ taking values in $\mathbb{R}^n \times \mathbb{M}$, where $\mathbb{M} = \{e_1, e_2, \dots, e_N\}$ is a finite set. In order to work with neutral topology we assume that for each $i = 1, \dots, N$, e_i is the i-th unit vector, $e_i \in \mathbb{R}^N$. Let $\{X_t, \theta_t\}$ be an $\mathbb{R}^n \times \mathbb{M}$-valued process given by the following stochastic differential equation of Ito-Skorohod type.

$$dX_t = a(X_t, \theta_t)dt + b(X_t, \theta_t)dW_t + \int_{\mathbb{R}^d} g_1(X_{t-}, \theta_{t-}, u)q_1(dt, du) + \int_{\mathbb{R}^d} g_2(X_{t-}, \theta_{t-}, u)p_2(dt, du), \tag{4.1}$$

$$d\theta_t = \int_{\mathbb{R}^d} c(X_{t-}, \theta_{t-}, u)p_2(dt, du). \tag{4.2}$$

Here:

(i) for $t = 0$, X_0 is a prescribed $\mathbb{R}^n$-valued random variable.

(ii) for $t = 0$, θ_0 is a prescribed $\mathbb{M}$-valued random variable.

(iii) W is an m-dimensional standard Wiener process.

(iv) $q_1(dt, du)$ is a martingale random measure associated to a Poisson random measure p_1 with intensity $dt \times m_1(du)$.

(v) $p_2(dt, du)$ is a Poisson random measure with intensity $dt \times m_2(du) = dt \times du_1 \times \bar{\mu}(d\underline{u})$, where $\bar{\mu}$ is a probability measure on $\mathbb{R}^{d-1}$, $u_1 \in \mathbb{R}$, $\underline{u}$ refers to all components of $u \in \mathbb{R}^d$ except the first one.

The coefficients are defined as follows

$$\begin{aligned} a &: \mathbb{R}^n \times \mathbb{M} \to \mathbb{R}^n \\ b &: \mathbb{R}^n \times \mathbb{M} \to \mathbb{R}^{n \times m} \\ g_1 &: \mathbb{R}^n \times \mathbb{M} \times \mathbb{R}^d \to \mathbb{R}^n \\ g_2 &: \mathbb{R}^n \times \mathbb{M} \times \mathbb{R}^d \to \mathbb{R}^n \\ \phi &: \mathbb{R}^n \times \mathbb{M} \times \mathbb{M} \times \mathbb{R}^{d-1} \to \mathbb{R}^n \\ \lambda &: \mathbb{R}^n \times \mathbb{M} \times \mathbb{M} \to \mathbb{R}_+ \\ c &: \mathbb{R}^n \times \mathbb{M} \times \mathbb{R}^d \to \mathbb{R}^N. \end{aligned}$$

Moreover, for all $k = 1, 2, \dots, N$ we define measurable mappings $\Sigma_k : \mathbb{R}^n \times \mathbb{M} \to \mathbb{R}_+$ in a following manner

$$\Sigma_k(x, e_i) = \begin{cases} \sum_{j=1}^k \lambda(x, e_i, e_j) & k > 0, \\ 0 & k = 0, \end{cases} \tag{4.3}$$

and function $c(\cdot, \cdot, \cdot)$ by

$$c(x, e_i, u) = \begin{cases} e_j - e_i & \text{if } u_1 \in (\Sigma_{j-1}(x, e_i), \Sigma_j(x, e_i)], \\ 0 & \text{otherwise }, \end{cases} \tag{4.4}$$

and function $g_2(\cdot, \cdot, \cdot)$ by

$$g_2(x, e_i, u) = \begin{cases} \phi(x, e_i, e_j, \underline{u}) & \text{if } u_1 \in (\Sigma_{j-1}(x, e_i), \Sigma_j(x, e_i)], \\ 0 & \text{otherwise }. \end{cases} \tag{4.5}$$

Let U_θ denote the projection of the support of function $\phi(\cdot,\cdot,\cdot,\cdot)$ on $\underline{U} = \mathbb{R}^{d-1}$. The jump size of X_t and the new value of θ_t at the jump times generated by Poisson random measure p_2 are determined by the functions (4.4) and (4.5) correspondingly. There are three different situations possible:

I. Simultaneous jump of X_t and θ_t

$$\begin{cases} c(\cdot,\cdot,u) \neq 0 & \text{if } u_1 \in (\Sigma_{j-1}(x,e_i), \Sigma_j(x,e_i)],\ i,j = 1,\dots,N \text{ and } j \neq i, \\ g_2(\cdot,\cdot,u) \neq 0 & \text{if } u_1 \in (\Sigma_{j-1}(x,e_i), \Sigma_j(x,e_i)],\ i,j = 1,\dots,N \text{ and } \underline{u} \in U_\theta. \end{cases}$$

II. Switch of θ_t only

$$\begin{cases} c(\cdot,\cdot,u) \neq 0 & \text{if } u_1 \in (\Sigma_{j-1}(x,e_i), \Sigma_j(x,e_i)],\ i,j = 1,\dots,N \text{ and } j \neq i, \\ g_2(\cdot,\cdot,u) = 0 & \text{if } u_1 \in (\Sigma_{j-1}(x,e_i), \Sigma_j(x,e_i)],\ i,j = 1,\dots,N \text{ and } \underline{u} \notin U_\theta. \end{cases}$$

III. Jump of X_t only

$$\begin{cases} c(\cdot,\cdot,u) = 0 & \text{if } u_1 \in (\Sigma_{j-1}(x,e_j), \Sigma_j(x,e_j)],\ j = 1,\dots,N, \\ g_2(\cdot,\cdot,u) \neq 0 & \text{if } u_1 \in (\Sigma_{j-1}(x,e_j), \Sigma_j(x,e_j)],\ j = 1,\dots,N, \ \text{ and } \underline{u} \in U_\theta. \end{cases}$$

We make the following assumptions on the coefficients of SDE (4.1)-(4.2).

(A1) There exists a constant l such that for each $i = 1,2,\dots,N$

$$|a(x,e_i)|^2 + |b(x,e_i)|^2 + \int_{\mathbb{R}^d} |g_1(x,e_i,u)|^2 m_1(du) \leq l(1+|x|^2).$$

(A2) for any $r > 0$ one can specify constant l_r such that for each $i = 1,2,\dots,N$

$$|a(x,e_i) - a(y,e_i)|^2 + |b(x,e_i) - b(y,e_i)|^2 + \int_{\mathbb{R}^d} |g_1(x,e_i,u) - g_1(y,e_i,u)|^2 m_1(du) \leq l_r |x-y|^2$$

for $|x| \leq r$, $|y| \leq r$.

(A3) Function c satisfies (4.3), (4.4) and for $i,j = 1,2,\dots,N$, $\lambda(e_i,e_j,\cdot)$ are bounded and measurable, $\lambda(e_i,e_j,\cdot) \geq 0$.

(A4) Function g_2 satisfies (4.3), (4.5) and for all $t > 0$, $i,j = 1,\dots,N$

$$\int_0^t \int_{\mathbb{R}^d} |\phi(x,e_i,e_j,u)| p_2(ds,du) < \infty, \quad P\text{-a.s.}$$

4.2. **Strong existence and uniqueness.**

Theorem 4.1. *Assume (A1)-(A4) and (4.3), (4.4), (4.5). Let p_1, p_2, W, X_0 and θ_0 be independent. Then SDE (4.1)-(4.2) has a unique strong solution which is a semimartingale.*

Proof. The switching jump diffusion $\{X_t, \theta_t\}$ governed by equations (4.1)-(4.2) can be seen as the $\mathbb{R}^{n+N}$-valued homogeneous jump diffusion $\{\xi_t\} \triangleq \{(X_t,\theta_t)^T\}$ governed by the stochastic differential equation

$$d\xi_t = \tilde{a}(\xi_t)dt + \tilde{b}(\xi_t)dW_t + \int_{\mathbb{R}^d} \tilde{f}_1(\xi_{t-},u) q_1(dt,du) + \int_{\mathbb{R}^d} \tilde{f}_2(\xi_{t-},u) p_2(dt,du) \tag{4.6}$$

with the following coefficients:

$$\begin{aligned}
\tilde{a} &: \mathbb{R}^{n+N} \to \mathbb{R}^{n+N} & \tilde{a}(\cdot) &\stackrel{\triangle}{=} [a(\cdot), O^N]^T \\
\tilde{b} &: \mathbb{R}^{n+N} \to \mathbb{R}^{(n+N)\times m} & \tilde{b}(\cdot) &\stackrel{\triangle}{=} [b(\cdot), O^{N\times m}]^T \\
\tilde{f}_1 &: \mathbb{R}^{n+N} \times \mathbb{R}^d \to \mathbb{R}^{n+N} & \tilde{f}_1(\cdot,\cdot) &= [g_1(\cdot,\cdot), O^N]^T \\
\tilde{f}_2 &: \mathbb{R}^{n+N} \times \mathbb{R}^d \to \mathbb{R}^{n+N} & \tilde{f}_2(\cdot,\cdot) &= [g_2(\cdot,\cdot), c(\cdot,\cdot)]^T
\end{aligned}$$

where by O^k and $O^{k\times s}$ we denote the k-dimensional zero vector and $k\times s$-dimensional zero matrix correspondingly.

Next we show that conditions (A1)-(A4) together with (4.3), (4.4), (4.5) imply the conditions of theorems 3.8 and 3.13 thus the equation (4.6) has an a.s. unique strong solution which implies that SDE (4.1)-(4.2) has an a.s. unique strong solution.

Let us verify all conditions.

Growth condition: by (A1) for every $\xi = (x, e_i)^T \in \mathbb{R}^{n+N}$ $i = 1, \dots, N$ we have

$$|\tilde{a}(\xi)|^2 + |\tilde{b}(\xi)|^2 + \int_{\mathbb{R}^d} |\tilde{f}_1(\xi, u)|^2 m_1(du) = |\tilde{a}(x, e_i)|^2 + |\tilde{b}(x, e_i)|^2 + \int_{\mathbb{R}^d} |\tilde{f}_1(x, e_i, u)|^2 m_1(du)$$

$$= |a(x, e_i)|^2 + |b(x, e_i)|^2 + \int_{\mathbb{R}^d} |g_1(x, e_i, u)|^2 m_1(du) \le l(1+|x|^2) \le l(1+|x|^2+|e_i|^2) = l(1+|\xi|^2).$$

Lipschitz condition: From (A1) and (A2) it follows that for any $r > 0$ one can specify a constant L_r such that for all $\xi = (x, e_i)^T \in \mathbb{R}^{n+N}$, $\zeta = (y, e_j)^T \in \mathbb{R}^{n+N}$ $i, j = 1, \dots, N$, and for $|x| < r$, $|y| < r$, i.e. $|\xi| \le \sqrt{r^2+1}$, $|\zeta| \le \sqrt{r^2+1}$, we have

$$\begin{aligned}
&|\tilde{a}(\xi) - \tilde{a}(\zeta)|^2 + |\tilde{b}(\xi) - \tilde{b}(\zeta)|^2 + \int_{\mathbb{R}^d} |\tilde{f}_1(\xi, u) - \tilde{f}_1(\zeta, u)|^2 m_1(du) \\
&= |a(x, e_i) - a(y, e_j)|^2 + |b(x, e_i) - b(y, e_j)|^2 + \int_{\mathbb{R}^d} |g_1(x, e_i, u) - g_1(y, e_j, u)|^2 m_1(du) \\
&\le 2\big(|a(x, e_i) - a(y, e_i)|^2 + |b(x, e_i) - b(y, e_i)|^2 + \int_{\mathbb{R}^d} |g_1(x, e_i, u) - g_1(y, e_i, u)|^2 m_1(du) \\
&+ |a(y, e_i) - a(y, e_j)|^2 + |b(y, e_i) - b(y, e_j)|^2 + \int_{\mathbb{R}^d} |g_1(y, e_i, u) - g_1(y, e_j, u)|^2 m_1(du)\big) \\
&\le 2\Big(l_r|x-y|^2 + 4\big(|a(y, e_i)|^2 + |b(y, e_i)|^2 + \int_{\mathbb{R}^d} |g_1(y, e_i, u)|^2 m_1(du)\big)\Big) \\
&\le 2\big(l_r|x-y|^2 + 4l(1+|y|^2)\big) \le 2\big(l_r|x-y|^2 + 4l(1+r^2)\big) = 2\big(l_r|x-y|^2 + 2l(1+r^2)|e_i - e_j|^2\big) \\
&\le L_r\big(|x-y|^2 + |e_i - e_j|^2\big) = L_r|\xi - \zeta|^2,
\end{aligned}$$

where $L_r = max(2l_r, 4l(1+r^2))$.

Let S be the support of $\tilde{f}_2$ and $S_u = S_{u_1} \times S_{\underline{u}}$ be the projection of S on $U = \mathbb{R}^d$. By (A3), (A4) and the fact that $\bar{\mu}$ is a probability measure, we have that $m_2(S_u) = m_L(S_{u_1}) \cdot \bar{\mu}(S_{\underline{u}}) < \infty$, where m_L is the Lebesgue measure.

By (A4) and definition of function c we have that for all $t > 0$, $i = 1, \dots N$

$$\int_0^t \int_{\mathbb{R}^d} |\tilde{f}_2(x, e_i, u)| p_2(ds, du) < \infty, \quad P\text{-a.s.}$$

We have shown that coefficients of equation (4.6) satisfy the conditions of theorems 3.8 and 3.13, thus equation (4.6) (correspondingly (4.1)-(4.2)) has an a.s. unique strong solution.

It is clear that under conditions of the theorem the solution $\{\xi_t\} = \{(X_t, \theta_t)^T\}$ admits the decomposition (2.1) with

$$A_t = \int_0^t \tilde{a}(\xi_s)ds + \int_0^t \int_U \tilde{f}_2(\xi_{s-}, u)p_2(ds, du) \in \mathcal{V}^n,$$
$$M_t = \int_0^t \tilde{b}(\xi_s)dW_s + \int_0^t \int_U \tilde{f}_1(\xi_{s-}, u)q_1(ds, du) \in \mathcal{M}^n_{loc},$$

hence it is a semimartingale. □

4.3. **Transformation of Blom** [4]**.** Following Blom 2003 [4] one can show that solution of (4.1)-(4.2) is indistinguishable from the solution of the following set of equations:

$$d\theta_t = \sum_{i=1}^N (e_i - \theta_{t-})p_2\left(dt, (\Sigma_{i-1}(X_{t-}, \theta_{t-}), \Sigma_i(X_{t-}, \theta_{t-})] \times \mathbb{R}^{d-1}\right), \tag{4.7}$$

$$dX_t = a(X_t, \theta_t)dt + b(X_t, \theta_t)dW_t + \int_{\mathbb{R}^d} g_1(X_{t-}, \theta_{t-}, u)q_1(dt, du) + \int_{\mathbb{R}^d} \phi(X_{t-}, \theta_{t-}, \theta_t, \underline{u})p_2\big(dt, \big(0, \Sigma_N(X_{t-}, \theta_{t-})\big] \times d\underline{u}\big). \tag{4.8}$$

Theorem 4.2. *Assume (A1)-(A4) and (4.3), (4.4), (4.5). Let p_1, p_2, W, X_0 and θ_0 be independent. Then SDE (4.7)-(4.8) has a unique strong solution which is a semimartingale.*

Proof. The proof consists of showing that the solution of (4.7)-(4.8) is indistinguishable from the solution of (4.1)-(4.2). Subsequently Theorem 4.2 is the consequence of Theorem 4.1.

Indeed, rewriting of (4.7) yields (4.2):

$$\begin{aligned} d\theta_t &= \sum_{i=1}^N (e_i - \theta_{t-})p_2\left(dt, (\Sigma_{i-1}(X_{t-}, \theta_{t-}), \Sigma_i(X_{t-}, \theta_{t-})] \times \mathbb{R}^{d-1}\right) \\ &= \int_{\mathbb{R}^d} \sum_{i=1}^N (e_i - \theta_{t-})\mathbf{1}_{(\Sigma_{i-1}(X_{t-}, \theta_{t-}), \Sigma_i(X_{t-}, \theta_{t-})]}(u_1)p_2(dt, du_1 \times d\underline{u}) \\ &= \int_{\mathbb{R}^d} c(X_{t-}, \theta_{t-}, u)p_2(dt, du). \end{aligned}$$

Next, since the first three right hand terms of (4.8) and (4.1) are equal, it remains to show that the fourth right hand term in (4.8) yields the fourth right hand term in (4.1) up to indistinguishability:

$$\int_{\mathbb{R}^d} \phi(X_{t-}, \theta_{t-}, \theta_t, \underline{u}) p_2\big(dt, \big(0, \Sigma_N(X_{t-}, \theta_{t-})\big] \times d\underline{u}\big)$$

$$= \int_{(0,\infty)} \int_{\mathbb{R}^{d-1}} \phi(X_{t-}, \theta_{t-}, \theta_t, \underline{u}) \mathbf{1}_{(0,\Sigma_N(X_{t-},\theta_{t-})]}(u_1) p_2(dt, du_1 \times d\underline{u})$$

$$= \int_{(0,\infty)} \int_{\mathbb{R}^{d-1}} \phi(X_{t-}, \theta_{t-}, \theta_t, \underline{u}) \sum_{i=1}^{N} \mathbf{1}_{(\Sigma_{i-1}(X_{t-},\theta_{t-}),\Sigma_i(X_{t-},\theta_{t-})]}(u_1) p_2(dt, du_1 \times d\underline{u})$$

$$= \int_{(0,\infty)} \int_{\mathbb{R}^{d-1}} \sum_{i=1}^{N} \big[\phi(X_{t-}, \theta_{t-}, \theta_t, \underline{u}) \mathbf{1}_{(\Sigma_{i-1}(X_{t-},\theta_{t-}),\Sigma_i(X_{t-},\theta_{t-})]}(u_1)\big] p_2(dt, du_1 \times d\underline{u})$$

$$= \int_{(0,\infty)} \int_{\mathbb{R}^{d-1}} \sum_{i=1}^{N} \big[\phi(X_{t-}, \theta_{t-}, \theta_{t-} + \Delta\theta_t, \underline{u}) \mathbf{1}_{(\Sigma_{i-1}(X_{t-},\theta_{t-}),\Sigma_i(X_{t-},\theta_{t-})]}(u_1)\big] p_2(dt, du_1 \times d\underline{u})$$

$$= \int_{(0,\infty)} \int_{\mathbb{R}^{d-1}} \sum_{i=1}^{N} \big[\phi(X_{t-}, \theta_{t-}, \theta_{t-} + (e_i - \theta_{t-}), \underline{u}) \times$$
$$\times \mathbf{1}_{(\Sigma_{i-1}(X_{t-},\theta_{t-}),\Sigma_i(X_{t-},\theta_{t-})]}(u_1)\big] p_2(dt, du_1 \times d\underline{u}) =$$

$$= \int_{(0,\infty)} \int_{\mathbb{R}^{d-1}} \sum_{i=1}^{N} \big[\phi(X_{t-}, \theta_{t-}, e_i, \underline{u}) \mathbf{1}_{(\Sigma_{i-1}(X_{t-},\theta_{t-}),\Sigma_i(X_{t-},\theta_{t-})]}(u_1)\big] p_2(dt, du_1 \times d\underline{u})$$

$$= \int_{\mathbb{R}^d} g_2(X_{t-}, \theta_{t-}, u) p_2(dt, du).$$

□

Remark 4.3. We notice the interesting aspect that presence of θ_t in ϕ (equation (4.8)) explicitly shows that jump of $\{X_t\}$ depends on the switch from θ_{t-} to θ_t, i.e., it is a hybrid jump.

5. General Stochastic Hybrid Processes as solutions of SDEs

In this section we will consider processes of somewhat more general type than those in previous sections. We want to construct an $\mathbb{R}^n \times \mathbb{M}$-valued switching jump diffusion which may have instantaneous jumps and switches when hitting boundaries of some given sets. In order to simplify analysis we initially assume that purely discontinuous martingale term is equal to zero (i.e. we take $g_1 \equiv 0$). For this we follow the approach in D2.2 (pp. 38-39).

5.1. **Sequence of processes.** Suppose for each $e_i \in \mathbb{M}$, $i = 1, \dots, N$ there is an open connected set $E^i \subset \mathbb{R}^n$, with boundary ∂E^i. Let

$$E = \{x \,|\, x \in E^i, i = 1, \dots, N\} = \bigcup_{i=1}^{N} E^i,$$

$$\partial E = \{x \,|\, x \in \partial E^i, i = 1, \dots, N\} = \bigcup_{i=1}^{N} \partial E^i.$$

The interior of the set E is the jump "destination" set. Suppose function g_2, defined by (4.5), in addition to requirement (**A4**) has the following property:
(B1) $(x + \phi(x, e_i, \underline{u})) \in E^i$ for each $x \in E^i$, $\underline{u} \in \mathbb{R}^{d-1}$, $i = 1, \dots, N$.

Following the approach in D2.2 (pp. 38-39) we consider an increasing sequence of stopping times τ_n^E and a sequence of jump-diffusions $\{X_t^n \; t \geq \tau_{n-1}^E\}$, $n = 1, 2, \dots$ governed by the following SDEs (in integral form):

$$X_t^n = X_{\tau_{n-1}^E}^n + \int_{\tau_{n-1}^E}^t a(X_s^n) ds + \int_{\tau_{n-1}^E}^t b(X_s^n) dW_s + \int_{\tau_{n-1}^E}^t \int_{\mathbb{R}^d} g_2(X_{s-}^n, \theta_{s-}^n, u) p_2(ds, du), \tag{5.1}$$

$$\theta_t^n = \theta_{\tau_{n-1}^E}^n + \int_{\tau_{n-1}^E}^t \int_{\mathbb{R}^d} c(X_{s-}^n, \theta_{s-}^n, u) p_2(ds, du), \tag{5.2}$$

$$X_{\tau_n^E}^{n+1} = g^x(X_{\tau_n^E}^n, \theta_{\tau_n^E}^n, \beta_{\tau_n^E}), \tag{5.3}$$

$$\theta_{\tau_n^E}^{n+1} = g^\theta(X_{\tau_n^E}^n, \theta_{\tau_n^E}^n, \beta_{\tau_n^E}). \tag{5.4}$$

More specifically, the stopping times are defined as follows

$$\tau_k^E \triangleq \inf\{t > \tau_{k-1}^E : X_t^k \in \partial E\}, \tag{5.5}$$

$$\tau_0^E \triangleq 0 \tag{5.6}$$

$k = 1, 2, \dots, N$, i.e. $\tau_0^E < \tau_1^E < \dots < \tau_k^E < \dots$ a.s.,

$$g^x : \partial E \times \mathbb{M} \times V \to \mathbb{R}^n, \tag{5.7}$$

$$g^\theta : \partial E \times \mathbb{M} \times V \to \mathbb{M}, \tag{5.8}$$

and $\{\beta_t, t \in [0, \infty)\}$ is the sequence of V-valued (one may take $V = \mathbb{R}^d$) i.i.d. random variables distributed according to some given distribution. The initial values X_0^1 and θ_0^1 are some prescribed random variables.

Remark 5.1. Property (**B1**) assures that the sequence of stopping times (5.5) is well defined and the boundary ∂E can be hit only by continuous part

$$X_t^{c,n} = X^n_{\tau^E_{n-1}} + \int_{\tau^E_{n-1}}^t a(X_s^n)ds + \int_{\tau^E_{n-1}}^t b(X_s^n)dW_s \tag{5.9}$$

of the processes $\{X_t^n\}$, $n = 1, 2, \dots,$ between the jumps and/or switching times generated by Poisson random measure p_2.

5.2. **Existence and uniqueness.** We define the process $\{X_t, \theta_t\}$ as follows

$$\left\{ \begin{array}{rcl} X_t(\omega) & = & \sum_{n=1}^\infty X_t^n(\omega)\mathbf{1}_{\left[\tau^E_{n-1}(\omega),\tau^E_n(\omega)\right)}(t) \\ \theta_t(\omega) & = & \sum_{n=1}^\infty \theta_t^n(\omega)\mathbf{1}_{\left[\tau^E_{n-1}(\omega),\tau^E_n(\omega)\right)}(t) \end{array} \right. \tag{5.10}$$

provided there exist solutions $\{X_t^n, \theta_t^n\}$ of SDEs (5.1)-(5.4). On open set E process $\{X_t, \theta_t\}$ (provided it exists) evolves as switching jump diffusion (4.1)-(4.2). At times τ_k^E there is a jump and/or switching determined by the mappings g^x and g^θ correspondingly, i.e. $X_{\tau_k^E} \neq X_{\tau_k^E-}$ and/or $\theta_{\tau_k^E} \neq \theta_{\tau_k^E-}$.

To ensure the existence of a strong unique solution of (5.10) we need assumption **B1** and the following two:

(B2) $d(\partial E, g^x(\partial E, \mathbb{M}, V)) > 0$, i.e. $\{X_t\}$ may jump only inside of open set E.

(B3) Process 5.10 hits the boundary ∂E a.s. finitely many times on any finite time interval.

Theorem 5.2. *Assume (A1)-(A4), (4.3), (4.4), (4.5) and (B1)-(B3). Let W, p_2, $\{\beta_t, t \in [0,\infty)\}$, X_0 and θ_0 be independent. Then process (5.10) exists for every $t \in \mathbb{R}_+$, it is strongly unique and it is a semimartingale.*

Proof. Let $\mathcal{F}_t$ be the σ-algebra generated by X_0, W_s, $p_2(ds, du)$, and β_s with $s \leq t$. Suppose $\tau_0^E < \tau_1^E < \dots$ is the sequence of all instantaneous jumps and/or switches at the boundary ∂E. By assumption (B3) the number of these jumps and/or switches is a.s. finite on every finite time interval and $\tau_k^E \uparrow \infty$ a.s. Similarly as in proofs of theorems 3.8 and 3.13 it suffices to establish the uniqueness and existence of the process (5.10) on interval $[\tau_{k-1}^E, \tau_k^E]$ with assumption that $\mathcal{F}_{\tau^E_{k-1}}$-measurable random variable $(X_{\tau^E_{k-1}}, \theta_{\tau^E_{k-1}})$ is given. Then we establish by induction that (5.10) exists and is unique on $\bigcup_{k=1}^\infty [\tau_{k-1}^E, \tau_k^E] = \mathbb{R}_+$.

Suppose $(X_{\tau^E_{k-1}}, \theta_{\tau^E_{k-1}}) = (X^k_{\tau^E_{k-1}}, \theta^k_{\tau^E_{k-1}})$ is $\mathcal{F}_{\tau^E_{k-1}}$-measurable. Then under conditions (A1)-(A4), and (4.3), (4.4), (4.5) and using the same arguments as in remark 3.12 it follows from theorem 4.1 that for $\tau^E_{k-1} \leq t < \tau^E_k$ there exists strongly unique process

$$\left\{ \begin{array}{rcl} X_t & = & X_t^k, \\ \theta_t & = & \theta_t^k. \end{array} \right. \tag{5.11}$$

It remains to show that $(X_{\tau_k^E}, \theta_{\tau_k^E})$ is $\mathcal{F}_{\tau_k^E}$-measurable and uniquely defined. By definition of the process (5.10) we have:

$$\left\{ \begin{array}{rcl} X_{\tau_k^E} & = & X^{k+1}_{\tau_k^E} = g^x(X^k_{\tau_k^E}, \theta^k_{\tau_k^E}, \beta_{\tau_k^E}), \\ \theta_{\tau_k^E} & = & \theta^{k+1}_{\tau_k^E} = g^\theta(X^k_{\tau_k^E}, \theta^k_{\tau_k^E}, \beta_{\tau_k^E}). \end{array} \right. \tag{5.12}$$

Since (X_t^k, θ_t^k) is $\mathcal{F}_t$-measurable, has no discontinuities of the second kind and by condition (B1) is continuous with probability 1 at the point τ_k^E (see remark 5.1), then $(X_{\tau_k^E}^k, \theta_{\tau_k^E}^k)$ is $\mathcal{F}_{\tau_k^E}$-measurable. $\beta_{\tau_k^E}$ is also $\mathcal{F}_{\tau_k^E}$-measurable. Thus the right hand side of (5.12) is $\mathcal{F}_{\tau_k^E}$-measurable, i.e. $(X_{\tau_k^E}, \theta_{\tau_k^E})$ is $\mathcal{F}_{\tau_k^E}$-measurable. From the strong uniqueness of $\{X_t, \theta_t\}_{t \in [\tau_{k-1}^E, \tau_k^E)}$ follows strong uniqueness of $(X_{\tau_k^E}, \theta_{\tau_k^E})$:

$$X_{\tau_k^E} = g^x(X_{\tau_k^E}^k, \theta_{\tau_k^E}^k, \beta_{\tau_k^E}) = g^x(X_{\tau_k^E-0}^k, \theta_{\tau_k^E-0}^k, \beta_{\tau_k^E}) = g^x(X_{\tau_k^E-0}, \theta_{\tau_k^E-0}, \beta_{\tau_k^E}),$$

$$\theta_{\tau_k^E} = g^\theta(X_{\tau_k^E}^k, \theta_{\tau_k^E}^k, \beta_{\tau_k^E}) = g^\theta(X_{\tau_k^E-0}^k, \theta_{\tau_k^E-0}^k, \beta_{\tau_k^E}) = g^\theta(X_{\tau_k^E-0}, \theta_{\tau_k^E-0}, \beta_{\tau_k^E}).$$

By induction we obtain that process (5.10) exists and is strongly unique on $\bigcup_{k=1}^{\infty}[\tau_{k-1}^E, \tau_k^E] = \mathbb{R}_+$. Moreover, it is a semimartingale, since by the theorem 4.1 each solution $\{X_t^k, \theta_t^k\}$, $k = 1, 2, \ldots$ is a semimartingale. □

6. Comparison of SDEs

This section presents a brief comparison of stochastic models developed in Blom 2003 [4], Ghosh and Bagchi 2004 [10] (also see Appendices A and B), and in the present report. Table 1 shows what type of jumps and switches are covered by each model.

Table 1. List of models and their main features

	θ	$X1$	$X2$	$\theta\&X2$	∂E
GB1, *Ghosh and Bagchi 2004*	✓	-	✓	✓	-
HB1, *Henk Blom 2003*	✓	-	✓	✓	-
KB1, *Krystul and Blom 2005*	✓	✓	✓	✓	-
GB2, *Ghosh and Bagchi 2004*	✓	-	-	-	✓
HB2, *Henk Blom 2003*	✓	-	✓	✓	✓
KB2, *Krystul and Blom 2005*	✓	-	✓	✓	✓

Notations:

HB1 - switching hybrid-jump diffusion (Blom 2003 [4]);

HB2 - switching hybrid-jump diffusion with hybrid jumps at the boundary (Blom 2003 [4]);

GB1 - switching jump diffusion (Ghosh and Bagchi 2004 [10] or appendix A);

GB2 - switching diffusion with hybrid jumps at the boundary (Ghosh and Bagchi 2004 [10] or appendix B);

KB1 - switching jump diffusion (Section 4);

KB2 - switching jump diffusion with hybrid jumps at the boundary (Section 5).

θ - independent random switching of θ_t;

$X1$ - independent random jump of X_t generated by compensated Poisson random measure;

$X2$ - independent random jump of X_t generated by Poisson random measure;

$\theta\&X2$ - simultaneous jump of X_t and θ_t generated by Poisson random measure;

∂E - simultaneous jump of X_t and θ_t at the boundary.

First, let us consider models GB1 and HB1. These two SDEs differ only in the integral term with respect to a Poisson random measure p, which determines the jumps of X_t component. In GB1 $p(dt, du)$ is a Poisson random measure with intensity $dt \times m(du)$, where m is the Lebesgue measure on $U = \mathbb{R}$. The projection of support of integrand function on $U = \mathbb{R}$ must be bounded. In HB1 $p(dt, du)$ is a Poisson random measure with intensity $dt \times m(du_1) \times \mu(\underline{u})$, where m is the Lebesgue measure on $U_1 = \mathbb{R}$ and μ is a probability measure on $\underline{U} = \mathbb{R}^{d-1}$. The projection of support of integrand function on $U_1 = \mathbb{R}$ is bounded by construction and projection on $\underline{U} = \mathbb{R}^{d-1}$ can be unbounded. Thus, it is clear that HB1 includes GB1 as a special case (GB1 $\subset$ HB1).

KB1 is almost the same as HB1 plus an extra integral term with respect compensated Poisson random measure q_1 (HB1 $\subset$ KB1).

HB2 is the HB1 plus the hybrid jumps at a boundary (HB1 $\subset$ HB2).

KB2 is the KB1 without integral term with respect to compensated Poisson random measure q_1, but with hybrid jumps at the boundary. Actually, KB2 and HB2 fall into one class of SDEs (HB2 = KB2).

In general GB2 is not a subclass of HB2 since in GB2 the state of the system (X_t, θ_t) takes values in $\bigcup_{k=1}^{\infty}(S_k, \mathbb{M}_k)$, where $\mathbb{M}_k = \{e_1, e_2, \ldots, e_{N_k}\}$ and $S_k \subset \mathbb{R}^{d_k}$ may be different for different k's. Let us denote by GB2* the set of GB2 models with $(S_k, \mathbb{M}_k) = (\mathbb{R}^n, \mathbb{M})$ for all $k \in \mathbb{N}$. Then GB2$^* \subset$ HB2. Indeed, the GB2* is a standard switching diffusion with hybrid jumps at the boundary, thus, it can be seen as a special case of HB2.

Finally, it is important to point out that assumptions adopted for HB1, HB2 and KB1, KB2 models, in order to ensure the existence and uniqueness of solutions, are more relaxed than in case of models GB1 and GB2 (GB2*) (see Appendices A and B).

We summarize the "hierarchy" of models in table 2.

Table 2. The hierarchy of models

$$\begin{array}{c} \text{GB1} \subset \text{HB1} \subset \text{KB1} \\ \cap \\ \text{GB2}^* \subset \text{HB2} = \text{KB2} \end{array}$$

7. Markov Property

Assume we are given [11]:

a) a measurable space $(S, \mathcal{B})$

b) a measurable space $(\Omega, \mathcal{G})$ and a family of σ-algebras $\{\mathcal{G}_t^s, 0 \le s \le t \le \infty\}$, such that $\mathcal{G}_t^s \subset \mathcal{G}_v^u \subset \mathcal{G}$ provided $0 \le u \le s \le t \le v$; $\mathcal{G}_t^s$ denotes a σ-algebra of events on time interval $[s,t]$; we write $\mathcal{G}_t$ in place of $\mathcal{G}_t^0$ and $\mathcal{G}^s$ in place of $\mathcal{G}_\infty^s$;

c) a probability measure $P_{s,x}$ for each pair $(s,x) \in [0,\infty) \times S$ on $\mathcal{G}^s$;

d) a function (stochastic process) $\xi_t(\omega) = \xi(t,\omega)$ defined on $[0,\infty) \times \Omega$ with values in S;

The system of objects described in a) - d) will be denoted by $\{\xi_t, \mathcal{G}_t^s, P_{s,x}\}$.

Definition 7.1. A system of objects $\{\xi_t, \mathcal{G}_t^s, P_{s,x}\}$ is called a *Markov process* provided:

1) for each $t \in [0,\infty)$ $\xi_t(\omega)$ is measurable mapping of $(\Omega, \mathcal{G})$ into $(S, \mathcal{B})$;

2) for arbitrary fixed s, t and B $(0 \le s \le t, B \in \mathcal{B})$ the function $P(s,x,t,B) = P_{s,x}(\xi_t \in B)$ is $\mathcal{B}$-measurable with respect to x;

3) $P_{s,x}(\xi_s = x) = 1$ for all $s \ge 0$ and $x \in S$;

4) $P_{s,x}(\xi_u \in B | \mathcal{G}_t^s) = P_{t,\xi_t}(\xi_u \in B)$ for all $s,t,u, 0 \le s \le t \le u < \infty$, $x \in S$ and $B \in \mathcal{B}$.

The measure $P_{s,x}$ should be considered as a probability law which determines probabilistic properties of the process $\xi_t(\omega)$ given that it starts at point x at the time s. Condition 4 expresses the Markov property of the processes. Let $\mathbb{E}_{s,x}$ denote the mathematical expectation with respect to measure $P_{s,x}$. For $\mathcal{G}^s$-measurable random variable $\xi(\omega)$

$$\mathbb{E}_{s,x}(\xi(\omega)) = \int \xi(\omega) P_{s,x}(d\omega).$$

It is not difficult to show that the Markov property (4) can be rewritten in terms of mathematical expectations as follows:

$$\mathbb{E}_{s,x}(f(\xi_u) | \mathcal{G}_t^s) = \mathbb{E}_{t,\xi_t}(f(\xi_u)),\ 0 \le s \le t \le u < \infty,$$

where f is an arbitrary $\mathcal{B}$-measurable bounded function.

Next, let us show that process

$$\left\{ \begin{array}{rcl} X_t(\omega) & = & \sum_{n=1}^\infty X_t^n(\omega) \mathbf{1}_{\left[\tau_{n-1}^E(\omega), \tau_n^E(\omega)\right)}(t) \\ \theta_t(\omega) & = & \sum_{n=1}^\infty \theta_t^n(\omega) \mathbf{1}_{\left[\tau_{n-1}^E(\omega), \tau_n^E(\omega)\right)}(t) \end{array} \right. \tag{7.1}$$

defined as a concatenation of solutions $\{X_t^n, \theta_t^n\}$ of the system of SDEs (5.1)-(5.4) (see sections 5, 5.2), is Markov. We follow the approach used in [12]. Let $\xi_t^{s,\eta} = (X_t^{s,x}, \theta_t^{s,\theta})$ denote the process (7.1) on $[s,\infty)$ satisfying initial condition $\xi_s^{s,\eta} = \eta = (X_s^{s,x}, \theta_s^{s,\theta})$. Assume that conditions of theorem 5.2 are satisfied. Let $\mathcal{F}_t^s$, $s < t$ be the σ-algebras generated by $\{W_u - W_s, p_2([s,u], dz), \beta_u, u \in [s,t]\}$, $\mathcal{F}_t^0 = \mathcal{F}_t$, $\mathcal{F}_\infty^s = \mathcal{F}^s$. For $s \le t$ the σ-algebras $\mathcal{F}_s$ and $\mathcal{F}^s$ are independent. Process $\xi_t^{s,\eta}$ is $\mathcal{F}^s$-measurable, hence, it is independent of σ-algebra $\mathcal{F}_s$. Let η_s be an arbitrary $\mathbb{R}^n \times \mathbb{M}$-valued $\mathcal{F}_s$ measurable random variable. Then ξ_t^{s,η_s}, $t \ge s$, is unique $\mathcal{F}_t$-measurable process on $[s,\infty)$ satisfying the initial condition $\xi_s^{s,\eta_s} = \eta_s$. Since for $u < s$ process $\xi_t^{u,y}$ is $\mathcal{F}_t$-measurable process on $[s,\infty)$ with initial condition $\xi_s^{u,y}$ then the following equality holds

$$\xi_t^{u,y} = \xi_t^{s,\xi_s^{u,y}},\ u < s < t. \tag{7.2}$$

Let φ be a bounded measurable function on $\mathbb{R}^n \times \mathbb{M}$, let ζ_s be an arbitrary bounded $\mathcal{F}_s$-measurable quantity. The independence of $\mathcal{F}_s$ and $\mathcal{F}^s$ and the Fubini's theorem imply that measure P on $\mathcal{F}_\infty$ is a product of measures P_s and P^s, where P_s is a restriction of P on $\mathcal{F}_s$, where P^s is a restriction of P on $\mathcal{F}^s$, and

$$\mathbb{E}(\varphi(\xi_t^{u,y})\zeta_s) = \mathbb{E}(\varphi(\xi_t^{s,\xi_s^{u,y}})\zeta_s) = \mathbb{E}\big(\zeta_s[\mathbb{E}(\varphi(\xi_t^{s,x}))]_{x=\xi_s^{u,y}}\big).$$

Since $\xi_s^{u,y}$ is $\mathcal{F}_s$-measurable then $\mathbb{E}(\varphi(\xi_t^{u,y})|\mathcal{F}_s) = \big[\mathbb{E}(\varphi(\xi_t^{s,x}))\big]_{x=\xi_s^{u,y}}$. Let

$$P(s,x,t,B) = P(\xi_t^{s,x} \in B),\ B \in \mathcal{B}_{\mathbb{R}^n\times\mathbb{M}}, \tag{7.3}$$

here $\mathcal{B}_{\mathbb{R}^n\times\mathbb{M}}$ is the σ-algebra of Borel sets on $\mathbb{R}^n \times \mathbb{M}$. Then, by taking $\varphi = I_B$, we obtain

$$P(\xi_t^{u,y} \in B|\mathcal{F}_s) = P(s, \xi_s^{u,y}, t, B). \tag{7.4}$$

If ξ_t is an arbitrary process defined by (7.1), by the same reasoning with help of which equalities (7.2) and (7.4) have been obtained, one can show that $\xi_t = \xi_t^{s,\xi_s}$ for $s < t$ and that

$$P(\xi_t \in B|\mathcal{F}_s) = P(s, \xi_s, t, B).$$

In this way processes defined by (7.1) are Markov processes with transition probability $P(s,x,t,B)$ defined by equality (7.4). To be precise, we have shown that the system of objects $\{(X_t,\theta_t), \mathcal{F}_t^s, P_{s,(x,\theta)}\}$, where $P_{s,(x,\theta)}\big((X_t,\theta_t) \in B\big) = P(s,(x,\theta),t,B) = P\big((X_t^{s,x}, \theta_t^{s,\theta}) \in B\big),\ B \in \mathcal{B}_{\mathbb{R}^n\times\mathbb{M}}$, is a Markov process.

8. Strong Markov property

Definition 8.1.

In this section we prove the Markov property

$$P_{s,x}(\xi_u \in B|\mathcal{G}_t^s) = P_{t,\xi_t}(\xi_u \in B), \quad s \le t \le u$$

remains valid also when a fixed time moment t is replaced by a stopping time.

Let $\{\xi_t(\omega), \mathcal{G}_t^s, P_{s,x}\}$ be a Markov process in the space $(S, \mathcal{B})$. Let $\mathcal{T}$ denote the σ-algebra of Borel sets on $[0, \infty)$.

Definition 8.2. A Markov process is called *strong Markov* if:

a) the transition probability $P(s, x, t, B)$ for a fixed B is a $\mathcal{T}\times\mathcal{B}\times\mathcal{T}$-measurable function of (s, x, t) on the set $0 \le s \le t < \infty$, $x \in S$;

b) it is progressively measurable;

c) for any $s \ge 0$, $t \ge 0$ and $\mathcal{B}$-measurable function $f(x)$ and an arbitrary stopping time τ equality

$$\mathbb{E}_{s,x}(f(\xi_{t+\tau})|\mathcal{G}_\tau^s) = \mathbb{E}_{\tau,\xi_\tau}(f(\xi_{t+\tau})) \tag{8.1}$$

is satisfied.

Remark 8.3. In order that equation (8.1) be satisfied, it is necessary that the random variable $g(\xi_\tau, \tau, t+\tau) = \mathbb{E}_{\tau,\xi_\tau}(f(\xi_{t+\tau}))$ be $\mathcal{G}_\tau^s$-measurable. For this reason assumptions a) and b) make part of the definition of the strong Markov property [11].

Now we return to the process $\xi_t = (X_t, \theta_t)$ defined in section 5.2. In previous section we have shown that it is Markov process. The following theorem proves that it is also a Strong Markov process.

Proposition 8.4. *Assume (A1)-(A4), (4.3), (4.4), (4.5) and (B1)-(B3). Let W, p_2, μ^E, X_0 and θ_0 be independent. Let $\mathcal{F}_t^s$, $s < t$ be the σ-algebras generated by $\{W_u - W_s, p_2(dz, [s, u]), \beta_u, u \in [s, t]\}$. For any bounded Borel function $f : \mathbb{R}^n \times \mathbb{M} \to \mathbb{R}$ and any $\mathcal{F}_t^s$-stopping time τ*

$$\mathbb{E}_{s,x}(f(\xi_{t+\tau})|\mathcal{F}_\tau^s) = \mathbb{E}_{\tau,\xi_\tau}(f(\xi_{t+\tau})).$$

Proof. Let $\{\sigma_k, k = 0, 1, \dots\}$ denote the ordered set of the stopping times $\{\tau_k^E, k = 1, 2, \dots\}$ and $\{\tau_k, k = 0, 1, \dots\}$. The latter set is the set of the stopping times generated by Poisson random measure p_2. Then on each time interval $[\sigma_{k-1}, \sigma_k)$, $k = 1, 2, \dots$ process ξ_t evolves as a diffusion staring at point $\xi_{\sigma_{k-1}}$ at the time σ_{k-1}. This means that on each time interval $[\sigma_{k-1}, \sigma_k)$ the Strong Markov property holds. Let $\mathcal{F}_\tau^s$ be the σ-algebra generated by the $\mathcal{F}_t^s$-stopping time τ. The sets

$\{\omega : \tau(\omega) \in [\sigma_{k-1}(\omega), \sigma_k(\omega))\}$, $k = 1, 2, ...$ are $\mathcal{F}^s_\tau$-measurable. Hence

$$\begin{aligned}
\mathbb{E}_{s,x}(f(\xi_{t+\tau})|\mathcal{F}^s_\tau) &= \sum_{k=0}^{\infty} \mathbf{1}_{[\sigma_{k-1},\sigma_k)}(\tau)\mathbb{E}_{s,x}\big(f(\xi_{t+\tau})|\mathcal{F}^s_\tau\big) \\
&= \sum_{k=0}^{\infty} \mathbb{E}_{s,x}\big(\mathbf{1}_{[\sigma_{k-1},\sigma_k)}(\tau)f(\xi_{t+\tau})|\mathcal{F}^s_\tau\big) \\
&= \sum_{k=0}^{\infty} \mathbb{E}_{\tau,\xi_\tau}\big(\mathbf{1}_{[\sigma_{k-1},\sigma_k)}(\tau)f(\xi_{t+\tau})\big) \\
&= \mathbb{E}_{\tau,\xi_\tau}\big(\sum_{k=0}^{\infty} \mathbf{1}_{[\sigma_{k-1},\sigma_k)}(\tau)f(\xi_{t+\tau})\big) \\
&= \mathbb{E}_{\tau,\xi_\tau}\big(f(\xi_{t+\tau})\big).
\end{aligned}$$

This completes the proof. □

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9. Concluding Remarks

The aim of this report was to significantly further the study of SDE's on a hybrid space, including characterizations of its solutions in terms of pathwise uniqueness, semimartingale and strong Markov process properties. We have used Jacod & Shyriayev 1987 [14] and Gihman & Skorohod 1982 (Russian) [12] to characterize jump-diffusion process solutions of SDE's. This yielded a valuable improvement over the Lepeltier & Marchal 1976 [16] regarding the understanding of semimartingale property and pathwise uniqueness of jump-diffusions. Next we have followed a similar path as taken by Blom 1990, 2003 [2, 4] in transferring this pathwise uniqueness and semimartingale understanding to the class of stochastic hybrid processes. This subsequently allowed to incorporate instantaneous jumps at a boundary within the same framework including pathwise uniqueness and semimartingale property. Finally we have introduced a novel approach in showing strong Markov property of solutions of SDEs.

Appendix A. Stochastic hybrid model 1 of Ghosh and Bagchi (2004)

The evolution of $\mathbb{R}^n \times \mathbb{M}$-valued Markov process $\{X_t, \theta_t\}$ is governed by the following equations:

$$dX_t = a(X_t, \theta_t)dt + b(X_t, \theta_t)dW_t + \int_{\mathbb{R}} g(X_{t-}, \theta_{t-}, u)p(dt, du), \tag{10.1}$$

$$d\theta_t = \int_{\mathbb{R}} h(X_{t-}, \theta_{t-}, u)p(dt, du). \tag{10.2}$$

Here:
(i) for $t = 0$, X_0 is a prescribed $\mathbb{R}^n$-valued random variable.

(ii) for $t = 0$, θ_0 is a prescribed $\mathbb{M}$-valued random variable.

(iii) W is an n-dimensional standard Wiener process.

(iv) $p(dt, du)$ is a Poisson random measure with intensity $dt \times m(du)$, where m is the Lebesgue measure on $\mathbb{R}$. p is assumed to be independent of W.

The coefficients are defined as follows

$$\begin{aligned} a &: \mathbb{R}^n \times \mathbb{M} \to \mathbb{R}^n \\ b &: \mathbb{R}^n \times \mathbb{M} \to \mathbb{R}^{n\times n} \\ g &: \mathbb{R}^n \times \mathbb{M} \times \mathbb{R} \to \mathbb{R}^n \\ h &: \mathbb{R}^n \times \mathbb{M} \times \mathbb{R} \to \mathbb{R}^N. \end{aligned}$$

Function h is defined as follows:

$$h(x, e_i, u) = \begin{cases} e_j - e_i & \text{if } u \in \Delta_{ij}(x) \\ 0 & \text{otherwise} \end{cases} \tag{10.3}$$

where for $i, j \in \{1, \dots, N\}$, $i \neq j$, $x \in \mathbb{R}^n$, $\Delta_{ij}(x)$ are the intervals of the real line defined in the following manner:

$$\begin{aligned} \Delta_{12}(x) &= [0, \lambda_{12}(x)) \\ \Delta_{13}(x) &= [\lambda_{12}(x), \lambda_{12}(x) + \lambda_{13}(x)) \\ &\ \vdots \\ \Delta_{1N}(x) &= \Big[\textstyle\sum_{j=2}^{N-1} \lambda_{1j}(x), \sum_{j=2}^{N} \lambda_{1j}(x)\Big) \\ \Delta_{21}(x) &= \Big[\textstyle\sum_{j=2}^{N} \lambda_{1j}(x), \sum_{j=2}^{N} \lambda_{1j}(x) + \lambda_{21}(x)\Big) \end{aligned}$$

and so on. In general,

$$\Delta_{ij}(x) = \Big[\sum_{i'=1}^{i-1} \sum_{\substack{j'=1 \\ j'\neq i'}}^{N} \lambda_{i'j'}(x) + \sum_{\substack{j'=1 \\ j'\neq i}}^{j-1} \lambda_{ij'}(x), \sum_{i'=1}^{i-1} \sum_{\substack{j'=1 \\ j'\neq i'}}^{N} \lambda_{i'j'}(x) + \sum_{\substack{j'=1 \\ j'\neq i}}^{j} \lambda_{ij'}(x)\Big).$$

For fixed x these are disjoint intervals, and the length of $\Delta_{ij}(x)$ is $\lambda_{ij}(x)$, $\lambda_{ij} : \mathbb{R}^n \to \mathbb{R}$, $i, j = 1, \dots, N$, $i \neq j$.

Let K_1 be the support of $g(\cdot, \cdot, \cdot)$ and let U_1 be the projection of K_1 on $\mathbb{R}$. It is assumed that U_1 is bounded. Let K_2 denote the support of $h(\cdot, \cdot, \cdot)$ and U_2 the projection of K_2 on $\mathbb{R}$. By definition of c, U_2 is a bounded set. One can define function $g(\cdot, \cdot, \cdot)$ so that the sets U_1 and U_2 form three nonempty sets: $U_1 \setminus U_2$, $U_1 \cap U_2$ and $U_2 \setminus U_1$ (see Figure 1). Then, we have the following:

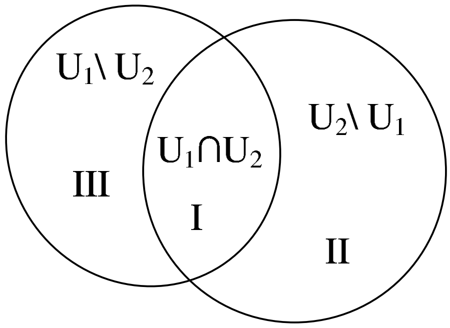

FIGURE 1. $U_1 \cup U_2$ is the projection of set $K_1 \cup K_2$ on $\mathbb{R}$

I: For $u \in U_1 \cap U_2$

$$\begin{cases} g(\cdot,\cdot,u) \neq 0 \\ h(\cdot,\cdot,u) \neq 0 \end{cases}$$

i.e., simultaneous jumps of X_t and switches of θ_t are possible.

II: For $u \in U_2 \setminus U_1$

$$\begin{cases} g(\cdot,\cdot,u) = 0 \\ h(\cdot,\cdot,u) \neq 0 \end{cases}$$

i.e., only random switches of θ_t are possible.

III: For $u \in U_1 \setminus U_2$

$$\begin{cases} g(\cdot,\cdot,u) \neq 0 \\ h(\cdot,\cdot,u) = 0 \end{cases}$$

i.e., only random jumps of X_t are possible.

Ghosh and Bagchi (2004) [10] proved that under the following conditions there exists an a.s. unique strong solution of SDE (10.1-10.2).

(C1) For each $e_i \in \mathbb{M}$, $i = 1, \ldots, N$, $a(\cdot, e_i)$ and $b(\cdot, e_i)$ are bounded and Lipschitz continuous.

(C2) For all $i, j \in \{1, \ldots, N\}$, $i \neq j$, functions $\lambda_{ij}(\cdot)$ are bounded and measurable, $\lambda_{ij}(\cdot) \geq 0$ for $i \neq j$ and $\sum_{j=1}^{N} \lambda_{ij}(\cdot) = 0$ for any $i \in \{1, \ldots, N\}$.

(C3) U_1, the projection of support of $g(\cdot,\cdot,\cdot)$ on $\mathbb{R}$, is bounded.

Let us compare SDEs (4.1-4.2) and (10.1-10.2).

The first two terms (i.e. the drift and the diffusion term) in (4.1) and in (10.1) are identical. However, when proving the existence of strong unique solution of SDE (10.1-10.2) Ghosh and Bagchi (2004) assume that the drift and the diffusion coefficients are bounded (condition C1). To prove the similar result for SDE (4.1-4.2) more general growth condition (A1) is adopted.

The jump part of X_t in (4.1) consists of two integral terms: one is an integral with respect to a compensated Poisson random measure q_1 and the second one is an integral with respect to a Poisson random measure p_2 on $\mathbb{R}_+ \times \mathbb{R}^d$ with intensity $dt \times m(du_1) \times \bar{\mu}(\underline{u})$ (see section 4). The jump part of X_t in (10.1) is described only by one integral term with respect to a Poisson random measure p on $\mathbb{R}_+ \times \mathbb{R}$ with intensity $dt \times m(du)$. The construction of the "switching" terms (4.2) and (10.2) is almost identical with some minor differences in defining the "rate" intervals. The conditions on "rate" functions $\lambda(e_i, e_j, \cdot)$ and $\lambda_{ij}(\cdot)$ are the same, i.e. these functions are assumed to be bounded and measurable for all $i, j = 1, \ldots, N$ (conditions A3 and C2).

There is a substantial difference in definitions of integrand functions g_2 and g which determine the jump size of X_t. In order to satisfy the existence and uniqueness conditions, U_1, the projection of support of function g on $U = \mathbb{R}$, must be bounded (condition C3), which is a kind of limitation. In case of function g_2 we have a bit more freedom. It has an extra argument $\underline{u} \in \underline{U} = \mathbb{R}^{d-1}$, and, since the intensity of p_2 with respect to $\underline{u}$ is a probability measure $\bar{\mu}$ (which is always finite), the projection of support of g_2 on $\underline{U} = \mathbb{R}^{d-1}$ can be unbounded. It is only required that g_2 be integrable with respect to $p_2(dt, du)$ (condition A4).

It is clear from the above that SDE (10.1-10.2) can be seen as a special case of (4.1-4.2).

Appendix B. Stochastic hybrid model 2 of Ghosh and Bagchi (2004)

The state of the system at time t, denoted by (X_t, θ_t), takes values in $\bigcup_{n=1}^{\infty}(S_n, \mathbb{M}_n)$, where $\mathbb{M}_n = \{e_1, e_2, \ldots, e_{N_n}\}$ and $S_n \subset \mathbb{R}^{d_n}$. Between the jumps of X_t the state equations are of the form

$$dX_t = a^n(X_t, \theta_t)dt + b^n(X_t, \theta_t)dW_t^n, \tag{11.1}$$

$$d\theta_t = \int_{\mathbb{R}} h^n(X_{t-}, \theta_{t-}, u)p(dt, du), \tag{11.2}$$

where for each $n \in \mathbb{N}$

$$a^n : S_n \times \mathbb{M}_n \to \mathbb{R}^{d_n}$$
$$b^n : S_n \times \mathbb{M}_n \to \mathbb{R}^{d_n \times d_n}$$
$$h^n : S_n \times \mathbb{M}_n \times \mathbb{R} \to \mathbb{R}^{N_n}.$$

Function h^n is defined in a similar way as (10.3) with rates $\lambda_{ij}^n : S_n \to \mathbb{R}$, $\lambda_{ij}^n \geq 0$ for $i \neq j$, and $\sum_{j=1}^{N_n} \lambda_{ij}^n(\cdot) = 0$ for any $i \in \{1, \ldots, N\}$. W^n is a standard d_n-dimensional Wiener process, p is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}$ with the intensity $dt \times m(du)$ as in the previous section.

For each $n \in \mathbb{N}$, let $A_n \subset S_n$, $D_n \subset S_n$. The set A_n is the set of instantaneous jump, whereas D_n is the destination set. It is assumed that for each $n \in \mathbb{N}$, A_n and D_n are closed sets, $A_n \cap D_n = \varnothing$ and $\inf_n d(A_n, D_n) > 0$. If at some random time X_t hits A_n, then it executes an instantaneous jump. The destination of (X_t, θ_t) at this juncture is determined by a map

$$g_n : A_n \times \mathbb{M}_n \to \cup_m (D_m \times \mathbb{M}_m).$$

After reaching the destination, the process $\{X_t, \theta_t\}$ follows the same evolutionary mechanism over and over again.

Let $\{\eta_t\}$ be an $\mathbb{N}$ valued process defined by

$$\eta_t = n \text{ if } (X_t, \theta_t) \in S_n \times \mathbb{M}_n. \tag{11.3}$$

The $\{\eta_t\}$ is a piecewise constant process, it changes from n to m when (X_t, θ_t) jumps from the regime $S_n \times \mathbb{M}_n$ to the regime $S_m \times \mathbb{M}_m$. Thus η_t is an indicator of a regime and a change in η_t means a switching in the regimes in which $\{X_t, \theta_t\}$ evolves.

Let

$$\tilde{S} = \{(x, e_i, n) | x \in S_n, e_i \in \mathbb{M}_n\},$$
$$\tilde{A} = \{(x, e_i, n) | x \in A_n, e_i \in \mathbb{M}_n\},$$
$$\tilde{D} = \{(x, e_i, n) | x \in D_n, e_i \in \mathbb{M}_n\}.$$

Then $\{X_t, \theta_t, \eta_t\}$ is an $\tilde{S}$-valued process. The set $\tilde{A}$ is the set where jumps occur and $\tilde{D}$ is the destination set for this process. The sets $\cup_n(S_n \times \mathbb{M}_n)$, $\cup_n(A_n \times \mathbb{M}_n)$ and $\cup_n(D_n \times \mathbb{M}_n)$ can be embedded in $\tilde{S}$, $\tilde{A}$ and $\tilde{D}$ respectively.

Let d^0 denote the injection map of $\cup_n(D_n \times \mathbb{M}_n)$ into $\tilde{D}$. Define three maps

$$\tilde{g}_i : \tilde{A} \to \tilde{D},\ i = 1, 2,$$
$$\tilde{h} : \tilde{A} \to \mathbb{N}.$$

$$\tilde{g}_1(x, e_i, n) = \text{ the first component in } d^0(g_n(x, e_i)),$$
$$\tilde{g}_2(x, e_i, n) = \text{ the second component in } d^0(g_n(x, e_i)),$$
$$\tilde{h}(x, e_i, n) = \text{ the third component in } d^0(g_n(x, e_i)).$$

Let τ_{m+1} be the stopping time defined by

$$\tau_{m+1} = \inf\{t > \tau_m | X_{t-}, \theta_{t-}, \eta_{t-} \in \tilde{A}\}.$$

The equations for $\{X_t, \theta_t, \eta_t\}$ may thus be summarized as follows:

$$(11.4) \quad dX_t = \Big(a(X_t, \theta_t, \eta_t) + \sum_{m=0}^{\infty} [\tilde{g}_1(X_{\tau_m -}, \theta_{\tau_m -}, \eta_{\tau_m -}) - X_{\tau_m -})]\delta(t - \tau_m)\Big)dt + b(X_t, \theta_t, \eta_t)dW_t^{\eta_t},$$

$$(11.5) \quad d\theta_t = \int_{\mathbb{R}} h(X_{t-}, \theta_{t-}, \eta_{t-}, u)p(dt, du) + \sum_{m=0}^{\infty} [\tilde{g}_2(X_{\tau_m -}, \theta_{\tau_m -}, \eta_{\tau_m -}) - \theta_{\tau_m -})]\delta(t - \tau_m)dt,$$

$$(11.6) \quad d\eta_t = \sum_{m=0}^{\infty} [\tilde{h}(X_{\tau_m -}, \theta_{\tau_m -}, \eta_{\tau_m -}) - \eta_{\tau_m -})]I_{\{\tau_m \leq t\}},$$

where δ is the Dirac measure and $a(x, e_i, n) = a^n(x, e_i)$, $b(x, e_i, n) = b^n(x, e_i)$, $h(x, e_i, n, u) = h^n(x, e_i, u)$.

To ensure the existence of strong unique solution of SDE (11.4-11.6) Ghosh and Bagchi (2004) [10] adopted the following assumptions:

(D1) For each $n \in \mathbb{N}$ and $e_i \in \mathbb{M}_i$, $a^n(\cdot, e_i)$ and $b^n(\cdot, e_i)$ are bounded and Lipschitz continuous.

(D2) For each $n \in \mathbb{N}$, $i, j = 1, \ldots, M_n$, $i \neq j$, functions $\lambda_{ij}^n(\cdot)$ are bounded and measurable, $\lambda_{ij}^n(\cdot) \geq 0$ for $i \neq j$ and $\sum_{j=1}^{N} \lambda_{ij}^n(\cdot) = 0$ for any $i \in \{1, \ldots, N\}$.

(D3) The maps g_n, $n \in \mathbb{N}$, are bounded and uniformly continuous.

(D4) $\inf_n d(A_n, D_n) > 0$.

The above model has similarities to the one we have considered in section 5. Let us see what are the main differences between SDE (5.1-5.4) and SDE (11.4-11.6).

Solutions of SDE (11.4-11.6) are the $\bigcup_{n=1}^{\infty}(S_n, \mathbb{M}_n)$-valued switching diffusions with hybrid jumps at the boundary. Before hitting the boundary $\{X_t, \theta_t\}$ evolves as an $(S_n, \mathbb{M}_n)$-valued switching diffusion in some regime $\eta_t = n \in \mathbb{N}$. The drift and the diffusion coefficients and the mapping determining a new starting point of the process after the hitting the boundary can be different for every different regime $n \in \mathbb{N}$.

Solutions of SDE (5.1-5.4) are the $(\mathbb{R}^n \times \mathbb{M})$-valued switching-jump diffusions with hybrid jumps at the boundary. The dimension of the state space and the coefficients of SDE are fixed. Hence, on this specific point, model 2 of Ghosh and Bagchi (2004) [10] is more general. However the jump term in SDE (5.1) is much more general than the jump term in equation (11.4).

Now let us have a look at assumptions D1-D4. Assumption D1 implies that our local assumptions A1 and A2 for SDE (4.1-4.2) are definitely satisfied. Assumptions

D2 and D3 imply that assumptions A3 and A4 for SDE (4.1-4.2) are satisfied. Assumption D4 implies that B1 and B2 adopted to SDE (5.1-5.4) are satisfied. It ensures that after the jump the process starts inside of some open set, but not on a boundary. The non-Zeno condition B3 of SDE (5.1-5.4) is missing in Ghosh and Bagchi (2004) [10].

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