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Distributed Control and Stochastic Analysis of Hybrid Systems
Supporting Safety Critical Real-Time Systems Design

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Report on Global Decentralized Conflict Resolution

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Report on Global Decentralized Conflict Resolution

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Contents

1 Global Decentralized Conflict Resolution Part 1: Holonomic Kinematics **5**
1.1 The case of Global Sensing Capabilities 5
1.1.1 Decentralized Navigation Functions(DNF's) 7
1.1.2 Control Strategy 9
1.1.3 Construction of the G function 9
1.1.4 The f function 12
1.1.5 Proof of Correctness 13
1.1.6 Input Constraints 15
1.2 The Case of Limited Sensing Capabilities 15
1.3 Proof Sketches 17
1.3.1 Proof of Lemma 1.2 17
1.3.2 Proof of Lemma 1.3 17
1.3.3 Proof of Lemma 1.4 18
1.3.4 Proof of Lemma 1.5 18
1.3.5 Proof of Lemma 1.6 21
1.3.6 Proof of Proposition 1.7 22
1.4 Simulations 28

2 Global Decentralized Conflict Resolution Part 2: Nonholonomic Kinematics **31**
2.1 The case of Global Sensing Capabilities 31
2.1.1 Decentralized Dipolar Navigation Functions(DDNF's) . . 32
2.1.2 Nonholonomic Control 33
2.2 The Case of Limited Sensing Capabilities 38
2.3 Simulations 38

3 Global Decentralized Conflict Resolution Part 3: Dynamic Models **40**
3.1 Holonomic Dynamics 40
3.1.1 Stability Analysis 41
3.2 Nonholonomic Dynamics 42
3.2.1 Elements from Nonsmooth Analysis 43
3.2.2 Nonholonomic Control and Stability Analysis 45

3.3 Simulations 48

4 Conclusions and Future Research Issues **51**

Introduction

This is the final deliverable of HYBRIDGE WP6, summarizing the work held under this work package on global decentralized conflict resolution. It presents the extension of navigation functions, which have been proven a very powerful tool for centralized navigation and collision avoidance, to decentralized navigation, for both the cases where the aircraft dynamics are considered holonomic or nonholonomic. The underlying work is held under the guidance of Task 6.2 of WP6.

Decentralized navigation approaches are more appealing to centralized ones, due to their reduced computational complexity and increased robustness with respect to agent failures. The main focus of work in this domain has been cooperative and formation control of multiple agents, where so much effort has been devoted to the design of systems with variable degree of autonomy ([15],[40],[34], [42]). There have been many different approaches to the decentralized motion planning problem. Open loop approaches use game theoretic and optimal control theory to solve the problem taking the constraints of vehicle motion into account; see for example [2],[13], [21],[18], [6], [41]. On the other hand, closed loop approaches use tools from classical Lyapunov theory and graph theory to design control laws and achieve the convergence of the distributed system to a desired configuration both in the concept of cooperative ([14], [24], [25], [22]) and formation control ([1], [17], [43], [32], [39]). A few approaches use computer science based tools to treat the problem;see for example [20], [29], [30]. However, the latter fail to guarantee convergence of the multi-agent system.

Closed loop strategies are apparently preferable to open loop ones, mainly because they provide robustness with respect to modelling uncertainties and agent failures and guaranteed convergence to the desired configurations. However, a common point of most work in this area is devoted to the case of point agents. Although this allows for variable degree of decentralization, it is far from realistic in real world applications. For example, in conflict resolution in Air Traffic Management, two aircraft are not allowed to approach each other closer than a specific “alert” distance. The construction of closed loop methods for distributed non-point multi-agent systems is both evident and appealing.

A closed loop approach for single robot navigation was proposed by Koditschek and Rimon in their seminal work [23]. Work under WP6 has extended this navigation functions' framework to the case of multiple non-point holonomic and nonholonomic agents.

In Air Traffic Management Systems, decentralized conflict detection and resolution involves reassignment of the control tasks from the central authority, i.e. the Air Traffic Controllers, to the agents, i.e. the cockpit. The level of decentralization depends on the knowledge an agent has on the other agents' actions and objectives. In the first approaches to the problem, the decentralization factor lied in the fact that each agent/aircraft had knowledge only of its own desired destination, but not of the desired destinations of the others. In this deliverable the method is extended to take into account the limited sensing capabilities of each aircraft. Specifically, each agent is capable of knowing the positions and/or velocities only of aircraft within its sensing zone at each time instant.

The rest of the report is organized as follows: chapter 1 presents the decentralized navigation functions' method for multiple holonomic agents for the cases of global and limited sensing capabilities. The counterparts for multiple nonholonomic agents are presented in chapter 2. Chapter 3 deals with dynamic models of vehicle movement. Nontrivial computer simulations are presented throughout the report to indicate the effectiveness of the methodology.

Chapter 1

Global Decentralized Conflict Resolution Part 1: Holonomic Kinematics

In this chapter, we review the decentralized conflict resolution algorithm developed under WP6 for the case when the dynamics of each aircraft are considered purely holonomic. We first present the fundamental approach using Decentralized Navigation Functions (DNF's) for agents with global sensing capabilities. We proceed by showing how this methodology has been successfully extended to take into account the limited sensing capabilities of each agent. A discussion on handling velocity constraints is also included.

1.1 The case of Global Sensing Capabilities

In this section, we consider the case where each agent has global knowledge of the positions of the others at each time instant. The decentralization factor lies in the assumption that each agent does not need to know the desired destinations of the others in order to navigate to its goal configuration. A provable way to extend this method to the case of limited sensing capabilities is presented in the next subsection.

Consider a system of N agents operating in the same workspace $W \subset \mathcal{R}^2$. Each agent i occupies a disc: $R = \{q \in \mathcal{R}^2 : \| q - q_i \| \leq r_i\}$ in the workspace where $q_i \in \mathcal{R}^2$ is the center of the disc and r_i is the radius of the agent. The configuration space is spanned by $q = [q_1, \ldots, q_N]^T$. The motion of each agent is described by the single integrator:

$$\dot{q}_i = u_i, i \in \mathcal{N} = [1, \ldots, N] \tag{1.1}$$

The desired destinations of the agents are denoted by the index d: $q_d = [q_{d1}, \ldots, q_{dN}]^T$ The following figure shows a three-agent conflict situation:

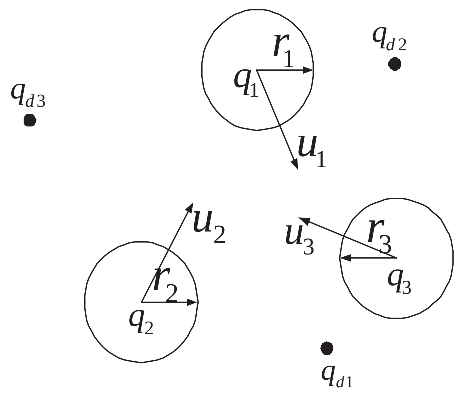

Figure 1.1: A conflict scenario with three agents.

The multi agent navigation problem can be stated as follows: " *Derive a set of control laws (one for each agent) that drives the team of agents from any initial configuration to a desired goal configuration avoiding, at the same time, collisions. Each agent has global knowledge of the team configuration but is unaware of the other agents desired destinations* ".

In this section we make the following assumptions:

- Each agent has global knowledge of the position of the others at each time instant.
- Each agent has knowledge only of its own desired destination but not of the others.
- We consider spherical agents.
- The workspace is bounded and spherical.

Our assumption regarding the spherical shape of the agents does not constrain the generality of this work since it has been proven that navigation properties are invariant under diffeomorphisms ([23]). Arbitrarily shaped agents diffeomorphic to spheres can be taken into account. Methods for constructing analytic diffeomorphisms are discussed in [38] for point agents and in [35] for rigid body agents.

The second assumption makes the problem decentralized. Clearly, in the centralized case a central authority has knowledge of everyones goals and positions at each time instant and it coordinates the whole team so that the desired specifications (destination convergence and collision avoidance) are fulfilled. In the current situation no such authority exists and we have to deal with the limited knowledge of each agent. This is of course the first step towards a variable degree of decentralization. The first assumption, regarding the global knowledge each agent has about the state space, is overcome in the next section, where we discuss how the methodology presented in the next subsections, can be extended to the case of limited sensing capabilities.

1.1.1 Decentralized Navigation Functions(DNF's)

Preliminaries

In this section we review the navigation function method introduced in the seminal paper of Koditscheck and Rimon [23] for single point robot navigation.

Navigation functions (NF's) are real valued maps realized through cost functions $\varphi(q)$, whose negated gradient field is attractive towards the goal configuration and repulsive wrt obstacles. It has been shown by Koditscheck and Rimon that strict global navigation (i.e. the system $\dot{q} = u$ under a control law of the form $u = -\nabla\varphi$ admits a globally attracting equilibrium state) is not possible, and a smooth vector field on any sphere world with a unique attractor, must have at least as many saddles as obstacles [23].

A navigation function can be defined as follows:

Definition 1.1 *Let $F \subset \mathcal{R}^{2N}$ be a compact connected analytic manifold with boundary. A map $\varphi : F \to [0, 1]$ is a navigation function if:(1) it is analytic on F, (2) it has only one minimum at $q_d \in int(F)$, (3) its Hessian at all critical points (zero gradient vector field) is full rank, and (4) $\lim_{q \to \vartheta F} \varphi(q) = 1$.*

Strictly speaking, the continuity requirements for the navigation functions are to be C^2. The first property of the above definition follows the intuition provided by the authors of [23], that is preferable to use closed form mathematical expressions to encode actuator commands instead of "patching together" closed form expressions on different portions of space, so as to avoid branching and looping in the control algorithm. Analytic navigation functions, through their gradient provide a direct way to calculate the actuator commands, and once constructed they provide a provably correct control algorithm for every environment that can be diffeomorphically transformed to a sphere world. In our approach, we further relax this requirement by using a non-analytic, merely C^1 navigation function. The discontinuity however, takes place outside of the region where critical points of the potential function occur, so it does not affect the navigation properties of the proposed function.

A function φ that has a unique minimum on F is called *polar*. By using a polar function on a compact connected manifold with boundary, all initial conditions will either be brought to a saddle point or to the unique minimum of the function.

A scalar valued function φ whose Hessian at all critical points is full rank is called *Morse*. The corresponding critical points are called *non-degenerate*. The requirement in Definition 1 that a navigation function must be a Morse function, establishes that the initial conditions that bring the system to saddle points are sets of measure zero [31]. In view of this property, all initial conditions away from sets of measure zero are brought to the unique minimum.

The last property of definition 1.1 guarantees that the resulting vector field is transverse to the boundary of F. The set F represents the free space of the agent movement, i.e. the subset which is free of collisions. This establishes that the system will be safely brought to q_d, avoiding collisions.

DNF's vs MRNF's

In [26], the navigation functions method has been extended to the case of multiple mobile robots with the use of Multi-Robot navigation functions (MRNF's).

In the form of a centralized setup [26], where a central authority has knowledge of the current positions and desired destinations of all agents, the sought control law is of the form: $u = -K\nabla\varphi(q)$ where K is a gain. In the decentralized case addressed in this work, each agent has knowledge of only the current positions of the others, and not of their desired destinations. Hence each agent has a different navigation law.

Following the procedure of [23],[26], we consider the following class of decentralized navigation functions(*DNF's*):

$$\varphi_i \triangleq \sigma_d \circ \sigma \circ \hat{\varphi}_i = \left(\frac{\gamma_i}{\gamma_i + G_i}\right)^{1/k} \tag{1.2}$$

which is a composition of $\sigma_d \triangleq x^{1/k}$, $\sigma \triangleq \frac{x}{1+x}$ and the cost function $\hat{\varphi}_i \triangleq \frac{\gamma_i}{G_i}$,where $\gamma_i^{-1}(0)$ denotes the desirable set(i.e. the goal configuration) and $G_i^{-1}(0)$ the set that we want to avoid(i.e. collisions with other agents).A suitable choice is:

$$\gamma_i = (\gamma_{di} + f_i)^k \tag{1.3}$$

where $\gamma_{di} = \| q_i - q_{di} \|^2$, is the squared metric of the current agent's configuration q_i from its destination q_{di}. The definition of the function f_i will be given later. Function G_i has as arguments the coordinates of all agents, i.e. $G_i = G_i(q)$, in order to express all possible collisions of agent i with the others. The proposed navigation function for agent i is

$$\varphi_i(q) = \frac{\gamma_{di} + f_i}{\left((\gamma_{di} + f_i)^k + G_i\right)^{1/k}} \tag{1.4}$$

By using the notation $\tilde{q}_i \triangleq [q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_N]^T$, the decentralized NF can be rewritten as

$$\varphi_i = \varphi_i(q_i, \tilde{q}_i) = \varphi_i(q_i, t)$$

that is, the potential function in hand contains a *time-varying* element which corresponds to the movement in time of all the other agents apart from i. This element is neglected in the case of a single agent moving in an environment of static obstacles ([23]), but in this case the term $\frac{\partial \varphi_i}{\partial t}$ is nonzero.

1.1.2 Control Strategy

The proposed feedback control strategy for agent i is defined as

$$u_i = -K_i \frac{\partial \varphi_i}{\partial q_i} \tag{1.5}$$

where $K_i > 0$ a positive gain.

1.1.3 Construction of the G function

In the proposed decentralized control law, each agent has a different G_i which represents its relative position with all the other agents. In contrast to the centralized case, in which a central authority has global knowledge of the positions and desired destinations of the whole team and plans a global G function accordingly, in the decentralized case, each member i of the team has its own G_i function, which encodes the different proximity relations with the rest. The main difference of the DNF's and the MRNF's in [26] from the NF's introduced in [23] lies in the structure of the function G. While there were attempts to prove convergence and collision avoidance to the straightforward extension of [23] to the multiple moving agents case, only collision avoidance properties were established. Furthermore simulation results motivated us to consider a different approach to [26] for the decentralized setup.

We review now the construction of the "collision" function G_i for each agent i. The "Proximity Function"between agents i and j is given by

$$\beta_{ij} = \|q_i - q_j\|^2 - (r_i + r_j)^2 \tag{1.6}$$

Consider now the situation in figure 1.2. There are 5 agents and we proceed to define the function G_R for agent R.

Definition 1.2 *A* relation *with respect to agent R is every possible collision scheme that can occur in a multiple agents scene with respect R.*

Definition 1.3 *A* binary relation *with respect to agent R is a relation between agent R and another.*

Definition 1.4 *The* relation level *in the number of binary relations in a relation.*

We denote by $(R_j)_l$ the jth relation of level-l with respect to agent R. With this terminology in hand, the collision scheme of figure (1.2a) is a level-1 relation (one binary relation) and that of figure (1.2b) is a level-3 relation (three binary relations), always with respect to the specific agent R. We use the notation

$$(R_j)_l = \{\{R, A\}, \{R, B\}, \{R, C\}, \ldots\}$$

to denote the set of binary relations in a relation with respect to agent R, where $\{A, B, C, ...\}$ the set of agents that participate in the specific relation. For example, in figure 1.2b:

$$(R_1)_3 = \{\{R, O_1\}, \{R, O_2\}, \{R, O_3\}\}$$

where we have set arbitrarily $j = 1$.

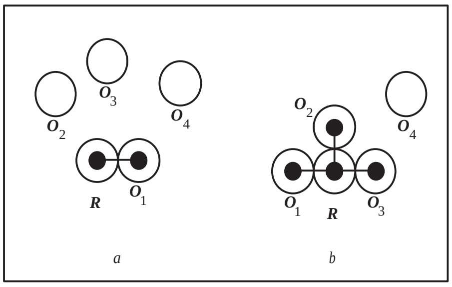

Figure 1.2: Part a represents a level-1 relation and part b a level-3 relation wrt agent R.

The complementary set $(R_j^C)_l$ of relation j is the set that contains all the relations of the same level apart from the specific relation j. For example in figure 1.2b:

$$\left(R_1^C\right)_3 = \{(R_2)_3, (R_3)_3, (R_4)_3\}$$

where

$$(R_2)_3 = \{\{R, O_1\}, \{R, O_2\}, \{R, O_4\}\}$$
$$(R_3)_3 = \{\{R, O_1\}, \{R, O_3\}, \{R, O_4\}\}$$
$$(R_4)_3 = \{\{R, O_2\}, \{R, O_3\}, \{R, O_4\}\}$$

A "Relation Proximity Function" (RPF) provides a measure of the distance between agent i and the other agents involved in the relation. Each relation has its own RPF. Let R_k denote the k^{th} relation of level l. The RPF of this relation is given by:

$$(b_{R_k})_l = \sum_{j \in (R_k)_l} \beta_{\{R,j\}} \tag{1.7}$$

where the notation $j \in (R_k)_l$ is used to denote the agents that participate in the specific relation of agent R. In the proofs, we also use the simplified notation $b_r = \sum_{j \in P_r} \beta_{ij}$ for simplicity, where r denotes a relation and P_r denotes the set of agents participating in the specific relation wrt agent i.

For example, in the relation of figure (1.2b) we have

$$(b_{R_1})_3 = \sum_{m \in (R_1)_3} \beta_{\{R,m\}} = \beta_{\{R,O_1\}} + \beta_{\{R,O_2\}} + \beta_{\{R,O_3\}}$$

A "Relation Verification Function" (RVF) is defined by:

$$(g_{R_k})_l = (b_{R_k})_l + \frac{\lambda (b_{R_k})_l}{(b_{R_k})_l + (B_{R_k^C})_l^{1/h}} \tag{1.8}$$

where λ, h are positive scalars and

$$(B_{R_k^C})_l = \prod_{m \in (R_k^C)_l} (b_m)_l$$

where as previously defined, $(R_k^C)_l$ is the complementary set of relations of level-l, i.e. all the other relations with respect to agent i that have the same number of binary relations with the relation R_k. Continuing with the previous example we could compute, for instance,

$$\left(B_{R_1^C}\right)_3 = (b_{R_2})_3 \cdot (b_{R_3})_3 \cdot (b_{R_4})_3$$

which refers to level-3 relations of agent R.

For simplicity we also use the notation $(B_{R_k^C})_l \equiv \tilde{b}_i = \prod_{m \in (R_k^C)_l} b_m$. The RVF can be written as $g_i = b_i + \frac{\lambda b_i}{b_i + \tilde{b}_i^{1/h}}$ It is obvious that for the highest level $l = n-1$ only one relation is possible so that $(R_k^C)_{n-1} = \emptyset$ and $(g_{R_k})_l = (b_{R_k})_l$ for $l = n-1$. The basic property that we demand from RVF is that it assumes the value of zero if a relation holds, while no other relations of the same or other levels hold. In other words it should indicate which of all possible relations holds. We have he following limits of RVF (using the simplified notation): (a) $\lim_{b_i \to 0} \lim_{\tilde{b}_i \to 0} g_i\left(b_i, \tilde{b}_i\right) = \lambda$ (b) $\lim_{\substack{b_i \to 0 \\ \tilde{b}_i \neq 0}} g_i\left(b_i, \tilde{b}_i\right) = 0$. These limits guarantee that RVF will behave in the way we want it to, as an indicator of a specific collision.

The function G_i is now defined as

$$G_i = \prod_{l=1}^{n_L^i} \prod_{j=1}^{n_{R_l}^i} (g_{R_j})_l \tag{1.9}$$

where n_L^i the number of levels and $n_{R_l}^i$ the number of relations in level-l with respect to agent i.

The definition of the G function in the multiple moving agents situation is slightly different than the one introduced by the authors in [23]. The collision scheme in that approach involved a single moving point agent in an environment with static obstacles. A collision with more than one obstacle was therefore impossible and the obstacle function was simply the product of the distances of the agent from each obstacle. In our case however, this is inappropriate, as can be seen in the next figure. The control law of agent A should distinguish when agent A is in conflict with B, C, or B and C simultaneously. Mathematically, the first two situations are level-1 relations and the third a level-2 relation with

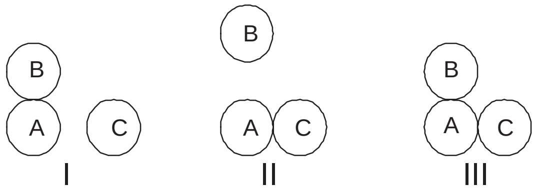

Figure 1.3: I,II are level-1 relations with respect to A, while III is level-2. The RVFs of the level-1 relations are nonzero in situation III.

respect to A. Whenever the latter occurs, the RVF of the level-2 relation tends to zero while the RVFs of the two separate level-1 relations (A,B and A,C) are nonzero. The key property of an RVF is that it tends to zero only when the corresponding relation holds. Hence it serves as an analytic switch that is activated (tends to zero) only when the relation it represents is realized.

An example

As an example, we will present steps to construct the function G with respect to a specific agent in a team of 4 agents indexed 1 through 4. We construct the function G_1 wrt agent 1. We begin by defining the Relation Proximity Functions in every level (Table 1):

Relation	Level 1	Level 2	Level 3
1	$(b_1)_1 = \beta_{12}$	$(b_1)_2 = \beta_{12} + \beta_{13}$	$(b_1)_3 = \beta_{12} +$ $+ \beta_{13} + \beta_{14}$
2	$(b_2)_1 = \beta_{13}$	$(b_2)_2 = \beta_{12} + \beta_{14}$	-
3	$(b_3)_1 = \beta_{14}$	$(b_3)_2 = \beta_{13} + \beta_{14}$	-

Table 1

It is now easy to calculate the Relation Verification Functions for each relation based on equation (1.8). For example, for the second relation of level 2, the complement (term $(B_{R_k^C})_l$ in eq.(1.8)) is given by $(B_{2^C})_2 = (b_1)_2 \cdot (b_3)_2$ and substituting in (1.8), we have

$$(g_2)_2 = (b_2)_2 + \frac{\lambda \, (b_2)_2}{(b_2)_2 + ((b_1)_2 \cdot (b_3)_2)^{1/h}}$$

The function G_1 is then calculated as the product of the Relation Verification Functions of all relations.

1.1.4 The f function

The key difference of the decentralized method with respect to the centralized case is that the control law of each agent ignores the destinations of the others. By using $\varphi_i = \frac{\gamma_{di}}{((\gamma_{di})^k + G_i)^{1/k}}$ as a navigation function for agent i, there is no potential for i to cooperate in a possible collision scheme when its initial condition coincides with its final destination. In order to overcome this limitation,we add a function f_i to γ_i so that the cost function φ_i attains positive values in proximity situations even when i has already reached its destination. A preliminary definition for this function was given in [12], [44]. Here, we modify the previous definitions to ensure that the destination point is a non-degenerate local minimum of φ_i with minimum requirements on assumptions. We define the function f_i by:

$$f_i(G_i) = \begin{cases} a_0 + \sum\limits_{j=1}^{3} a_j G_i^j, \; G_i \leq X \\ 0, \; G_i > X \end{cases} \tag{1.10}$$

where $X, Y = f_i(0) > 0$ are positive parameters the role of which will be made clear in the following. The parameters a_j are evaluated so that f_i is maximized when $G_i \to 0$ and minimized when $G_i = X$. We also require that f_i is continuously differentiable at X. Therefore we have:

$$a_0 = Y, a_1 = 0, a_2 = \frac{-3Y}{X^2}, a_3 = \frac{2Y}{X^3}$$

The parameter X serves as a sensing parameter that activates the f_i function whenever possible collisions are bound to occur. The only requirement we have for X is that it must be small enough to guarantee that f_i vanishes whenever the system has reached its equilibrium, i.e. when everyone has reached its destination. In mathematical terms:

$$X < G_i\left(q_{d1}, \ldots, q_{dN}\right) \; \forall i \tag{1.11}$$

That's the minimum requirement we have regarding knowledge of the destinations of the team.

The resulting navigation function is no longer analytic but merely C^1 at $G_i = X$. However, by choosing X large enough, the resulting function is analytic in a neighborhood of the boundary of the free space so that the characterization of its critical points can be made by the evaluation of its Hessian. Hence, the parameter X must be chosen small enough in order to satisfy (1.11) but large enough to include the region described above. Clearly, this is a tradeoff the control design has to pay in order to achieve decentralization. Intuitively, the destinations should be far enough from one another.

1.1.5 Proof of Correctness

Let $\varepsilon > 0$. Define $B_{j,l}^i(\varepsilon) \equiv \{q : 0 < (g_{R_j}^i)_l < \varepsilon\}$. Following [23],[26] we discriminate the following topologies for the function φ_i:

1. The destination point: q_{di}
2. The free space boundary: $\partial F(q) = G_i^{-1}(\delta), \delta \to 0$

3. The set near collisions: $F_0(\varepsilon) = \bigcup_{l=1}^{n_L^i} \bigcup_{j=1}^{n_{R,l}^i} B_{j,l}^i(\varepsilon) - \{q_{di}\}$

4. The set away from collisions: $F_1(\varepsilon) = F - (\{q_{di}\} \cup \partial F \cup F_0(\varepsilon))$

The following theorem allows us to derive results for the function φ_i by examining the simpler function $\hat{\varphi}_i(q) = \frac{\gamma_i}{G_i}$:
Theorem 1.1 [23] *Let I_1, I_2 be intervals, $\hat{\varphi} : F \to I_1$ and $\sigma : I_1 \to I_2$ be analytic. Define the composition $\varphi : F \to I_2$ to be $\varphi = \sigma \circ \hat{\varphi}$. If σ is monotonically increasing on I_1, then the set of critical points of φ and $\hat{\varphi}$ coincide and the (Morse) index of each critical point is identical.*

A key point in the discrimination between centralized and decentralized navigation functions is that the latter contain a time-varying part which depends on the movement of the other agents. Using the same procedure as in [23],[26] we first prove that the construction of each φ_i guarantees collision avoidance:

Proposition 1.1 *For each fixed t, the function $\varphi_i(q_i, \cdot)$ is a navigation function if the parameters h, k assume values bigger than a finite lower bound.*

Proof Sketch: For the complete proof see [8]. The set of critical points of φ_i is defined as $C_{\varphi_i} = \{q : \partial\varphi_i/\partial q_i = 0\}$. A critical point is non-degenerate if $\partial^2\varphi_i/\partial^2 q_i$ has full rank at that point.The statement of the proposition is guaranteed by the following Lemmas:

Lemma 1.2 *If the workspace is valid, the destination point q_{di} is a non-degenerate local minimum of φ_i.*

Lemma 1.3 *All critical points of φ_i are in the interior of the free space.*

Lemma 1.4 *For every $\varepsilon > 0$, there exists a positive integer $N(\varepsilon)$ such that if $k > N(\varepsilon)$ then there are no critical points of $\hat{\varphi}_i$ in $F_1(\varepsilon)$.*

Lemma 1.5 *There exists an $\varepsilon_0 > 0$ such that $\hat{\varphi}_i$ has no local minimum in $F_0(\varepsilon)$, as long as $\varepsilon < \varepsilon_0$.*

Lemma 1.6 *There exist $\varepsilon_1 > 0$ and $h_1 > 0$, such that the critical points of $\hat{\varphi}_i$ are non-degenerate as long as $\varepsilon < \varepsilon_1$ and $h > h_1$.*

The complete proofs of the Lemmas can be found in [8]. Sketches of the proofs are found in section 1.4. Lemmas 1.2-1.5 guarantee the polarity of the proposed DNF, whilst Lemma 1.6 guarantees the non-degeneracy of the critical points. By choosing k, h that satisfy the above Lemmas, the statement of Proposition 1.1 is proved.

This however does not guarantee global convergence of the system state to the destination configuration. This is achieved by using a Lyapunov function for the *whole* system which is *time invariant* that is a function that depends on the positions of all the agents. The candidate Lyapunov function that we use in this paper is simply the sum of the DNF's of all agents. Specifically we prove the following:

Proposition 1.7 *The time-derivative of $\varphi = \sum_{i=1}^{N} \varphi_i$ is negative definite across the trajectories of the system up to a set of initial conditions of measure zero if the parameters h, k assume values bigger than a finite lower bound.*

A rather detailed proof based on matrix calculus be found in [8] while a proof sketch in section 1.4.

1.1.6 Input Constraints

Handling of input constraints can also be incorporated in the proposed set-up. Suppose that the velocity specifications for each agent are given by:

$$\|u_i\| \le V_{max} \forall i \in \mathcal{N} = [1, ..., N] \tag{1.12}$$

It is straightforward to see that this is equivalent to $K_i \le \frac{V_{max}}{\|\vartheta\varphi_i/\vartheta q_i\|_{max}}$. Hence, a small enough K_i can always be chosen to meet the desired specifications. What remains is to show that $\|\frac{\vartheta\varphi_i}{\vartheta q_i}\|_{max}$ is finite. This is guaranteed by the following Lemma:

Lemma 1.8 *The term $\|\frac{\vartheta\varphi_i}{\vartheta q_i}\|$ admits a finite upper bound.*

Proof: Following the proof of Proposition 1.7 in [8] we have

$$\frac{\partial\varphi_i}{\partial q_i} = \frac{G_i\frac{\partial\gamma_{di}}{\partial q_i} + \sigma_i\frac{\partial G_i}{\partial q_i}}{\left((\gamma_{di} + f_i)^k + G_i\right)^{1+1/k}} \Rightarrow \left\|\frac{\partial\varphi_i}{\partial q_i}\right\|_{\max} = \frac{\left(G_i\frac{\partial\gamma_{di}}{\partial q_i} + \sigma_i\frac{\partial G_i}{\partial q_i}\right)_{\max}}{\left(\left((\gamma_{di} + f_i)^k + G_i\right)^{1+1/k}\right)_{min}}$$

The denominator admits a strictly positive lower bound because even when $\gamma_{di} = 0, G_i \to 0,\ f_i \to Y$ which is strictly positive. For the nominator we have

$$\left(G_i\frac{\partial\gamma_{di}}{\partial q_i} + \sigma_i\frac{\partial G_i}{\partial q_i}\right)_{\max} \le (G_i)_{\max}\|q_i - q_{di}\|_{\max} + |\sigma_i|_{\max}\left(\frac{\partial G_i}{\partial q_i}\right)_{\max}$$

The first term is always bounded in a bounded workspace while the second term is also bounded by virtue of Lemma 2.3 in [8]. This establishes the boundedness of $\|\frac{\vartheta\varphi_i}{\vartheta q_i}\|$. $\diamondsuit$

1.2 The Case of Limited Sensing Capabilities

In the previous section, it was shown how with a suitable choice of the parameters h, k the proposed control law can satisfy the collision avoidance and destination convergence properties in a bounded workspace. The decentralization feature of the whole scheme lied in the fact that each agent didn't have knowledge of the desired destinations of the rest of the team. On the other hand, each one had global knowledge of the positions of the others at each time instant. This is far from realistic in real world applications.

In this section we provide the necessary machinery to take the limited sensing capabilities of each agent into account. Specifically, we alter the definition of inter-agent proximity functions in order to cope with the limited sensing range of each agent.

We consider a bounded workspace with n agents. Each agent has only local knowledge of the positions of the others at each time instant. Specifically, it only knows the position of agents which are in a cyclic neighborhood of specific radius d_C around its center. Therefore the Proximity Function between two agents has to be redefined in this case. We propose the following nonsmooth function:

$$\beta_{ij} = \begin{cases} \|q_i - q_j\|^2 - (r_i + r_j)^2, \text{for } \|q_i - q_j\| \leq d_C \\ d_C^2 - (r_i + r_j)^2, \text{for } \|q_i - q_j\| > d_C \end{cases} \tag{1.13}$$

The whole scheme is now modelled as a (deterministic) switched system in which

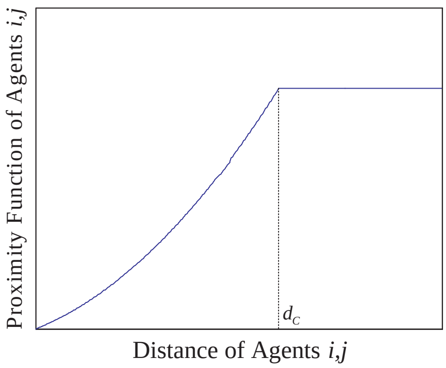

Figure 1.4: The function β_{ij} for $r_i + r_j = 1, d_C = 4$.

switches occur whenever a agent enters or leaves the neighborhood of another. In [8], we have used $\varphi = \sum_{i=1}^{n} \varphi_i$ as a Lyapunov function for the whole system. In this case this function is continuous everywhere, but nonsmooth whenever a switching occurs, i.e. whenever $\|q_i - q_j\| = d_c$ for some i, j. We define the *switching surface* as:

$$S = \{q : \exists i, j, i \neq j | \|q_i - q_j\| = d_c\} \tag{1.14}$$

We have proved that the system converges whenever $q \notin S$. On the switching surface the Lyapunov function is no longer smooth so classic stability theory for smooth systems is no longer adequate.

In [7], we prove the validity of Proposition 2 under the nonsmooth modification of the Proximity Functions. We make use of tools form nonsmooth stability theory ([5],[36]). It is shown than the nonsmooth alternative of the navigation function does not affect the stability and convergence properties of the system.

The prescribed control strategy is another step towards decentralization of the navigation functions' methodology. Although each agent must be aware of the *number* of agents in the entire workspace, it only has to know the *positions* of agents located in its neighborhood. The next step towards global decentralization is to consider the case where each agent is unaware of the global number of agents in the workspace, but only knows what is going on in its neighborhood.

1.3 Proof Sketches

Before proceeding with our proof, we introduce some simplifications concerning terminology. To simplify notation we denote by q instead of q_i the current agent configuration, by q_d instead of q_{di} its goal configuration, by G instead of its "G" function and by q_j the configurations of the other agents. In the proof sketches of Lemmas 1.2-1.6 we use the notation $\frac{\partial}{\partial q_i}(\cdot) \triangleq \nabla(\cdot)$ and $\frac{\partial^2}{\partial q_i^2}(\cdot) \triangleq \nabla^2(\cdot)$

1.3.1 Proof of Lemma 1.2

At steady state, the function f vanishes due to the constraint $X < G_i(q_{d1}, \ldots, q_{dN})$ $\forall i$. Taking the gradient of the definition of φ we have:

$$\nabla\varphi(q_d) = \frac{\left(\gamma_d^k + G\right)^{1/k} \nabla\gamma_d - \gamma_d \nabla\left(\gamma_d^k + G\right)^{1/k}}{\left(\gamma_d^k + G\right)^{2/k}} = 0$$

since both γ_d and $\nabla(\gamma_d)$ vanish by definition at q_d. The Hessian at q_d is

$$\begin{aligned}&\nabla^2\varphi(q_d) = \frac{\left(\gamma_d^k+G\right)^{1/k}\nabla^2\gamma_d - \gamma_d\nabla^2\left(\gamma_d^k+G\right)^{1/k}}{\left(\gamma_d^k+G\right)^{2/k}} = \\ &= G^{-1/k}\cdot\nabla^2(\gamma_d) = 2G^{-1/k}I\end{aligned}$$

which is non-degenerate.$\diamondsuit$

1.3.2 Proof of Lemma 1.3

Let q_0 be a point in ϑF and suppose that $(g_{R_a})_b(q_0) = 0$ for some relation a of level b. If the workspace is valid:$\left(g_{R_j}\right)_l(q_0) > 0$ for any level-l and $j \neq a$ since only one RVF can hold at a time. Using the terminology previously defined, and setting $g_i \equiv (g_{R_a})_b(q_0) = 0$, it follows that $\bar{g}_i > 0$. Taking the gradient of φ at q_0 , we obtain:

$$\begin{aligned}&\nabla\varphi(q_0) = \left.\frac{\left((\gamma_d+f)^k+G\right)^{1/k}\nabla(\gamma_d+f) - (\gamma_d+f)\nabla\left((\gamma_d+f)^k+G\right)^{1/k}}{\left((\gamma_d+f)^k+G\right)^{2/k}}\right|_{q_0} \\ &\overset{G(q_0)=0}{=} \frac{(\gamma_d+f)\nabla(\gamma_d+f) - (\gamma_d+f)\nabla(\gamma_d+f) - \frac{1}{k}(\gamma_d+f)^{2-k}\nabla G}{(\gamma_d+f)^2} = \\ &= -\frac{1}{k}(\gamma_d+f)^{-k}\nabla G = -\frac{1}{k}(\gamma_d+f)^{-k}\bar{g}_i\nabla g_i \neq 0\end{aligned}$$

1.3.3 Proof of Lemma 1.4

At a critical point $q \in C_{\hat{\varphi}} \bigcap F_1(\varepsilon)$ we have:

$$\begin{aligned}
&\hat{\varphi} = \tfrac{\gamma}{G} \Rightarrow \nabla\hat{\varphi} = \tfrac{1}{G^2}\left(G\nabla\gamma - \gamma\nabla G\right) \\
&\overset{\nabla\hat{\varphi}=0}{\Rightarrow} G\nabla\gamma = \gamma\nabla G \Rightarrow G\nabla\left(\gamma_d + f\right)^k = \left(\gamma_d + f\right)^k \nabla G \\
&\Rightarrow kG\nabla\left(\gamma_d + f\right) = \left(\gamma_d + f\right)\nabla G
\end{aligned}$$

Taking the magnitude of both sides yields:

$$kG\left\|\nabla\left(\gamma_d + f\right)\right\| = \left(\gamma_d + f\right)\left\|\nabla G\right\|$$

A sufficient condition for the above equality not to hold is given by:

$$\frac{\left(\gamma_d + f\right)\left\|\nabla G\right\|}{G\left\|\nabla\left(\gamma_d + f\right)\right\|} < k, \forall q \in F_1(\varepsilon)$$

An upper bound for the left side is given by:

$$\begin{aligned}
&\tfrac{(\gamma_d+f)\|\nabla G\|}{G\|\nabla(\gamma_d+f)\|} < \tfrac{(\gamma_d+f)}{\|\nabla(\gamma_d+f)\|} \cdot \sum_{l=1}^{n_L}\sum_{j=1}^{n_{R,l}} \tfrac{\overline{G}_{j,l}}{G}\left\|\nabla\left(g_{R_j}\right)_l\right\| < \\
&< \tfrac{1}{\varepsilon} \cdot \frac{\left(\max\limits_W\{\gamma_d\}+\max\limits_W\{f\}\right)\cdot\sum\limits_{l=1}^{n_L}\sum\limits_{j=1}^{n_{R,l}}\max\limits_W\left\|\nabla\left(g_{R_j}\right)_l\right\|}{\min\limits_W\|\nabla(\gamma_d+f)\|} = \\
&= \tfrac{1}{\varepsilon} \cdot \frac{\left(\max\limits_W\{\gamma_d\}+Y\right)\cdot\sum\limits_{l=1}^{n_L}\sum\limits_{j=1}^{n_{R,l}}\max\limits_W\left\|\nabla\left(g_{R_j}\right)_l\right\|}{\min\limits_W\|\nabla(\gamma_d+f)\|}
\end{aligned}$$

since: $\left(g_{R_j}\right)_l \geq \varepsilon.\diamondsuit$

1.3.4 Proof of Lemma 1.5

If $q \in F_0(\varepsilon) \cap C_{\hat{\varphi}}$, where $C_{\hat{\varphi}}$ is the set of critical points, then $q \in B_i^L(\varepsilon)$ for at least one set $\{L, i\}, i \in \{1...n_{R,L}\}, L \in \{1...n_L\}$, with n_L the number of levels and $n_{R,L}$ the number of relations in level L. We will use a unit vector as a test direction to demonstrate that $\left(\nabla^2\hat{\varphi}\right)(q)$ has at least one negative eigenvalue. At a critical point,

$$\left(\nabla\hat{\varphi}\right)(q) = \frac{kG\left(\gamma_d + f\right)^{k-1}\nabla\left(\gamma_d + f\right) - \left(\gamma_d + f\right)^k\nabla G}{G^2} = 0$$

Hence,

$$k \cdot G \cdot \nabla\left(\gamma_d + f\right) = \left(\gamma_d + f\right) \cdot \nabla G \tag{1.15}$$

The Hessian at a critical point is:

$$\left(\nabla^2\hat{\varphi}\right)(q) = \frac{1}{G^2}\left(G \cdot \nabla^2\left(\gamma_d + f\right)^k - \left(\gamma_d + f\right)^k \cdot \nabla^2 G\right)$$

and expanding

$$\begin{array}{l}\left(\nabla^2\hat{\varphi}\right)(q) = \frac{(\gamma_d+f)^{k-2}}{G^2}\cdot \\ \cdot\left\{kG\left[\begin{array}{l}(\gamma_d+f)\,\nabla^2(\gamma_d+f)+ \\ (k-1)\,\nabla(\gamma_d+f)\,\nabla(\gamma_d+f)^T\end{array}\right]-(\gamma_d+f)^2\,\nabla^2 G\right\}\end{array}$$

Taking the outer product of both sides of equation (A.1), we get:

$$(kG)^2\,\nabla(\gamma_d+f)\,\nabla(\gamma_d+f)^T = (\gamma_d+f)^2\,\nabla G\nabla G^T \tag{1.16}$$

Substituting equation (A.2) in equation (A.1), we get:

$$\left(\nabla^2\hat{\varphi}\right)(q) = \frac{(\gamma_d+f)^{k-1}}{G^2}\left\{\begin{array}{l}kG\nabla^2(\gamma_d+f)+ \\ +\left(1-\frac{1}{k}\right)\frac{(\gamma_d+f)}{G}\nabla G\nabla G^T- \\ -(\gamma_d+f)\,\nabla^2 G\end{array}\right\}$$

We choose the test vector (unit magnitude) to be:$\hat{u} = \frac{\nabla b_i(q_c)^{\perp}}{\left\|\nabla b_i(q_c)^{\perp}\right\|}$. By its definition $\hat{u}$ is orthogonal to ∇b_i at a critical point q_c, and so the following properties hold:$\hat{u}^T\cdot\nabla b_i = 0$ and $\nabla b_i^T\cdot\hat{u} = 0$. With $\nabla^2(\gamma_d+f) = 2\cdot\mathrm{I}+\nabla^2 f$, we form the quadratic form:

$$\begin{array}{l}\frac{G^2}{(\gamma_d+f)^{k-1}}\hat{u}^T\left(\nabla^2\hat{\varphi}\right)(q)\,\hat{u} = 2kG + kG\hat{u}^T\nabla^2 f\hat{u} \\ +\left(1-\frac{1}{k}\right)\frac{(\gamma_d+f)}{G}\hat{u}^T\nabla G\nabla G^T\hat{u} - (\gamma_d+f)\,\hat{u}^T\nabla^2 G\hat{u}\end{array}$$

After many nontrivial calculation we get

$$\begin{array}{l}\frac{G^2}{(\gamma_d+f)^{k-1}}\hat{u}^T\left(\nabla^2\hat{\varphi}\right)(q)\,\hat{u} = \\ \bar{g}_i c_i\left(1+\frac{a_0}{\gamma_d}\right)\left(\frac{1}{2}\nabla b_i^T\nabla\gamma_d - \upsilon_i\gamma_d\right) \\ +g_i\left\{\begin{array}{l}k\bar{g}_i\hat{u}^T\nabla^2 f\hat{u} + (\gamma_d+f)\,\eta_i - (\gamma_d+f)\,\psi_i + \frac{z_2}{2\gamma_d} \\ -\upsilon_i\bar{g}_i c_i\left(\sum\limits_{j=1}^{3} a_j g_i^{j-1}\bar{g}_i^j\right) - \zeta_i\end{array}\right\}\end{array} \tag{1.17}$$

where $c_i = 1+\frac{\lambda}{b_i+\tilde{b}_i^{1/h}}$,$\upsilon_i = 2\cdot l$,$l$ the relation level,

$$\eta_i = \left(1-\frac{1}{k}\right)\left[\begin{array}{l}\frac{\hat{u}^T\nabla\bar{g}_i\nabla\bar{g}_i^T\hat{u}}{\bar{g}_i} - 2\lambda\frac{\hat{u}^T\nabla\bar{g}_i\left(\nabla\tilde{b}_i^{1/h}\right)^T\hat{u}}{c_i\left(b_i+\tilde{b}_i^{1/h}\right)^2}+ \\ +\lambda^2\bar{g}_i\frac{\hat{u}^T\nabla\tilde{b}_i^{1/h}\left(\nabla\tilde{b}_i^{1/h}\right)^T\hat{u}}{c_i^2\left(b_i+\tilde{b}_i^{1/h}\right)^4}\end{array}\right]$$

$$\begin{array}{l}\psi_i = \hat{u}^T\cdot\nabla^2\bar{g}_i\cdot\hat{u} + \frac{\bar{g}_i}{c_i}\cdot\hat{u}^T\cdot B_i\cdot\hat{u}- \\ -2\frac{\lambda}{c_i\left(b_i+\tilde{b}_i^{1/h}\right)^2}\cdot\hat{u}^T\cdot\nabla\tilde{b}_i^{1/h}\cdot\nabla\bar{g}_i\cdot\hat{u}\end{array}$$

$$B_i = \lambda\left[\begin{array}{l}2\frac{\left(\nabla b_i+\nabla\tilde{b}_i^{1/h}\right)\left(\nabla b_i+\nabla\tilde{b}_i^{1/h}\right)^T}{\left(b_i+\tilde{b}_i^{1/h}\right)^3}- \\ -\frac{\left(\nabla^2 b_i+\nabla^2\tilde{b}_i^{1/h}\right)}{\left(b_i+\tilde{b}_i^{1/h}\right)^2}\end{array}\right]$$

$$z_2\left(g_i,\bar{g}_i,\nabla g_i,\nabla\bar{g}_i\right)=\gamma_d\nabla\bar{g}_i^T\nabla\gamma_d+f\nabla\bar{g}_i^T\nabla\gamma_d+\dots$$
$$-k\bar{g}_i\left(2\nabla\gamma_d^T\cdot\nabla f-\nabla f^T\cdot\nabla f\right)$$

$$\zeta_i=\frac{\lambda\bar{g}_i}{2c_i\left(b+\tilde{b}^{1/h}\right)^2}\left(\nabla b+\nabla\tilde{b}^{1/h}\right)^T\cdot\nabla\gamma_d$$

Setting:

$$\tilde{\mu}_i=\left(1+\frac{a_0}{\gamma_d}\right)\cdot\mu_i$$

where:

$$\mu_i=\frac{1}{2}\nabla b_i^T\nabla\gamma_d-\upsilon_i\cdot\gamma_d$$

equation (A.3) becomes:

$$\frac{G^2}{(\gamma_d+f)^{k-1}}\hat{u}^T\left(\nabla^2\hat{\varphi}\right)(q)\,\hat{u}=\bar{g}_i\cdot c_i\cdot\tilde{\mu}_i+ \\ g_i\left\{\begin{array}{l} k\cdot\bar{g}_i\cdot\hat{u}^T\cdot\nabla^2 f\cdot\hat{u}+(\gamma_d+f)\cdot\eta_i-(\gamma_d+f)\cdot\psi_i \\ +\frac{z_2}{2\gamma_d}-\upsilon_i\bar{g}_i c_i\left(\sum\limits_{j=1}^{3}a_j g_i^{j-1}\bar{g}_i^j\right)-\zeta_i\end{array}\right\} \tag{1.18}$$

The second term is proportional to g_i and can be made arbitrarily small by a suitable choice of ε but can still be positive, so the first term should be strictly negative.

From the result of Lemma 7 in [8], we have:

$$\max_{q\in F_0}\{\mu_i\}= \\ =\frac{2}{l}\left(\begin{array}{l}\frac{1}{l}\sqrt{\left\|\sum q_j\right\|^2-l\sum\left\|q_j\right\|^2+l\left(\sum(r+r_j)^2+\varepsilon\right)} \\ -\left\|lq_d-\sum q_j\right\|\end{array}\right)\cdot \\ \cdot\left\|lq_d-\sum q_j\right\|$$

For ε small enough, $\max\limits_{q\in F_0}\{\mu_i\}$ is negative. Moreover, the term $\left(1+\frac{a_0}{\gamma_d}\right)$ is always greater than one, since we have assumed that $a_0>0$, and $\gamma_d>0$ for $q\in F_0(\varepsilon)$. Thus for ε small enough, $\tilde{\mu}_i$ is also negative. So, for $\tilde{\mu}_i$, according to Lemma 1, it is sufficient to make sure that:

$$\frac{1}{l}\cdot\sqrt{\left\|\sum q_j\right\|^2-l\cdot\sum\left\|q_j\right\|^2+l\cdot\left(\sum(r+r_j)^2+\varepsilon\right)}< \\ <\left\|l\cdot q_d-\sum q_j\right\|\Rightarrow\varepsilon<l\cdot\left\|l\cdot q_d-\sum q_j\right\|^2+\sum\left\|q_j\right\|^2- \\ -\frac{1}{l}\cdot\left\|\sum q_j\right\|^2-\sum(r+r_j)^2\equiv\varepsilon_0$$

An other constraint arises from the fact that $\varepsilon>0$. . So for a valid workspace it will be:

$$l\cdot\left\|l\cdot q_d-\sum q_j\right\|^2+\sum\left\|q_j\right\|^2-\frac{1}{l}\cdot\left\|\sum q_j\right\|^2>\sum(r+r_j)^2$$

$\diamondsuit$

1.3.5 Proof of Lemma 1.6

From the proof of the previous Lemma, we have at a critical point

$$\frac{G^2}{(\gamma_d+f)^{k-1}}\left(\nabla^2\hat{\varphi}\right) = kG\nabla^2\left(\gamma_d+f\right) + \\ \left(1-\frac{1}{k}\right)\frac{\gamma_d+f}{G}\nabla G\nabla G^T - \left(\gamma_d+f\right)\nabla^2 G$$

We also have

$$\nabla f = \left(\underbrace{\sum_{j=1}^{3} j a_j G_i^{j-1}}_{\sigma(G)}\right)\nabla G$$

and

$$\nabla^2 f = \sigma\nabla^2 G + \sigma^*\nabla G\nabla G^T, \sigma^* = \sum_{j=2}^{3} j(j-1)a_j G^{j-2}$$

At a critical point:

$$\begin{aligned} & kG\nabla\left(\gamma_d+f\right) = \left(\gamma_d+f\right)\nabla G \Rightarrow \\ & kG\nabla\gamma_d = \left(\gamma_d+f\right)\nabla G - kG\nabla f \Rightarrow \\ & kG\nabla\gamma_d = \left(\gamma_d+f-kG\sigma(G)\right)\nabla G \Rightarrow \\ & G\nabla\gamma_d = \left\{\underbrace{\frac{\gamma_d+f}{k} - G\sigma(G)}_{-\sigma_i}\right\}\nabla G \end{aligned}$$

Taking the magnitude from both sides we have

$$2kG = \frac{k\left|\sigma_i\right|^2}{2G\gamma_d}\left\|\nabla G\right\|^2$$

Choosing $\tilde{u} = \widehat{\nabla b_i}$ as a test direction and after some manipulation we have

$$\frac{G^2}{k(\gamma_d+f)^{k-1}}\tilde{u}^T\left(\nabla^2\hat{\varphi}\right)\tilde{u} = \underbrace{\frac{\left|\sigma_i\right|^2}{2G\gamma_d}\left\|\nabla G\right\|^2}_{L} + \\ \underbrace{\xi\tilde{u}^T\nabla G\nabla G^T\tilde{u}}_{M} + \underbrace{\sigma_i\tilde{u}^T\nabla^2 G\tilde{u}}_{N}$$

where

$$\xi = \left(1-\frac{1}{k}\right)\frac{\gamma_d+Y}{kG} + \sum_{j=2}^{3}\left\{kj(j-1) + \left(1-\frac{1}{k}\right)\right\}\frac{a_j}{k}G^{j-1}$$

After some manipulation, we have

$$
\begin{aligned}
&L+M+N \geq \tfrac{|\sigma_i|^2}{2G\gamma_d}\left\{\begin{array}{l} g_i^2\left\|\nabla \bar{g}_i\right\|^2+\bar{g}_i^2\left\|\nabla g_i\right\|^2- \\ 2G\left\|\nabla \bar{g}_i\right\|\left\|\nabla g_i-2\left(\tilde{u}^T \nabla g_i\right) \tilde{u}\right\| \end{array}\right\} \\
&+2\left(\tfrac{|\sigma_i|^2}{\gamma_d}+\xi G+\sigma_i\right)\left(\tilde{u}^T \nabla g_i\right)\left(\nabla \bar{g}_i \tilde{u}\right) \\
&+\xi \bar{g}_i^2\left(\tilde{u}^T \nabla g_i\right)^2+\sigma_i \tilde{u}^T\left(g_i \nabla^2 \bar{g}_i+\bar{g}_i \nabla^2 g_i\right) u
\end{aligned}
$$

But $\left\|\nabla g_i-2\left(\tilde{u}^T \nabla g_i\right) \tilde{u}\right\|^2=\left\|\nabla g_i\right\|^2$ so that

$$
\begin{aligned}
&g_i^2\left\|\nabla \bar{g}_i\right\|^2+\bar{g}_i^2\left\|\nabla g_i\right\|^2- \\
&2G\left\|\nabla \bar{g}_i\right\|\left\|\nabla g_i-2\left(\tilde{u}^T \nabla g_i\right) \tilde{u}\right\|=\left(g_i\left\|\nabla \bar{g}_i\right\|-\bar{g}_i\left\|\nabla g_i\right\|\right)^2
\end{aligned}
$$

so that

$$
\begin{aligned}
&L+M+N \geq 2\left(\tfrac{|\sigma_i|^2}{\gamma_d}+\xi G+\sigma_i\right)\left(\tilde{u}^T \nabla g_i\right)\left(\nabla \bar{g}_i \tilde{u}\right) \\
&+\xi \bar{g}_i^2\left(\tilde{u}^T \nabla g_i\right)^2+\sigma_i \tilde{u}^T\left(g_i \nabla^2 \bar{g}_i+\bar{g}_i \nabla^2 g_i\right) u
\end{aligned}
$$

It is shown in [8] that the second term, which is strictly positive, dominates the third and the first term for sufficiently small ε.

1.3.6 Proof of Proposition 1.7

In the proof sketch of Proposition 1.7, the terms $\nabla(\cdot)$, $\nabla^2(\cdot)$ have their usual meaning and refer to the whole state space and not a single agent, namely $\nabla(\cdot) \triangleq \left[\frac{\partial}{\partial q_1}(\cdot), \ldots, \frac{\partial}{\partial q_N}(\cdot)\right]^T$ and $\nabla^2(\cdot) \triangleq \left[\frac{\partial^2}{\partial q_{ij}}(\cdot)\right]$.

We immediately note that the following proof is existential rather than computational. We show that a finite k that renders the system almost everywhere asymptotically stable *exists*, but we do not provide an analytical expression for this lower bound. However, practical values of k have been provided in the simulation section.
Let us recall that the Proximity function between agents i and j is given by:

$$
\beta_{ij}(q)=\left\|q_i-q_j\right\|^2-\left(r_i+r_j\right)^2=q^T D_{ij} q-\left(r_i+r_j\right)^2
$$

where the $2N \times 2N$ matrix D_{ij} is defined in [26]:

$$
D_{ij}= \left[\begin{array}{ccccc}
 & & O_{2(i-1)\times 2N} & & \\
O_{2\times 2(i-1)} & I_{2\times 2} & O_{2\times 2(j-i-1)} & -I_{2\times 2} & O_{2\times 2(N-j)} \\
 & & O_{2(j-i-1)\times 2N} & & \\
O_{2\times 2(i-1)} & -I_{2\times 2} & O_{2\times 2(j-i-1)} & I_{2\times 2} & O_{2\times 2(N-j)} \\
 & & O_{2(N-j)\times 2N} & &
\end{array}\right]
$$

We can also write $b_r^i=q^T P_r^i q-\sum\limits_{j\in P_r}\left(r_i+r_j\right)^2$,where $P_r^i=\sum\limits_{j\in P_r} D_{ij}$, and P_r denotes the set of binary relations in relation r. It can easily be seen that

$\nabla b_r^i = 2P_r^i q, \nabla^2 b_r^i = 2P_r^i$. We also use the following notation for the r-th relation wrt agent i:

$$g_r^i = b_r^i + \frac{\lambda b_r^i}{b_r^i + \left(\tilde{b}_r^i\right)^{1/h}}, \tilde{b}_r^i = \prod_{\substack{s \in S_r \\ s \neq r}} b_s^i,$$
$$\nabla \tilde{b}_r^i = \sum_{\substack{s \in S_r \\ s \neq r}} \underbrace{\prod_{\substack{t \in S_r \\ t \neq s,r}} b_i^t}_{\tilde{b}_{s,r}^i} \cdot 2P_s^i q$$

where S_r denotes the set of relations in the same level with relation r. An easy calculation shows that

$$\nabla g_r^i = \ldots = 2\left[d_r^i P_r^i - w_r^i \tilde{P}_r^i\right] q \triangleq Q_r^i q, \tilde{P}_r^i \triangleq \sum_{\substack{s \in S_r \\ s \neq r}} \tilde{b}_{s,r}^i P_s^i$$

where $d_r^i = 1 + (1 - \frac{b_r^i}{b_r^i + (\tilde{b_r^i})^{1/h}}) \frac{\lambda}{b_r^i + (\tilde{b_r^i})^{1/h}}$, $w_r^i = \frac{\lambda b_r^i (\tilde{b_r^i})^{\frac{1}{h}-1}}{h(b_r^i + (\tilde{b_r^i})^{1/h})^2}$. The gradient of the G_i function is given by:

$$G_i = \prod_{r=1}^{N_i} g_r^i \Rightarrow \nabla G_i = \sum_{r=1}^{N_i} \underbrace{\prod_{\substack{l=1 \\ l \neq r}}^{N_i} g_l^i}_{\tilde{g}_r^i} \nabla g_r^i = \sum_{r=1}^{N_i} \tilde{g}_r^i Q_r^i q \triangleq Q_i q$$

We define $\nabla G \triangleq \begin{bmatrix} \nabla G_1 \\ \vdots \\ \nabla G_N \end{bmatrix} = \begin{bmatrix} Q_1 \\ \vdots \\ Q_N \end{bmatrix} q \triangleq Qq$

Remembering that $u_i = -K_i \frac{\partial \varphi_i}{\partial q_i}$ and that $\varphi_i = \frac{\gamma_{di} + f_i}{\left((\gamma_{di} + f_i)^k + G_i\right)^{1/k}}, f_i = \sum_{j=0}^{3} a_i G_i^j$ the closed loop dynamics of the system are given by:

$$\dot{q} = \begin{bmatrix} -K_1 A_1^{-(1+1/k)} \left\{ G_1 \frac{\partial \gamma_{d1}}{\partial q_1} + \sigma_1 \frac{\partial G_1}{\partial q_1} \right\} \\ \vdots \\ -K_N A_N^{-(1+1/k)} \left\{ G_N \frac{\partial \gamma_{dN}}{\partial q_N} + \sigma_N \frac{\partial G_N}{\partial q_N} \right\} \end{bmatrix} = \ldots$$
$$= -A_K G \left(\partial \gamma_d\right) - A_K \Sigma Q q$$

where $(\partial \gamma_d) = \left[\frac{\partial \gamma_{d1}}{\partial q_1} \ldots \frac{\partial \gamma_{dN}}{\partial q_N}\right]^T$, $\sigma_i = G_i \sigma(G_i) - \frac{\gamma_{di} + f_i}{k}$, $\sigma(G_i) = \sum_{j=1}^{3} j a_j G_i^{j-1}$, $A_i = (\gamma_{di} + f_i)^k + G_i$ and the matrices

$$G \triangleq \underbrace{diag\left(G_1, G_1, \ldots, G_N, G_N\right)}_{2N \times 2N}$$

$$A_K \triangleq \underbrace{diag\left(\begin{array}{l} K_1 A_1^{-(1+1/k)}, K_1 A_1^{-(1+1/k)}, \ldots, \\ K_N A_N^{-(1+1/k)}, K_N A_N^{-(1+1/k)} \end{array}\right)}_{2N\times 2N}$$

$$\Sigma \triangleq \underbrace{\left[\begin{array}{ccc} \underbrace{\Sigma_1}_{2N\times 2N}, & \ldots, & \underbrace{\Sigma_N}_{2N\times 2N} \end{array}\right]}_{2N\times 2N^2},$$

$$\Sigma_i = diag\left(0, 0, \ldots, \underbrace{\sigma_i, \sigma_i}_{2i-1, 2i}, \ldots, 0, 0\right)$$

By using $\varphi = \sum\limits_i \varphi_i$ as a candidate Lyapunov function we have

$$\begin{array}{l} \varphi = \sum\limits_i \varphi_i \Rightarrow \dot{\varphi} = \left\{\sum\limits_i (\nabla\varphi_i)^T\right\}\dot{q}, \\ \nabla\varphi_i = A_i^{-(1+1/k)}\left\{G_i \nabla\gamma_{di} + \sigma_i \nabla G_i\right\} \end{array}$$

and after some trivial calculation

$$\sum_i (\nabla\varphi_i)^T = \ldots = (\partial\gamma_d)^T A_G + q^T Q^T A_\Sigma$$

where

$$A_G = \underbrace{diag\left(\begin{array}{l} G_1 A_1^{-(1+1/k)}, G_1 A_1^{-(1+1/k)}, \ldots, \\ G_N A_N^{-(1+1/k)}, G_N A_N^{-(1+1/k)} \end{array}\right)}_{2N\times 2N}$$

$$A_\Sigma = \underbrace{\left[\begin{array}{c} \underbrace{A_{\Sigma_1}}_{2N\times 2N} \\ \vdots \\ \underbrace{A_{\Sigma_N}}_{2N\times 2N} \end{array}\right]}_{2N^2\times 2N}, A_{\Sigma_i} = \underbrace{diag\left(\begin{array}{l} A_i^{-(1+1/k)}\sigma_i, \ldots, \\ A_i^{-(1+1/k)}\sigma_i \end{array}\right)}_{2N\times 2N}$$

So we have

$$\begin{array}{l} \dot{\varphi} = \left\{\sum\limits_i (\nabla\varphi_i)^T\right\}\dot{q} = \ldots = \\ = -\left[\begin{array}{cc} (\partial\gamma_d)^T & q^T \end{array}\right] \underbrace{\left[\begin{array}{cc} M_1 & M_2 \\ M_3 & M_4 \end{array}\right]}_{M} \left[\begin{array}{c} \partial\gamma_d \\ q \end{array}\right] \end{array}$$

where $M_1 = A_G A_K G$, $M_2 = A_G A_K \Sigma Q$, $M_3 = Q^T A_\Sigma A_K G$, $M_4 = Q^T A_\Sigma A_K \Sigma Q$.

In [8], we provide an analytic expression for the elements of the matrix Q.

We examine the positive definiteness of the matrix M by use of the following theorems:

Theorem 1.9 [19]*: Given a matrix $A \in \Re^{n\times n}$ then all its eigenvalues lie in the union of n discs:*

$$\bigcup_{i=1}^{n}\left\{z : |z-a_{ii}| \le \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}|\right\} \triangleq \bigcup_{i=1}^{n} R_i(A) \triangleq R(A)$$

Each of these discs is called a Gersgorin disc of A.

Corollary 1.10 [19]*: Given a matrix $A \in \Re^{n\times n}$ and n positive real numbers $p_1,\ldots,p_n$ then all its eigenvalues of A lie in the union of n discs:*

$$\bigcup_{i=1}^{n}\left\{z : |z-a_{ii}| \le \frac{1}{p_i}\sum_{\substack{j=1\\ j\neq i}}^{n} p_j\,|a_{ij}|\right\}$$

A key point of Corollary 1.10 is that if we bound the first $n/2$ Gersgorin discs of a matrix A sufficiently away from zero, then an appropriate choice of the numbers $p_1,\ldots,p_n$ renders the remaining $n/2$ discs sufficiently close to the corresponding diagonal elements. Hence, by ensuring the positive definiteness of the eigenvalues of the matrix M corresponding to the first $n/2$ rows, then we can render the remaining ones sufficiently close to the corresponding diagonal elements. This fact will be made clearer in the analysis that follows.

Some useful bounds are obtained by the following lemma:

Lemma 1.11 *: The following bounds hold for the terms $Q^i_{ii}, Q^j_{ii}, \sigma_i$*

$$\sigma_i(\varepsilon) \in \begin{cases} \left[-Y\left(\frac{1}{k}+\frac{8}{9}\right)-\frac{\gamma_{di}}{k},\ \underbrace{-\frac{Y}{k}-\frac{\gamma_{di}}{k}}_{\sigma_i(0)}\right], 0 \le \varepsilon \le \varepsilon^* \\ \left[-Y\left(\frac{1}{k}+\frac{8}{9}\right)-\frac{\gamma_{di}}{k},\ \underbrace{-\frac{\gamma_{di}}{k}}_{\sigma_i(X)}\right], X \ge \varepsilon \ge \varepsilon^* \end{cases}$$

$$0 < Q^i_{ii} < \left|Q^i_{ii}\right|_{\max} < \infty$$

and

$$0 < Q^j_{ii} < \left|Q^j_{ii}\right|_{\max} < \infty$$

Proof: See [8].

Let us examine the Gersgorin discs of the first half rows of the matrix M. We denote this procedure as $M_1 - M_2$, as the main diagonal elements of M_1 are "compared" with the corresponding raw elements of M_2. Note that the submatrices M_1, M_2 are both diagonal, therefore the only nonzero elements of raw i of the $4N \times 4N$ matrix M are the elements $M_{ii}, M_{i,2N+i}$ where of course $1 \leq i \leq 2N$ as we calculate the Gersgorin discs of the first half rows of the matrix M. We have:

$$\begin{array}{l} |z - M_{ii}| \leq \frac{1}{p_i} \sum\limits_{j \neq i} p_j \, |M_{ij}|, 1 \leq i \leq 2N \Rightarrow \\ \left| z - A_i^{-2(1+1/k)} K_i G_i^2 \right| \leq \frac{p_{2N+i}}{p_i} \left| A_i^{-2(1+1/k)} \sigma_i K_i G_i Q_{ii}^i \right| \Rightarrow \\ \Rightarrow z \geq A_i^{-2(1+1/k)} K_i G_i^2 - \frac{p_{2N+i}}{p_i} \left| A_i^{-2(1+1/k)} \sigma_i K_i G_i Q_{ii}^i \right| \end{array}$$

We examine the following three cases:

- $G_i < \varepsilon$ At a critical point in this region, the corresponding eigenvalue tends to zero, so that the derivative of the Lyapunov function could achieve zero values. However, the result of Lemma 1.6 indicates that φ_i is a Morse function, hence its critical points are isolated [23]. Thus the set of initial conditions that lead to saddle points are sets of measure zero[31].

- $G_i > X$ The corresponding eigenvalue is guaranteed to be positive as long as:
$$\begin{array}{l} z > 0 \Leftarrow A_i^{-2(1+1/k)} K_i \left(G_i - \frac{p_{2N+i}}{p_i} \left| \sigma_i Q_{ii}^i \right| \right) > 0 \Leftarrow \\ G_i \geq X > \frac{p_{2N+i}}{p_i} \left| \sigma_i Q_{ii}^i \right| = \frac{\gamma_{di}}{k} \frac{p_{2N+i}}{p_i} \left| Q_{ii}^i \right| \Leftarrow \\ \Leftarrow k > \frac{(\gamma_{di})_{\max}}{X} \frac{p_{2N+i}}{p_i} \left| Q_{ii}^i \right|_{\max} \end{array}$$

- $0 < \varepsilon \leq G_i \leq X$
$$\begin{array}{l} z > 0 \Leftarrow \varepsilon > \left\{ Y \left| \frac{1}{k} + \frac{8}{9} \right| + \left| \frac{\gamma_{di}}{k} \right| \right\} \frac{p_{2N+i}}{p_i} \left| Q_{ii}^i \right|_{\max} \Leftarrow \\ \Leftarrow \varepsilon > 2 \max \left\{ \begin{array}{c} 2 \max \left\{ \frac{Y}{k}, \frac{8Y}{9} \right\}, \\ \left| \frac{(\gamma_{di})_{\max}}{k} \right| \end{array} \right\} \frac{p_{2N+i}}{p_i} \left| Q_{ii}^i \right|_{\max} \\ \overset{Y \leq \frac{\Theta_1}{k}}{\Longleftarrow} k > 2 \max \left\{ \begin{array}{c} 2 \sqrt{\frac{\Theta_1}{\varepsilon}}, \frac{16 \Theta_1}{9 \varepsilon}, \\ \frac{(\gamma_{di})_{\max}}{\varepsilon} \end{array} \right\} \frac{p_{2N+i}}{p_i} \left| Q_{ii}^i \right|_{\max} \end{array}$$

A key point is that there is no restriction on how to select the terms $\frac{p_{2N+i}}{p_i}$. This will help us in deriving bounds that guarantee the positive definiteness of the matrix M.

Let us examine the Gersgorin discs of the second half rows of the matrix M. Likewise, we denote this procedure as $M_3 - M_4$. The discs of Corollary 1.10 are evaluated:

$$\begin{array}{l} |z - M_{ii}| \leq \sum\limits_{j \neq i} \frac{p_j}{p_i} \, |M_{ij}|, 2N + 1 \leq i \leq 4N, 1 \leq j \leq 4N \Rightarrow \\ \Rightarrow |z - (M_4)_{ii}| \leq R_i(M_3) + R_i(M_4) \end{array}$$

where

$$(M_4)_{ii} = \sum_j K_i A_i^{-(1+1/k)} A_j^{-(1+1/k)} \sigma_j \sigma_i Q_{ii}^i Q_{ii}^j$$

and

$$R_i(M_3) = \sum_{j=1}^{2N} \frac{p_j}{p_i} \left| (M_3)_{ij} \right| =$$
$$= \sum_{j=1}^{2N} \frac{p_j}{p_i} \left| \sum_l A_l^{-(1+1/k)} \sigma_l A_j^{-(1+1/k)} K_j G_j Q_{ij}^l \right|$$

$$R_i(M_4) = \sum_{\substack{j=2N+1 \\ j \neq i}}^{4N} \frac{p_j}{p_i} \left| (M_4)_{ij} \right| =$$
$$= \sum_{j \neq i} \frac{p_j}{p_i} \left| \sum_l \left(A_l A_j\right)^{-(1+1/k)} \sigma_l \sigma_j K_j Q_{ij}^l Q_{jj}^j \right|$$

A sufficient condition for the positive definiteness of the corresponding eigenvalue for raw i is then:

$$(M_4)_{ii} > R_i(M_3) + R_i(M_4) \Leftarrow$$
$$\Leftarrow (M_4)_{ii} > \max\left\{2R_i(M_3), 2R_i(M_4)\right\}$$

We first show that we always have $R_i(M_3) \geq R_i(M_4)$. By taking into account the relations $Q_{jk}^i = Q_{kj}^i = 0, Q_{ij}^i = -Q_{jj}^i, j \neq i \neq k \neq j$ and expanding it is easy to see that

$$R_i(M_3) = -\frac{1}{p_i} \sum_{j=1}^{2N} p_j \left\{ \begin{array}{l} A_j^{-2(1+1/k)} \sigma_j K_j G_j Q_{ii}^j + \\ \left(A_j A_i\right)^{-(1+1/k)} \sigma_i K_j G_j Q_{jj}^i \end{array} \right\} =$$
$$= -\sum_{\substack{j=1 \\ j \neq i}}^{2N} \frac{p_j}{p} \left\{ \begin{array}{l} \underbrace{A_j^{-2(1+1/k)} \sigma_j K_j G_j Q_{ii}^j}_{(I)} + \\ \underbrace{\left(A_j A_i\right)^{-(1+1/k)} \sigma_i K_j G_j Q_{jj}^i}_{(II)} \end{array} \right\}$$
$$-2\frac{p_i}{p} A_i^{-2(1+1/k)} \sigma_i K_i G_i Q_{ii}^i$$

where without loss of generality we choose $p_i = p, 2N+1 \leq i \leq 4N$.We also have

$$R_i(M_4) = \sum_{j \neq i} \left\{ \begin{array}{l} \underbrace{A_j^{-2(1+1/k)} \sigma_j^2 K_j Q_{ii}^j Q_{jj}^j}_{(I)} + \\ \underbrace{\left(A_i A_j\right)^{-(1+1/k)} \sigma_i \sigma_j K_j Q_{jj}^i Q_{jj}^j}_{(II)} \end{array} \right\}$$

By comparing the terms (I) and (II) in the last two equations we have:

$$
\begin{aligned}
&(I): -\tfrac{p_j}{p} A_j^{-2(1+1/k)} \sigma_j K_j G_j Q_{ii}^j \geq A_j^{-2(1+1/k)} \sigma_j^2 K_j Q_{ii}^j Q_{jj}^j \\
&\Leftarrow -\tfrac{p_j}{p} \sigma_j G_j \geq \sigma_j^2 Q_{jj}^j \Leftrightarrow \sigma_j \left(\sigma_j Q_{jj}^j + \tfrac{p_j}{p} G_j\right) \leq 0 \\
&\overset{\sigma_j \leq 0}{\Leftrightarrow} \sigma_j Q_{jj}^j + \tfrac{p_j}{p} G_j \geq 0 \\
&(II): -\tfrac{p_j}{p} \left(A_j A_i\right)^{-(1+1/k)} \sigma_i K_j G_j Q_{jj}^i \geq \\
&\geq \left(A_i A_j\right)^{-(1+1/k)} \sigma_i \sigma_j K_j Q_{jj}^i Q_{jj}^j \\
&\Leftarrow -\tfrac{p_j}{p} \sigma_i G_j \geq \sigma_i \sigma_j Q_{jj}^j \Leftrightarrow \sigma_i \left(\sigma_j Q_{jj}^j + \tfrac{p_j}{p} G_j\right) \leq 0 \\
&\overset{\sigma_i \leq 0}{\Leftrightarrow} \sigma_j Q_{jj}^j + \tfrac{p_j}{p} G_j \geq 0
\end{aligned}
$$

Thus, the condition $\sigma_j Q_{jj}^j + \frac{p_j}{p} G_j \geq 0$ guarantees that $R_i(M_3) \geq R_i(M_4) \forall i$. Hence it suffices to show that $(M_4)_{ii} > 2R_i(M_3)$. The fact that $\sigma_j Q_{jj}^j + \frac{p_j}{p} G_j \geq 0$ is a direct conclusion of the results of procedure $M_1 - M_2$. For example, by the last bound on k we have:

$$
\begin{aligned}
&k > 2\max\left\{2\sqrt{\tfrac{\Theta_1}{\varepsilon}}, \tfrac{16\Theta_1}{9\varepsilon}, \tfrac{(\gamma_{dj})_{\max}}{\varepsilon}\right\} \tfrac{p}{p_j} \left|Q_{jj}^j\right|_{\max} \\
&\underset{G_j \geq \varepsilon}{\overset{Y \leq \frac{\Theta_1}{k}}{\Rightarrow}} G_j > 2\max\left\{2\max\left\{\tfrac{Y}{k}, \tfrac{8Y}{9}\right\}, \left|\tfrac{(\gamma_{dj})_{\max}}{k}\right|\right\} \tfrac{p}{p_j} \left|Q_{jj}^j\right|_{\max} \\
&\Rightarrow G_j > \left\{Y\left|\tfrac{1}{k} + \tfrac{8}{9}\right| + \left|\tfrac{\gamma_{dj}}{k}\right|\right\} \tfrac{p}{p_j} \left|Q_{jj}^j\right|_{\max} \\
&\Rightarrow \tfrac{p_j}{p} G_j > \left|\sigma_j\right|_{\max} \left|Q_{jj}^j\right|_{\max} \Rightarrow \sigma_j Q_{jj}^j + \tfrac{p_j}{p} G_j > 0
\end{aligned}
$$

The fact that $(M_4)_{ii} > 0$ is guaranteed by Lemma 5.4. This lemma also guarantees that there is always a finite upper bound on the terms

$$
\left|(M_3)_{ij}\right| = \left|\sum_l A_l^{-(1+1/k)} \sigma_l A_j^{-(1+1/k)} K_j G_j Q_{ij}^l\right|
$$

We have

$$
\begin{aligned}
&(M_4)_{ii} > 2R_i(M_3) = 2\sum_{j=1}^{2N} \tfrac{p_j}{p} \left|(M_3)_{ij}\right| \Leftarrow \\
&p > \tfrac{4N}{(M_4)_{ii}} \max_j \left\{p_j \left|(M_3)_{ij}\right|\right\}, \\
&2N+1 \leq i \leq 4N, 1 \leq j \leq 2N
\end{aligned}
$$

♢

1.4 Simulations

To demonstrate the navigation properties of our decentralized approach, we present two simulations of multiple holonomic agents that have to navigate from an initial to a final configuration, avoiding collision with each other and satisfying velocity bounds. The chosen configurations constitute non-trivial setups since the straight-line paths connecting initial and final positions of each agent are obstructed by other agents.

The first simulation involves 8 holonomic agents with global sensing and the second four agents with local sensing capabilities.

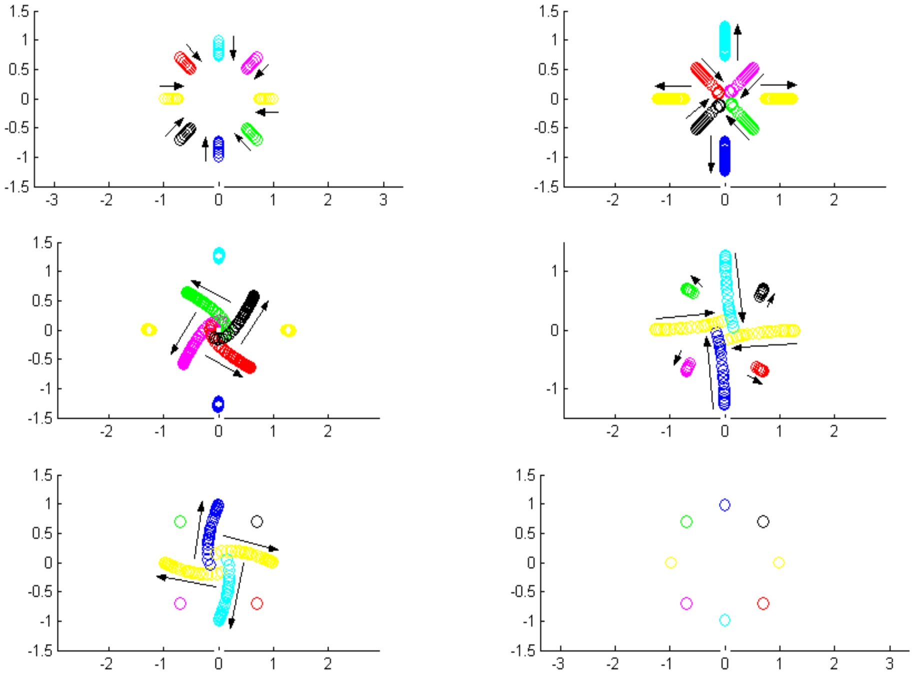

Figure 1.5: 8 holonomic agents

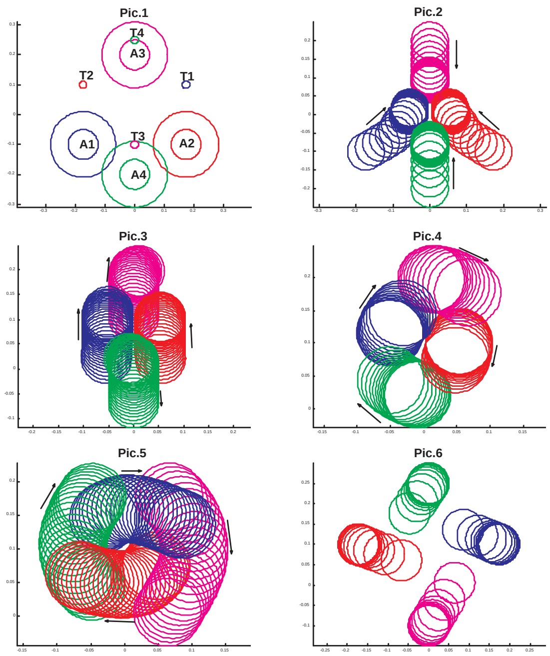

Figure 1.6: 4 holomonic agents with limited sensing capabilities

Chapter 2

Global Decentralized Conflict Resolution Part 2: Nonholonomic Kinematics

In this chapter, we review the decentralized conflict resolution algorithm developed under WP6 for the case when the dynamics of each aircraft are considered nonholonomic. Nonholonomic constraints cannot be written as an algebraic constraint in the configuration space. When the constraints are explicitely integrable, then they can take the form of an algebraic constraint. Hence one can relate the words holonomic and nonholonomic to integrable and non-integrable respectively. We first present the fundamental approach using Decentralized Navigation Functions (DNF's) for agents with global sensing capabilities. We proceed by showing how this methodology has been successfully extended to take into account the limited sensing capabilities of each agent. A discussion on handling velocity constraints is made in the end of this chapter.

2.1 The case of Global Sensing Capabilities

In this section, we consider the case where each agent has global knowledge of the positions and velocities of the others at each time instant. The decentralization factor lies in the assumption that each agent does not need to know the desired destinations of the others in order to navigate to its goal configuration. The means to extend this method to the case of limited sensing capabilities is presented in the next subsection.

Consider the following system of N nonholonomic vehicles:

$$\begin{aligned} \dot{x}_i &= u_i \cos\theta_i \\ \dot{y}_i &= u_i \sin\theta_i \\ \dot{\theta}_i &= \omega_i \end{aligned} \tag{2.1}$$

with $i \in \{1 \ldots N\}$. (x_i, y_i, θ_i) are the position and orientation of each robot, u_i and w_i are the translational and rotational velocities respectively.

The problem can be now stated as follows: *"Given the N nonholonomic systems, derive a control law that steers every system from any feasible initial configuration to its goal configuration avoiding collisions. The control law must be decentralized in the sense that each system has no knowledge of the targets of other systems."*

In this section we make the following assumptions:

- Each agent has global knowledge of the position and velocity of the others at each time instant.
- Agents have no information about other agents targets.
- Around the target of each agent $\mathcal{A}$ there is a region called the agent's $\mathcal{A}$ safe region
- Agent's $\mathcal{A}$ safe region is only accessible by agent $\mathcal{A}$, while regarded as an obstacle by other agents.

2.1.1 Decentralized Dipolar Navigation Functions(DDNF's)

In this section, we show how the DNF's of the previous section have been redefined in [27] in order to provide trajectories suitable for nonholonomic navigation. This is accomplished by a enhancing a dipolar structure [37] to the navigation functions. Dipolar potential fields have been proven a very effective tool for stabilization [38] of nonholonomic systems as well as for centralized coordination of multiple agents with nonholonomic constraints [28]. The key advantage of this class of potential fields is that they drive the controlled agent to its destination with desired orientation.

The navigation function of the previous section is modified in the following manner in order to be able to produce a dipolar potential field:

$$\varphi_i = \frac{\gamma_{di}}{(\gamma_{di}^k + H_{nh_i} \cdot G_i \cdot b_{t_i})^{1/k}} \tag{2.2}$$

where $b_{t_i} = \prod_{j \neq i}(\|q_i - q_{d_j}\|^2 - (\varepsilon + r_i)^2)$. The term $\varepsilon > 0$ is the radius of the safe region of its agent. H_{nh_i} has the form of a pseudo-obstacle and is defined as

$$H_{nh_i} = \varepsilon_{nh} + \eta_{nh_i}$$

with $\varepsilon_{nh} > 0$, $\eta_{nh_i} = \|(q_i - q_{di}) \cdot n_{di}\|^2$ and $n_{di} = [\cos(\theta_{di}), \sin(\theta_{di})]^T$. Moreover $\gamma_{di} = \|q_i - q_{di}\|^2$, i.e. the heading angle is not incorporated in the distance to the destination metric. The next figure shows a 2D dipolar navigation function.

An important feature that should be noticed is the fact that this navigation function does not have to include the f_i function as each agent treats the other agents' targets as static obstacles.

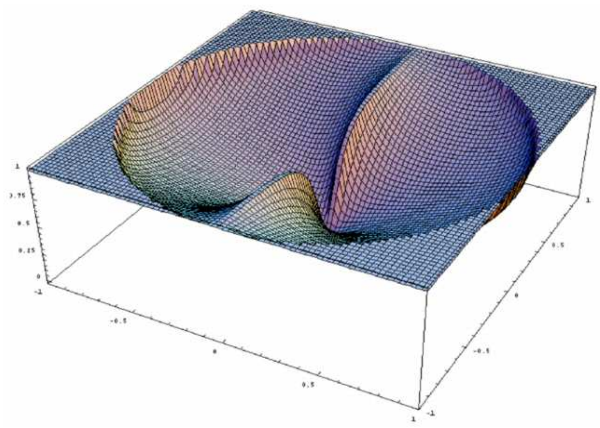

Figure 2.1: A dipolar potential field

2.1.2 Nonholonomic Control

Thus far we have established that the dipolar function φ_i has navigation properties. We consider convergence of the multi-agent system as a two-stage process: In the first stage agents converge to a ball of radius ε called safe region, containing the desired destination of each agent. Each agent can get in its own safe region but not in others. The safe region of one agent is regarded as an obstacle from the other agents. Once an agent gets in its own safe region, it remains in the set and asymptotically converges to the origin.

Before defining the control we need some preliminary definitions: We define by $\frac{\partial^2}{\partial q_i^2}\varphi_i(q_i,t) = {}^i\nabla^2\varphi_i(q_i,t)$ the Hessian of φ_i. Let $\lambda_{\min}, \lambda_{\max}$ be the minimum and maximum eigenvalues of the Hessian and $\hat{\upsilon}_{\lambda_{\min}}, \hat{\upsilon}_{\lambda_{\max}}$ the unit eigenvectors corresponding to the minimum and maximum eigenvalues of the Hessian. Since navigation functions are Morse functions [31], their Hessian at critical points is never degenerate, i.e. their eigenvalues have always nonzero values.

As discussed before,φ_i is a dipolar navigation function. The flows of the dipolar navigation field provide feasible directions for nonholonomic navigation. What we need now is to extract this information from the dipolar function. To this extend we define the "nonholonomic angle":

$$\theta_{nh_i} = \begin{cases} \arg\left(\frac{\partial\varphi_i}{\partial x_i}\cdot s_i + \mathrm{i}\frac{\partial\varphi_i}{\partial y_i}\cdot s_i\right), & \neg P_1 \\ \arg\left(d_i \cdot s_i(\upsilon^x_{\lambda_{\min}} + \mathrm{i}\upsilon^y_{\lambda_{\min}}\right), & P_1 \end{cases}$$

where condition P_1 is used to identify sets of points that contain measure zero sets whose positive limit sets are saddle points:

$$P_1 = (\lambda_{\min} < 0) \wedge (\lambda_{\max} > 0) \wedge \left(\left|\hat{\upsilon}_{\lambda_{\min}} \cdot {}^i\nabla\varphi_i\right| < \varepsilon_1\right)$$

where

$$\varepsilon_1 < \min_{C=\{q_i:\|q_i-q_{di}\|=\varepsilon\}}\left(\left\|{}^i\nabla\varphi_i(C)\right\|\right), s_i = \mathrm{sgn}((q_i - q_{di})\cdot\eta_{di})$$

$$d_i = \text{sgn}(\upsilon_{\lambda_{\min_i}} \cdot {}^i\nabla\varphi_i), \eta_{di} = [\cos(\theta_{di})\ \sin(\theta_{di})\ ]^T$$
$$\eta_i = [\cos(\theta_i)\ \sin(\theta_i)\ ]^T$$

Before proceeding we need the following:

Lemma 2.1 *If* $\left|\hat{\upsilon}_{\lambda_{\min}} \cdot {}^i\nabla\varphi_i\right| = 0$ *then* P_1 *consists of the measure zero set of initial conditions that lead to saddle points.*

For a proof of this lemma the reader is referred to [27].

In view of Lemma 2.1, ε_1 can be chosen to be arbitrarily small so the sets defined by P_1 eventually consist of thin sets containing sets of initial conditions that lead to saddle points.

The following provides a suitable nonholonomic controller for the first stage:

Proposition 2.2 *The system under the control law*

$$\begin{array}{l} u_i = -\text{sgn}\left(\frac{\partial\varphi_i}{\partial x_i}\cos\theta_i + \frac{\partial\varphi_i}{\partial y_i}\sin\theta_i\right)\cdot \\ \cdot\left(K_{u_i}K_{z_i} + c_i\frac{|\partial\varphi_i/\partial t|}{\left|\frac{\partial\varphi_i}{\partial x_i}\cos\theta_i + \frac{\partial\varphi_i}{\partial y_i}\sin\theta_i\right|}\tanh\left(\left|\frac{\partial\varphi_i}{\partial x_i}\cos\theta_i + \frac{\partial\varphi_i}{\partial y_i}\sin\theta_i\right|^2\right)\right) \\ \omega_i = \dot{\theta}_{nh_i} + (\theta_{nh_i} - \theta_i)\left(K_{\theta_i} + c_i\frac{|\partial\varphi_i/\partial t|}{2\left(\theta_{nh_i} - \theta_i\right)^2}\tanh\left(|\theta_{nh_i} - \theta_i|^3\right)\right) \end{array} \tag{2.3}$$

converges to the set

$$B_i = \{p_i : \|q_i - q_{di}\| \le \varepsilon - \delta, \theta_i \in (-\pi, \pi]\}$$

up to a set of measure zero of initial conditions where $0 < \delta < \varepsilon$. *Here* $K_{z_i} = \left\|{}^i\nabla\varphi_i\right\|^2 + \|q_i - q_{di}\|^2$, K_{u_i}, K_{θ_i} *are positive constants,* $c_i > \frac{\varepsilon_2+1}{\varepsilon_2}$ *where* $\varepsilon_2 = 2\pi^3\varepsilon_1^2\left(4\varepsilon_1 + \sqrt{2}\pi^{3/2}\right)^{-2}$ *and*

$$\frac{\partial\varphi_i}{\partial t} = \sum_{j\neq i}\left\{\left(\frac{\partial\varphi_i}{\partial x_j}\cos\theta_j + \frac{\partial\varphi_i}{\partial y_j}\sin\theta_j\right)\cdot u_j\right\} \tag{2.4}$$

.

Proof[27]:We form the following Lyapunov function:

$$V_i = \varphi_i\left(x_i, y_i, t\right) + \left(\theta_{nh_i}\left(x_i, y_i, t\right) - \theta_i\right)^2$$

and take it's time derivative:

$$\begin{array}{l} \dot{V}_i = \frac{\partial\varphi_i}{\partial t} + u_i\eta_i{}^i\nabla\varphi_i + \\ \quad +2\left(\theta_{nh_i} - \theta_i\right)\left(-w_i + \frac{\partial\theta_{nh_i}}{\partial t} + u_i\eta_i\cdot{}^i\nabla\theta_{nh_i}\right) \end{array}$$

After substituting the control law u_i and w_i, we get:

$$\begin{array}{l} \dot{V}_i = \frac{\partial\varphi_i}{\partial t} - \left|{}^i\nabla\varphi_i\cdot\eta_i\right|\left(K_{z_i} + c_i\frac{|\partial\varphi_i/\partial t|}{|{}^i\nabla\varphi_i\cdot\eta_i|}\tanh\left(\left|{}^i\nabla\varphi_i\cdot\eta_i\right|^2\right)\right) \\ -2\left(\theta_{nh_i} - \theta_i\right)^2\left(K_{\theta_i} + c_i\frac{|\partial\varphi_i/\partial t|}{2\left(\theta_{nh_i} - \theta_i\right)^2}\tanh\left(|\theta_{nh_i} - \theta_i|^3\right)\right) \le \\ \le \frac{\partial\varphi_i}{\partial t} - c_i\left|\frac{\partial\varphi_i}{\partial t}\right|\left(\tanh\left(\left|{}^i\nabla\varphi_i\cdot\eta_i\right|^2\right) + \tanh\left(|\theta_{nh_i} - \theta_i|^3\right)\right) \end{array}$$

Since the set P_1 is by construction repulsive for ε_1 sufficiently small, we only need to consider the set $\neg P_1$. Then: $\left|{}^i\nabla\varphi_i \cdot \eta_i\right|^2 = \left\|{}^i\nabla\varphi_i\right\|^2 \cos^2\left(\theta_{nh_i} - \theta_i\right)$. Let $\Delta\theta = |\theta_{nh_i} - \theta_i|$. After substituting we get:

$$\dot{V}_i \leq \frac{\partial\varphi_i}{\partial t} - c_i \left|\frac{\partial\varphi_i}{\partial t}\right| \left(\tanh\left(\left\|{}^i\nabla\varphi_i\right\|^2 \cos^2(\Delta\theta)\right) + \tanh\left(\Delta\theta^3\right)\right)$$

Before proceeding we need the following:

Lemma 2.3 *The following inequalities hold:*

1. $\tanh(x) \geq \frac{x}{x+1}$, $x \geq 0$
2. $\frac{x}{x+1} + \frac{y}{y+1} \geq \frac{x+y}{x+y+1}$, $x, y \geq 0$
3. $\cos^2\Delta\theta \geq \frac{8}{\pi^3}\left(\left|\Delta\theta - \frac{\pi}{2}\right|\right)^3 \qquad \Delta\theta \in \left[0, \frac{\pi}{2}\right]$

Proof:

1. For $x \geq 0$ we have that $e^{2x} - 1 - 2x \geq 0$. Hence $(x+1)\left(e^x - e^{-x}\right) \geq x\left(e^x + e^{-x}\right)$ and we get the result: $\tanh(x) \geq \frac{x}{x+1}$. The equality holds at $x = 0$.
2. With $x, y \geq 0$ we have : $\frac{x}{x+1} + \frac{y}{y+1} = \frac{2xy+x+y}{xy+x+y+1} \geq \frac{xy+x+y}{xy+x+y+1} \geq \frac{x+y}{x+y+1}$ and the equality holds at $x = y = 0$
3. Denote $A(\Delta\theta) = \cos^2\Delta\theta$ and $B(\Delta\theta) = \frac{8}{\pi^3}\left(\left|\Delta\theta - \frac{\pi}{2}\right|\right)^3$. Solving $A(\Delta\theta) = B(\Delta\theta)$, for $\Delta\theta \in \left[0, \frac{\pi}{2}\right]$ we get $\Delta\theta = 0$ for $A = B = 1$ and $\Delta\theta = \frac{\pi}{2}$ for $A = B = 0$. But at $\frac{\partial A}{\Delta\theta}_{|\Delta\theta=0} = 0 > -\frac{6}{\pi} = \frac{\partial \mathrm{B}}{\Delta\theta}_{|\Delta\theta=0}$ and since A and B have no other intersection for $\Delta\theta \in \left[0, \frac{\pi}{2}\right]$ it follows that $A(\Delta\theta) \geq B(\Delta\theta)$, for $\Delta\theta \in \left[0, \frac{\pi}{2}\right]$.

By use of Lemma 2.3.1 we get: $\dot{V}_i \leq \frac{\partial\varphi_i}{\partial t} - \left|\partial\varphi_i/\partial t\right| \left(c_i \frac{\left\|{}^i\nabla\varphi_i\right\|^2\cos^2\Delta\theta}{\left\|{}^i\nabla\varphi_i\right\|^2\cos^2\Delta\theta+1} + c_i\frac{\Delta\theta^3}{\Delta\theta^3+1}\right)$.
By use of Lemma 2.3.2 we get: $\dot{V}_i \leq \frac{\partial\varphi_i}{\partial t} - \left|\partial\varphi_i/\partial t\right| c_i \left(\frac{\left\|{}^i\nabla\varphi_i\right\|^2\cos^2\Delta\theta+\Delta\theta^3}{\left\|{}^i\nabla\varphi_i\right\|^2\cos^2\Delta\theta+\Delta\theta^3+1}\right)$
and from Lemma 2.3.3 we get: $\dot{V}_i \leq \frac{\partial\varphi_i}{\partial t} - \left|\partial\varphi_i/\partial t\right| c_i \frac{\left\|{}^i\nabla\varphi_i\right\|^2\frac{8}{\pi^3}\left(\left|\Delta\theta-\frac{\pi}{2}\right|\right)^3+\Delta\theta^3}{\left\|{}^i\nabla\varphi_i\right\|^2\frac{8}{\pi^3}\left(\left|\Delta\theta-\frac{\pi}{2}\right|\right)^3+\Delta\theta^3+1}$.
In view of the fact that the function $\frac{f(x)}{f(x)+1}$ has the same extremal points with $f(x) \geq 0$ (see [23] for a proof), the minimum of $[\frac{\left\|{}^i\nabla\varphi_i\right\|^2\frac{8}{\pi^3}\left(\left|\Delta\theta-\frac{\pi}{2}\right|\right)^3+\Delta\theta^3}{\left\|{}^i\nabla\varphi_i\right\|^2\frac{8}{\pi^3}\left(\left|\Delta\theta-\frac{\pi}{2}\right|\right)^3+\Delta\theta^3+1}$ coincides with the minimum of $m = \left\|{}^i\nabla\varphi_i\right\|^2\frac{8}{\pi^3}\left(\left|\Delta\theta - \frac{\pi}{2}\right|\right)^3 + \Delta\theta^3$. Trying to minimize m, we get: $\frac{\partial m}{\partial\left\|{}^i\nabla\varphi_i\right\|} = \frac{16}{\pi^3}\left\|{}^i\nabla\varphi_i\right\|\left(\left|\Delta\theta - \frac{\pi}{2}\right|\right)^3 \geq 0$ which means that m is strictly increasing in the direction of $\left\|{}^i\nabla\varphi_i\right\|$. Examining $\frac{\partial m}{\partial\Delta\theta} = 3\cdot\Delta\theta^2 + \frac{24}{\pi^3}\left\|{}^i\nabla\varphi_i\right\|^2\cdot\left(\Delta\theta - \frac{\pi}{2}\right)^2\cdot sign\left(\Delta\theta - \frac{\pi}{2}\right)$ and requiring $\frac{\partial m}{\partial\Delta\theta} = 0$

for an extremum in the direction of $\Delta\theta$, we get:

$$\Delta\theta = \begin{cases} \frac{2\left\|{}^i\nabla\varphi_i\right\|\pi}{4\|{}^i\nabla\varphi_i\|\pm\sqrt{2}\cdot\pi^{3/2}} & \Delta\theta \le \pi/2 \\ \frac{2\left\|{}^i\nabla\varphi_i\right\|\pi}{4\|{}^i\nabla\varphi_i\|\pm\mathbf{i}\sqrt{2}\cdot\pi^{3/2}} & \Delta\theta > \pi/2 \end{cases}$$

The only feasible solution is: $\Delta\theta = \frac{2\left\|{}^i\nabla\varphi_i\right\|\pi}{4\|{}^i\nabla\varphi_i\|+\sqrt{2}\cdot\pi^{3/2}}$. Substituting the solution in m we get: $\min_{\Delta\theta}(m) = \frac{2\left\|{}^i\nabla\varphi_i\right\|^2\pi^3}{\left(4\|{}^i\nabla\varphi_i\|+\sqrt{2}\cdot\pi^{3/2}\right)^2}$. Minimizing the last we get: $\frac{\partial \min_{\Delta\theta}(m)}{\partial\|{}^i\nabla\varphi_i\|} = \frac{4\sqrt{2}\left\|{}^i\nabla\varphi_i\right\|\pi^{9/2}}{\left(4\|{}^i\nabla\varphi_i\|+\sqrt{2}\cdot\pi^{3/2}\right)^3} \ge 0$. Activating the constraint $\left\|{}^i\nabla\varphi_i\right\| \ge \varepsilon_1$ we get: $\varepsilon_2 = \min(m) = \frac{2\varepsilon_1^2\pi^3}{\left(4\varepsilon_1+\sqrt{2}\cdot\pi^{3/2}\right)^2}$. Substituting in the time derivative of the Lyapunov function, we have that: $\dot{V}_i \le \frac{\partial\varphi_i}{\partial t} - \left|\partial\varphi_i/\partial t\right| c\frac{\varepsilon_2}{\varepsilon_2+1}$, so choosing $c > \frac{\varepsilon_2+1}{\varepsilon_2}$ we get that

$$\dot{V}_i \le \left|\partial\varphi_i/\partial t\right| \left(sign\left(\partial\varphi_i/\partial t\right) - k\right) \le 0$$

since

$$k = c\frac{\varepsilon_2}{\varepsilon_2+1} > 1$$

The equality holds when

$$(q_i = q_{d_i}) \wedge \left(\partial\varphi_i/\partial t = 0\right)$$

We assume that the system's initial conditions are in the set $\mathcal{W}_i\backslash\mathcal{S}_i$ where the set $\mathcal{S}_i = \left\{p_i : \left\|{}^i\nabla\varphi_i\right\| < \varepsilon_1\right\}$. ε_1 can be chosen to be arbitrarily small such that the set $\mathcal{S}_i$ includes arbitrarily small regions only around the saddle points and the target. Since we are considering convergence to the set B_i, we have that

$$\dot{V}_i < 0, \forall q_i \in W_{free}\backslash\left\{\bar{B}_i \cup \left\{q_i : \left\|{}^i\nabla\varphi_i(q_i)\right\| < \varepsilon_1\right\}\right\}$$

, where the bar denotes the set internal. ◇

For the second stage each agent is isolated from the rest of the system. The dipolar navigation function for this case becomes:

$$\varphi_{int_i}(x_i, y_i, \theta_i) = \frac{\gamma_{d,\theta_i}}{\left(\gamma_{d,\theta_i}^k + H_{nh_i}\cdot\beta_{int_i}\right)^{1/k}} \tag{2.5}$$

where $\beta_{int_i} = \varepsilon^2 - \|q_i - q_{d_i}\|^2$, and $\gamma_{d,\theta_i} = \|q_i - q_{d_i}\|^2 + (\theta - \theta_{d_i})^2$. Define

$$\Delta_i = K_{\theta_i} \cdot {}^{\partial\varphi_{\text{int}_i}}/_{\partial\theta_i} \cdot (\theta_{inh_i} - \theta_i) - K_{u_i} \cdot K_{z_i} \cdot \left|{}^i\nabla\varphi_{\text{int}_i} \cdot \eta_i\right|$$

and

$$\theta_{inh_i} = \arg\left(\frac{\partial\varphi_{\text{int}_i}}{\partial x_i} \cdot s_i + \mathbf{i}\frac{\partial\varphi_{\text{int}_i}}{\partial y_i} \cdot s_i\right)$$

Then for each aircraft in isolation we have the following:

Proposition 2.4 *Each subsystem under the control law*

$$\begin{array}{l} u_i = -\text{sgn}\left(\frac{\partial\varphi_{\text{int}_i}}{\partial x_i}\cos\theta_i + \frac{\partial\varphi_{\text{int}_i}}{\partial y_i}\sin\theta_i\right)K_{u_i}K_{z_i} \\ \omega_i = K_{\theta_i}\left(\theta_{inh_i} - \theta_i\right), \Delta_i < 0 \\ \omega_i = -K_{\theta_i}\frac{\partial\varphi_{\text{int}_i}}{\partial\theta_i}, \Delta_i \geq 0 \end{array} \tag{2.6}$$

converges to p_{d_i}

Proof: Taking $V_i = \varphi_{\text{int}_i}$ as a Lyapunov function candidate, we have for the time derivative:

$$\dot{V}_i = \dot{\mathbf{x}} \cdot \nabla\varphi_{\text{int}_i} = u_i\left({}^i\nabla\varphi_{\text{int}_i} \cdot \eta_i\right) + w_i{}^{\partial\varphi_{\text{int}_i}}/_{\partial y_i}$$

. We can now discriminate two cases, depending on the level of Δ_i:

1. $\Delta_i < 0$. Then $\dot{V}_i = -K_{u_i}K_{z_i}\left|{}^i\nabla\varphi_{\text{int}_i} \cdot \eta_i\right| + K_{\theta_i}\left(\theta_{inh_i} - \theta_i\right){}^{\partial\varphi_{\text{int}_i}}/_{\partial y_i} = \Delta_i < 0$
2. $\Delta_i \geq 0$. Then $\dot{V}_i = -K_{u_i}K_{z_i}\left|{}^i\nabla\varphi_{\text{int}_i} \cdot \eta_i\right| - K_{\theta_i}\left({}^{\partial\varphi_{\text{int}_i}}/_{\partial y_i}\right)^2 \leq 0$, with the equality holding only at the origin.

$\diamondsuit$

The fact that each agent remains in its safe region after the first stage is established by the following lemma which is a direct application of the properties of the navigation function:

Lemma 2.5 *For each subsystem i under the control law (2.6)the set*

$$B_{\text{int}_i} = \{p_i : \|q_i - q_{di}\| \leq \varepsilon, \theta_i \in (-\pi, \pi]\}$$

is positive invariant.

Proof : The boundary of (2.5) is the set $\mathcal{B}_{int_i} = \{p_i : \beta_{int_i}(q_i) = 0\} = \{p_i : \|q_i - q_{d_i}\| = \varepsilon\} = \partial B_{\text{int}_i}$, i.e. the workspace boundary, which is positive invariant for a navigation function [23],[8]. $\diamondsuit$

2.2 The Case of Limited Sensing Capabilities

In the previous section, we presented the nonholonomic control scheme for multiple agents with global sensing capabilities. In this section we modify this in order to cope with the limited sensing range of each agent.

It is obvious that each agent takes into account the other agents only on the first stage. The inter-agent proximity functions are modified according to (1.13). However each agent has also only local knowledge of the velocities of the rest of the team. Therefore the term $\frac{\partial \varphi_i}{\partial t}$ must be modified according to:

$$\frac{\partial \varphi_i}{\partial t} = \sum_{j:\|q_i - q_{di}\| \leq d_C} \left\{ \left(\frac{\partial \varphi_i}{\partial x_j} \cos\theta_j + \frac{\partial \varphi_i}{\partial y_j} \sin\theta_j \right) \cdot u_j \right\} \tag{2.7}$$

where d_C is again the radius of the sensing zone of each agent. Hence each agent has to take into account only the positions and velocities of agents that are within each sensing zone at each time instant.

This modification of the control law (2.3) does not affect the stability results of the previous section as the nodes of the deterministic switched system admit a common Lyapunov function. Using arguments from established results on stability for hybrid systems([3],[33]) the convergence in the first stage is guaranteed for each agent in this case as well. The interested reader can refer to [10] for more details.

2.3 Simulations

To demonstrate the navigation properties of our decentralized approach, we present three simulations of four nonholonomic agents that have to navigate from an initial to a final configuration, avoiding collisions. The chosen configurations constitute non-trivial setups since the straight-line paths connecting initial and final positions of each agent are obstructed by other agents. The following sequence of figures verifies the collision avoidance and global convergence properties of our algorithm. In each figure the circles denote the targets of each agent while the ring around each target represents the corresponding transition guard where the transition from the first to the second stage takes place.

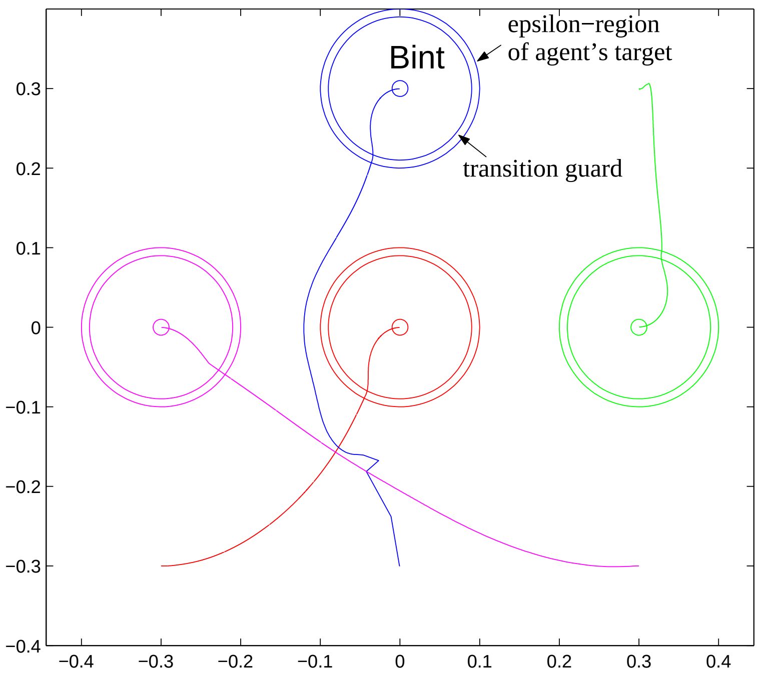

Figure 2.2: 4 nonholonomic agents

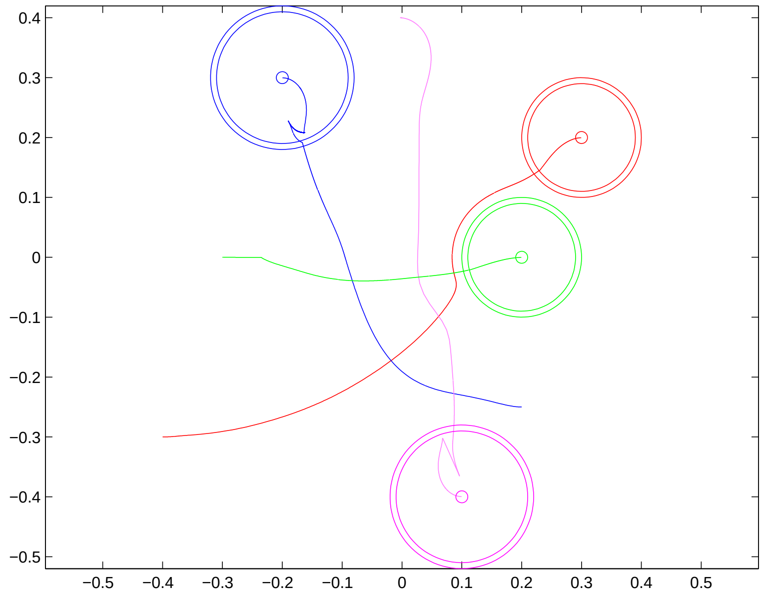

Figure 2.3: 4 nonholonomic agents

Chapter 3

Global Decentralized Conflict Resolution Part 3: Dynamic Models

The mathematical models of the moving vehicles/agents in the previous chapters were considered purely kinematic. In practice however, real mechanical systems are controlled through their acceleration. It is therefore evident that second order models are considered as well in the navigation functions' approach. The next two sections present the extension of the DNF's approach of the previous paragraphs to the cases of dynamic models for holonomic and nonholonomic systems, respectively.

3.1 Holonomic Dynamics

In this section, we present the decentralized control scheme for a multi-agent system with double integrator dynamics. The following discussion is based on [9].

We consider the following system of n agents with double integrator dynamics:

$$\begin{array}{l} \dot{q}_i = v_i \\ \dot{v}_i = u_i \end{array}, i \in \{1, \dots, N\} \tag{3.1}$$

We will show that the system is asymptotically stabilized under the control law

$$u_i = -K_i \frac{\partial \varphi_i}{\partial q_i} + \theta_i \left(v_i, \frac{\partial \varphi_i}{\partial t} \right) - g_i v_i \tag{3.2}$$

where $K_i, g_i > 0$ are positive gains,

$$\theta_i \left(v_i, \frac{\partial \varphi_i}{\partial t} \right) \triangleq -\frac{c v_i}{\tanh \left(\|v_i\|^2 \right)} \left| \frac{\partial \varphi_i}{\partial t} \right|$$

and

$$\frac{\partial \varphi_i}{\partial t} = \sum_{j \neq i} \frac{\partial \varphi_i}{\partial q_j} \dot{q}_j$$

The first term of equation (3.2) corresponds to the potential field (decentralized navigation function) described in chapter 1. The second term exploits the knowledge each agent has of the velocities of the others, and is designed to guarantee convergence of the whole team to the desired configurations. The last term serves as a damping element that ensures convergence to the destination point by suppressing oscillatory motion around it.

By using the notation $x = \left[x_1^T, \ldots, x_N^T\right]^T$, $x_i^T = \left[\begin{array}{cc} q_i^T & v_i^T \end{array}\right]$ the closed loop dynamics of the system can be rewritten as

$$\dot{x} = \xi(x) = \left[\xi_1^T(x), \ldots, \xi_N^T(x)\right]^T \tag{3.3}$$

with

$$\xi_i(x) = \left[\begin{array}{c} v_i \\ -K_i \frac{\partial \varphi_i}{\partial q_i} - \frac{c v_i}{\tanh\left(\|v_i\|^2\right)} \left|\frac{\partial \varphi_i}{\partial t}\right| - g_i v_i \end{array}\right]$$

We will use the function $V = \sum\limits_i K_i \varphi_i + \frac{1}{2} \sum\limits_i \|v_i\|^2$ as a candidate Lyapunov function to show that the agents converge to their destinations points . We will check the stability of the multi-agent system with LaSalle's Invariance Principle.

3.1.1 Stability Analysis

In the following we prove the following theorem:

Theorem 3.1 *The system (4) is asymptotically stabilized to* $\left[\begin{array}{cc} q_d^T & 0 \end{array}\right]$*,*$q_d = \left[q_{d1}, \ldots, q_{dN}\right]^T$ *up to a set of initial conditions of measure zero if the exponent* k *assumes values bigger than a finite lower bound and* $c > \max_i(K_i)$*.*

Proof: The candidate Lyapunov Function we use is $V = \sum\limits_i K_i \varphi_i + \frac{1}{2} \sum\limits_i \|v_i\|^2$ and by taking its derivative we have

$$\begin{array}{l}
V = \sum\limits_i K_i \varphi_i + \frac{1}{2} \sum\limits_i \|v_i\|^2 \Rightarrow \\
\dot{V} = \sum K_i \dot{\varphi}_i + \sum v_i^T \dot{v}_i = \sum K_i \left(\frac{\partial \varphi_i}{\partial t} + v_i^T \frac{\partial \varphi_i}{\partial q_i}\right) \\
+ \sum v_i^T \left(-K_i \frac{\partial \varphi_i}{\partial q_i} + \theta_i \left(v_i, \frac{\partial \varphi_i}{\partial t}\right) - g_i v_i\right) \\
\Rightarrow \dot{V} = \sum \left(K_i \frac{\partial \varphi_i}{\partial t} + v_i^T \theta_i \left(v_i, \frac{\partial \varphi_i}{\partial t}\right) - g_i \|v_i\|^2\right)
\end{array}$$

Using the notation $B_i \triangleq K_i \frac{\partial \varphi_i}{\partial t} + v_i^T \theta_i \left(v_i, \frac{\partial \varphi_i}{\partial t} \right)$ we first show that $\sum_i B_i \leq 0$ if $c > \max_i(K_i)$:

$$\begin{array}{l}
\frac{\partial \varphi_i}{\partial t} > 0 : \\
c > \max_i (K_i) \Rightarrow c > K_i \frac{\tanh\left(\|v_i\|^2\right)}{\|v_i\|^2} \\
\Rightarrow K_i > \frac{c\|v_i\|^2}{\tanh\left(\|v_i\|^2\right)} \operatorname{sgn}\left(\frac{\partial \varphi_i}{\partial t} \right) \\
\Rightarrow K_i \frac{\partial \varphi_i}{\partial t} + v_i^T \theta_i \left(v_i, \frac{\partial \varphi_i}{\partial t} \right) < 0 \forall i : \frac{\partial \varphi_i}{\partial t} > 0 \\
\frac{\partial \varphi_i}{\partial t} < 0 : \\
c > 0 \Rightarrow c > -K_i \frac{\tanh\left(\|v_i\|^2\right)}{\|v_i\|^2} \\
\Rightarrow K_i > \frac{c\|v_i\|^2}{\tanh\left(\|v_i\|^2\right)} \operatorname{sgn}\left(\frac{\partial \varphi_i}{\partial t} \right) \\
\Rightarrow K_i \frac{\partial \varphi_i}{\partial t} + v_i^T \theta_i \left(v_i, \frac{\partial \varphi_i}{\partial t} \right) < 0 \forall i : \frac{\partial \varphi_i}{\partial t} < 0
\end{array}$$

Of course, $K_i \frac{\partial \varphi_i}{\partial t} + v_i^T \theta_i \left(v_i, \frac{\partial \varphi_i}{\partial t} \right) = 0$ for $\frac{\partial \varphi_i}{\partial t} = 0$. In the preceding equations we used the fact that $0 \leq \frac{\tanh(x)}{x} \leq 1 \forall x \geq 0$. So we have $\sum_i B_i \leq 0$ with equality holding only when $\frac{\partial \varphi_i}{\partial t} = 0 \forall i$. We have

$$\dot{V} = \sum_i B_i - \sum_i g_i \left\| v_i \right\|^2 \leq 0$$

Hence, by LaSalle's Invariance Principle, the state of the system converges to the largest invariant set contained in the set

$$\begin{array}{l}
S = \left\{ q, v : \left(\frac{\partial \varphi_i}{\partial t} = 0 \right) \wedge (v_i = 0) \, \forall i \right\} = \\
= \{ q, v : (v_i = 0) \, \forall i \}
\end{array}$$

because by definition the set $\left\{ q, v : \left(\frac{\partial \varphi_i}{\partial t} = 0 \right) \forall i \right\}$ is contained in the set $\{ q, v : (v_i = 0) \, \forall i \}$. For this subset to be invariant we need

$$\dot{v}_i = 0 \Rightarrow \frac{\partial \varphi_i}{\partial q_i} = 0 \forall i$$

The analysis of chapter 1 revealed that this situation occurs whenever the potential functions either reach the destination or a saddle point. By bounding the parameters k, h from below by a finite number, φ_i becomes a navigation function, hence its critical points are isolated ([23]). Thus the set of initial conditions that lead to saddle points are sets of measure zero ([31]). Hence the largest invariant set contained in the set $\frac{\partial \varphi_i}{\partial q_i} = 0 \forall i$ is simply $q_d \diamondsuit$

3.2 Nonholonomic Dynamics

In chapter 2, we presented the decentralized navigation functions methodology for multiple agents with nonholonomic kinematics. Although each agent had no specific knowledge about the destinations of the others, it treated a spherical region around the target of each agent as a static obstacle. In this section we modify the proposed control law in order to allow each agent to neglect the destinations of the others. Furthermore, the control inputs are the acceleration and rotational velocity of each vehicle, coping in this way with realistic classes of mechanical systems. The following discussion is based on [11].

We consider the following system of n nonholonomic agents with the following dynamics

$$\begin{aligned} \dot{x}_i &= v_i \cos\theta_i \\ \dot{y}_i &= v_i \sin\theta_i \\ \dot{\theta}_i &= \omega_i \\ \dot{v}_i &= u_i \end{aligned} \quad , i \in \{1, \ldots, N\} \tag{3.4}$$

where v_i, ω_i are the translational and rotational velocities of agent i respectively, and u_i its acceleration.

The problem we treat in this paper can be now stated as follows:" *Given the N nonholonomic agents (3.4),consider the rotational velocity ω_i and the acceleration u_i as control inputs for each agent and derive a control law that steers every agent from any feasible initial configuration to its goal configuration avoiding, at the same, collisions.*"

We make the following assumptions:

- Each agent has global knowledge of the position of the others at each time instant.
- Each agent has knowledge only of its own desired destination but not of the others.
- We consider spherical agents.
- The workspace is bounded and spherical.

3.2.1 Elements from Nonsmooth Analysis

In this section, we review some elements from nonsmooth analysis and Lyapunov theory for nonsmooth systems that we use in the stability analysis of the next section.

We consider the vector differential equation with discontinuous right-hand side:

$$\dot{x} = f(x) \tag{3.5}$$

where $f : R^n \to R^n$ is measurable and essentially locally bounded.

Definition 3.1 [16]*: In the case when n is finite, the vector function $x(.)$ is called a solution of (3.5) in $[t_0, t_1]$ if it is absolutely continuous on $[t_0, t_1]$ and there exists $N_f \subset R^n, \mu(N_f) = 0$ such that for all $N \subset R^n, \mu(N) = 0$ and for almost all $t \in [t_0, t_1]$*

$$\dot{x} \in K[f](x) \equiv \overline{co}\{\lim_{x_i \to x} f(x_i) | x_i \notin N_f \cup N\}$$

The above definition along with the assumption that f is measurable guarantees the uniqueness of solutions of (3.5) [16].

Lyapunov stability theorems have been extended for nonsmooth systems in [36],[4]. The authors use the concept of *generalized gradient* which for the case of finite-dimensional spaces is given by the following definition:

Definition 3.2 [5]*: Let $V : R^n \rightarrow R$ be a locally Lipschitz function. The generalized gradient of V at x is given by*

$$\partial V(x) = \overline{co}\{\lim_{x_i \rightarrow x} \nabla V(x_i) | x_i \notin \Omega_V\}$$

where Ω_V is the set of points in R^n where V fails to be differentiable.

Lyapunov theorems for nonsmooth systems require the energy function to be *regular*. Regularity is based on the concept of *generalized derivative* which was defined by Clarke as follows:

Definition 3.3 [5]*: Let f be Lipschitz near x and v be a vector in R^n. The generalized directional derivative of f at x in the direction v is defined*

$$f^0(x; v) = \lim_{y \rightarrow x} \sup_{t \downarrow 0} \frac{f(y + tv) - f(y)}{t}$$

Regularity of a function is defined:

Definition 3.4 [5]*: The function $f : R^n \rightarrow R$ is called regular if*
1) $\forall v$, the usual one-sided directional derivative $f'(x; v)$exists and
2) $\forall v, f'(x; v) = f^0(x; v)$

The following chain rule provides a calculus for the time derivative of the energy function in the nonsmooth case:

Theorem 3.2 [36]*: Let x be a Filippov solution to $\dot{x} = f(x)$ on an interval containing t and $V : R^n \rightarrow R$ be a Lipschitz and regular function. Then $V(x(t))$ is absolutely continuous, $(d/dt)V(x(t))$ exists almost everywhere and*

$$\frac{d}{dt}V(x(t)) \in^{a.e.} \dot{\tilde{V}}(x) := \bigcap_{\xi \in \partial V(x(t))} \xi^T K[f](x(t))$$

We shall use the following nonsmooth version of LaSalle's invariance principle to prove the convergence of the prescribed system:

Theorem 3.3 [36] *Let Ω be a compact set such that every Filippov solution to the autonomous system $\dot{x} = f(x), x(0) = x(t_0)$ starting in Ω is unique and remains in Ω for all $t \geq t_0$. Let $V : \Omega \rightarrow R$ be a time independent regular function such that $v \leq 0 \forall v \in \dot{\tilde{V}}$ (if $\dot{\tilde{V}}$ is the empty set then this is trivially satisfied). Define $S = \{x \in \Omega | 0 \in \dot{\tilde{V}}\}$. Then every trajectory in Ω converges to the largest invariant set,M, in the closure of S.*

3.2.2 Nonholonomic Control and Stability Analysis

We will show that the system is asymptotically stabilized under the control law

$$\begin{array}{c} u_i = -v_i\{|\nabla_i\varphi_i\cdot\eta_i| + M_i\} - g_i v_i - \frac{v_i}{\tanh(|v_i|)}K_{v_i}K_{z_i} \\ \omega_i = -K_{\theta_i}(\theta_i - \theta_{di} - \theta_{nh_i}) + \dot{\theta}_{nh_i} \end{array} \tag{3.6}$$

where $K_{v_i}, K_{\theta_i}, g_i > 0$ are positive gains,

$$\theta_{nh_i} = \arg(\frac{\partial\varphi_i}{\partial x_i}\cdot s_i + \mathbf{i}\frac{\partial\varphi_i}{\partial y_i}\cdot s_i)$$

$$s_i = \mathrm{sgn}((q_i - q_{di})\cdot\eta_{di})$$

$$\eta_i = \left[\begin{array}{cc} \cos\theta_i & \sin\theta_i \end{array}\right]^T$$

$$\eta_{di} = \left[\begin{array}{cc} \cos\theta_{di} & \sin\theta_{di} \end{array}\right]^T$$

$$K_{zi} = \|\nabla_i\varphi_i\|^2 + \|q_i - q_{di}\|^2$$

$$M_i > |\sum_{j\neq i}\nabla_i\varphi_j\cdot\eta_i|_{max}$$

$$\nabla_i\varphi_j = \left[\begin{array}{cc} \frac{\partial\varphi_j}{x_i} & \frac{\partial\varphi_j}{y_i} \end{array}\right]$$

In particular, we prove the following theorem:

Theorem 3.4 *Under the control law (3.6), the system is asymptotically stabilized to* $p_d = [p_{d1},\ldots,p_{dN}]^T$.

Proof: Let us first consider the case $|v_i| > 0 \forall i$. We use

$$V = \sum V_i, V_i = \varphi_i + |v_i| + \frac{1}{2}(\theta_i - \theta_{di} - \theta_{nhi})^2$$

as a Lyapunov function candidate. For $|v_i| > 0$ we have

$$\dot{V} = \sum_i \dot{V}_i = \sum_i \left\{ \begin{array}{c} \sum_j v_j\left(\nabla_j\varphi_i\right)\cdot\eta_j + \mathrm{sgn}(v_i)\dot{v}_i + \\ +\left(\theta_i - \theta_{di} - \theta_{nhi}\right)\left(\dot{\theta}_i - \dot{\theta}_{nhi}\right) \end{array} \right\}$$

and substituting

$$\begin{array}{l} \dot{V} = \sum_i\left\{\sum_j v_j\left(\nabla_j\varphi_i\right)\cdot\eta_j - |v_i|\left(|(\nabla_i\varphi_i)\cdot\eta_i| + M_i\right)\right\} \\ -\sum_i \frac{|v_i|}{\tanh(|v_i|)}K_{v_i}K_{z_i} - \sum_i g_i|v_i| \\ -\sum_i K_{\theta_i}\left(\theta_i - \theta_{di} - \theta_{nhi}\right)^2 \end{array}$$

The first term of the right hand side of the last equation can be rewritten as

$$\sum_i \left\{ \sum_j v_j \left(\nabla_j \varphi_i\right) \cdot \eta_j - |v_i| \left(|(\nabla_i \varphi_i) \cdot \eta_i| + M_i \right) \right\} =$$
$$= \sum_i \left\{ \begin{array}{l} v_i \left(\nabla_i \varphi_i\right) \cdot \eta_i + v_i \sum_{j \neq i} \left(\nabla_i \varphi_j\right) \cdot \eta_i - \\ - |v_i| \left(|(\nabla_i \varphi_i) \cdot \eta_i| + M_i \right) \end{array} \right\} \leq 0$$

so that

$$\dot{V} \leq -\sum_i K_{v_i} K_{z_i} - \sum_i g_i |v_i| - \sum_i K_{\theta_i} \left(\theta_i - \theta_{di} - \theta_{nhi}\right)^2$$

where the inequality $\frac{x}{\tanh x} \geq 1$ for $x \geq 0$.

The candidate Lyapunov function is nonsmooth whenever $v_i = 0$ for some i. The generalized gradient of V is given by

$$\partial V = \left[\begin{array}{c} \sum_i \nabla_1 \varphi_i \\ \vdots \\ \sum_i \nabla_N \varphi_i \\ \partial |v_1| \\ \vdots \\ \partial |v_N| \\ \frac{1}{2} \nabla_{\theta_1} \left(\theta_1 - \theta_{d1} - \theta_{nh1}\right)^2 \\ \vdots \\ \frac{1}{2} \nabla_{\theta_N} \left(\theta_N - \theta_{dN} - \theta_{nhN}\right)^2 \\ \frac{1}{2} \nabla_{\theta_{nh1}} \left(\theta_1 - \theta_{d1} - \theta_{nh1}\right)^2 \\ \vdots \\ \frac{1}{2} \nabla_{\theta_{nhN}} \left(\theta_N - \theta_{dN} - \theta_{nhN}\right)^2 \end{array} \right]$$

and the Filippov set of the closed loop system by

$$K[f] = \begin{bmatrix} v_1 \cos\theta_1 \\ v_1 \sin\theta_1 \\ \vdots \\ v_N \cos\theta_N \\ v_N \sin\theta_N \\ u_1 \\ \vdots \\ u_N \\ \omega_1 \\ \vdots \\ \omega_N \\ \dot{\theta}_{nh1} \\ \vdots \\ \dot{\theta}_{nhN} \end{bmatrix} = \begin{bmatrix} v_1 \cos\theta_1 \\ v_1 \sin\theta_1 \\ \vdots \\ v_N \cos\theta_N \\ v_N \sin\theta_N \\ K[u_1] \\ \vdots \\ K[u_N] \\ \omega_1 \\ \vdots \\ \omega_N \\ \dot{\theta}_{nh1} \\ \vdots \\ \dot{\theta}_{nhN} \end{bmatrix}$$

We denote by

$$D \triangleq \{x : \exists i \in \{1, \ldots N\} \, \text{s.t.} v_i = 0\}$$

the "discontinuity surface" and

$$D_S \triangleq \{i \in \{1, \ldots N\} \, \text{s.t.} v_i = 0\}$$

the set of indices of agents that participate in D. We then have

$$\begin{aligned}
&\dot{\tilde{V}} = \bigcap_{\xi \in \partial V} \xi^T K[f] = \\
&v_1 \left(\sum_i \nabla_1 \varphi_i \right) \cdot \eta_1 + \ldots + v_N \left(\sum_i \nabla_N \varphi_i \right) \cdot \eta_N \\
&+ \bigcap_{\xi \in \partial |v_1|} \xi^T K[u_1] + \ldots + \bigcap_{\xi \in \partial |v_N|} \xi^T K[u_N] \\
&+ \sum_i (\theta_i - \theta_{di} - \theta_{nhi}) \left(\omega_i - \dot{\theta}_{nhi} \right) \Rightarrow
\end{aligned}$$

$$\begin{aligned}
\dot{\tilde{V}} &= \sum_{i \notin D_S} \left\{ v_i \left(\sum_i \nabla_i \varphi_j \right) \cdot \eta_i + \text{sgn}(v_i) \, u_i \right\} \\
&+ \sum_{i \in D_S} \bigcap_{\xi \in \partial |v_i|} \xi^T K[u_i] - \sum_i K_{\theta_i} (\theta_i - \theta_{di} - \theta_{nhi})^2
\end{aligned}$$

For $i \in D_S$ we have $\partial |v_i|\big|_{v_i = 0} = [-1, 1]$ and

$$K[u_i]\big|_{v_i = 0} = [-|K_{vi} K_{zi}|, |K_{vi} K_{zi}|]$$

so that

$$\bigcap_{\xi \in \partial |v_i|} \xi^T K[u_i] = 0$$

From the previous analysis we also derive that

$$\sum_{i \notin D_S} \left\{ v_i \left(\sum_i \nabla_i \varphi_j \right) \cdot \eta_i + \operatorname{sgn}(v_i)\, u_i \right\} \leq \\ - \sum_{i \notin D_S} \left\{ K_{vi} K_{zi} + g_i \left| v_i \right| \right\}$$

Going back to Theorem 3.4 it is easy to see that $v \leq 0 \forall v \in \dot{\tilde{V}}$. Each function V_i is regular as the sum of regular functions ([36]) and V is regular for the same reason. The level sets of V are compact so we can apply this theorem. We have that $S = \{x | 0 \in \dot{\tilde{V}}\} = \{x : (v_i = 0 \forall i) \bigwedge (\theta_i - \theta_{di} = \theta_{nhi} \forall i)\}$. The trajectory of the system converges to the largest invariant subset of S. For this subset to be invariant we must have

$$\dot{v}_i = 0 \Rightarrow K_{vi} K_{zi} = 0 \Rightarrow (\nabla_i \varphi_i = 0) \wedge (q_i = q_{di}) \, \forall i$$

For $\nabla_i \varphi_i = 0$ we have $\theta_{nhi} = 0$ so that $\theta_i = \theta_{di}$. $\diamondsuit$

3.3 Simulations

The navigation properties of the proposed control scheme are verified in the dynamic case as well through the following non-trivial simulations involving four holonomic and nonholonomic agents respectively. The simulations of dynamic models of the next two figures have their own importance as they deal with mathematical models of real world applications, such as aircraft and mechanical systems. The simulation of figure 3.2 is more closely related to realistic aircraft movement than the simulation of the kinematic nonholonomic model of figures 2.3,2.4 both from the modelling (dynamic model) as well as from the curvature constraints viewpoint. A formal proof of the last derivation is a topic of ongoing research.

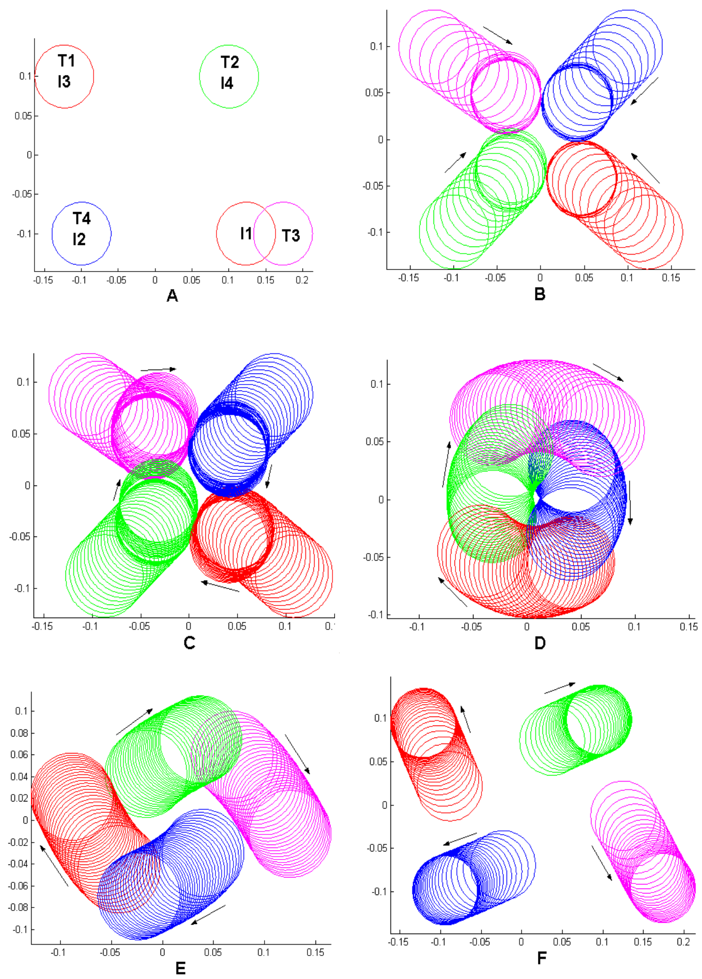

Figure 3.1: 4 dynamic holonomic agents

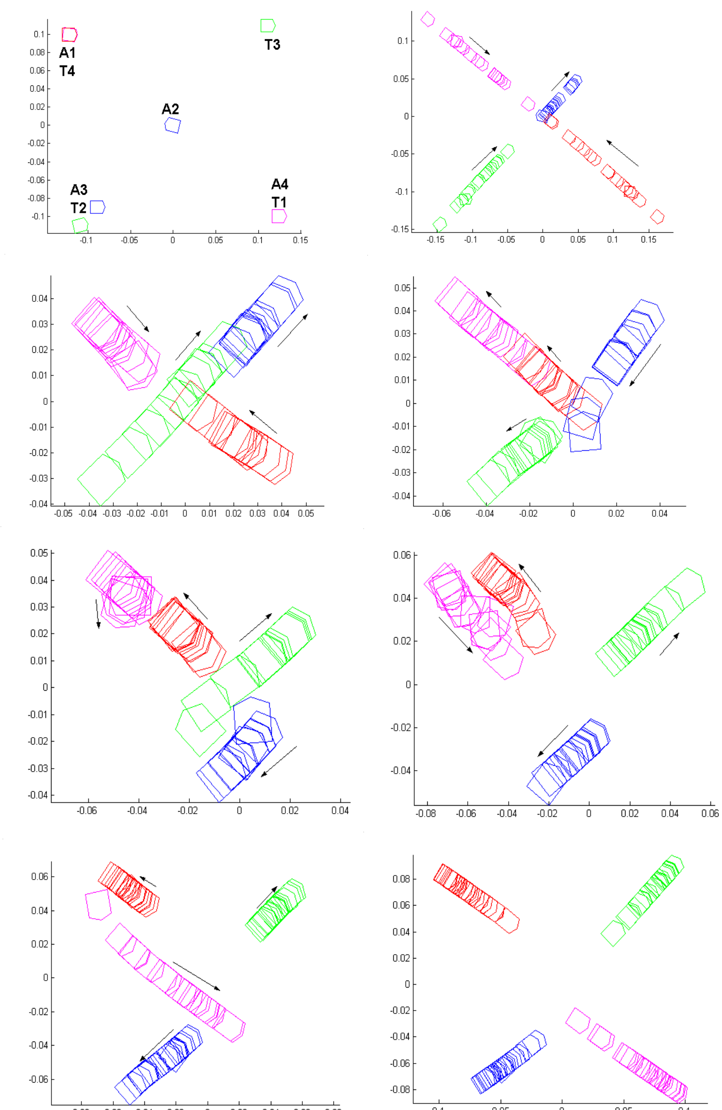

Figure 3.2: 4 dynamic nonholonomic agents

Chapter 4

Conclusions and Future Research Issues

This report summarized the work held on Global Decentralized Conflict Resolution under task 6.2 of HYBRIDGE WP6. This work package has established the first globally convergent algorithm for decentralized collision avoidance and destination convergence of multiple non-point agents. The mathematical models of agents' movement have been considered to be holonomic kinematics (Chapter 1), nonholonomic kinematics (Chapter 2), holonomic dynamics (Chapter 3.1) and nonholonomic dynamics (Chapter 3.2). Global and limited sensing capabilities of each agent have been taken into account, and a globally convergent feedback control strategy allowing each agent to take into account only nearby agents (i.e. agents within its sensing zone) in the conflict resolution procedure has been presented.

A major advantage of the adopted methodology with respect to existing methods is the fact that the collision avoidance procedure takes into account the real volume of the moving vehicles. That's a crucial factor in ATM where a collision occurs whenever two or more aircraft come closer than a minimum alert distance in the workspace. Furthermore, its closed loop nature enforces robustness with respect to modelling uncertainties and agent failures. Hence uncertainty is confronted by the robustness of the closed loop algorithms.

The fact that the decentralized algorithms have been extended to take into account nonholonomic models of vehicle motion shows its applicability to ATM. The dynamic nonholonomic model taken into account in section 3.2 is the closest to realistic aircraft movement from a modelling viewpoint. Furthermore, simulation results have shown that the proposed model satisfies in most cases curvature constraints of real aircraft. Taking into account that the limited sensing capabilities of each agent are being considered, the adopted methodology can be a milestone for future implementation of a (at least partially) distributed ATM system.

Current research involves extension of the proposed feedback control scheme towards many directions that make its applicability to real world applications even more apparent. The extension to three-dimensional models is rather straightforward for the holonomic case, whereas it could lead to even more satisfactory aircraft movement in the nonholonomic case. Furthermore, a formal control design for the satisfaction of curvature constraints in the dynamic nonholonomic case(section 3.2), which is the closest to real aircraft motion, is currently under development.

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