HYBRIDGE

Distributed Control and Stochastic Analysis of Hybrid Systems
Supporting Safety Critical Real-Time Systems Design

WP7: Error Evolution Control

Hybrid Observer Design Methodology

Elena De Santis, Maria D. Di Benedetto,
Stefano Di Gennaro, Giordano Pola[1]

19 August 2003

Version: *0.5*
Task number: *7.2*
Deliverable number: *D7.2*
Contract: *IST-2001-32460 of European Commission*

[1] University of L'Aquila

DOCUMENT CONTROL SHEET

Title of document:	*Hybrid observer design methodology*
Authors of document:	*Elena De Santis, Maria D. Di Benedetto, Stefano Di Gennaro, Giordano Pola*
Deliverable number:	*D7.2*
Contract:	*IST-2001-32460 of European Commission*
Project:	*Distributed Control and Stochastic Analysis of Hybrid Systems Supporting Safety Critical Real-Time Systems Design (HYBRIDGE)*

DOCUMENT CHANGE LOG

Version #	Issue Date	Sections affected	Relevant information
0.1	24 June 2003	All	First draft
0.2	7 July 2003	All	Second draft
0.3	23 July 2003	All	Third draft
0.4	15 August 2003	All	Fourth draft
0.5	19 August 2003	All	Fifth draft

Author(s) and Reviewers		Organisation	Signature/Date
Authors	E. De Santis	AQUI	
	M.D. Di Benedetto	AQUI	
	S. Di Gennaro	AQUI	
	G. Pola	AQUI	
Internal reviewers	A. Balluchi	PARADES	
	H. Blom	NLR	
	A. D'Innocenzo	AQUI	
	J. Lygeros	UCAM/ Univ. Patras	

Abstract

This report is the second deliverable D7.2 of Work Package WP7 of the HYBRIDGE Project. The main contributions of D7.2 is a review of the concept of observability for hybrid systems and the presentation of a new approach to observability and observer design for hybrid systems. The first part of the report deals with a formal definition of hybrid system that is used to present in a unified framework the special cases that have been addressed in the literature on hybrid systems. This report presents the research efforts in the area of stochastic hybrid systems in the unified framework. Then it focuses on classes of hybrid systems for which some results on observability are available. Using the limitations of these approaches as a spring board, the report deals with a new approach to the definition of observability for hybrid systems that is at the same time more general and consistent with the classical results on continuous–time dynamical systems.

Contents

Chapter 1

Introduction

The objective of Air Traffic Management is to ensure the safe and efficient operation of aircrafts. The stress placed on the present systems by the ever-increasing air traffic has forced the authorities to plan for an overhaul of ATM to make them safer, more reliable and efficient. A move in this direction could be obtained by more automation and a more sophisticated monitoring and control system. Automation and control require in turn a precise formulation of the problem. In this context, variables that can be measured or estimated have to be identified together with safety indices and objective functions. To make things more complex, the behaviour of ATM depends critically on the actions of humans who control the operations that are very difficult to observe, measure, model, and predict.

The purpose of Work Package WP7, "Error Evolution Control", of the HYBRIDGE project is to develop algorithms with guaranteed performances for assisting human operators in avoiding the propagation of errors and other non-nominal events in distributed systems. Estimation methods and observer design techniques are essential in this regard for the design of a control strategy for error propagation avoidance and/or error recovery. The first task of Work-Package 7 (documented in Deliverable 7.1 [19]) dealt with a review of some of the psychological models that are in use for the study of air traffic management.

The objectives of Task 7.2 are:

1. Identification of a stochastic hybrid model that describes the dynamics involved in error evolution control and captures the essential features studied in Task 7.1.

2. Development of estimation methods and observer design techniques for this class of stochastic hybrid systems.

The first objective of Task 7.2 has been pursued in collaboration with the University of

Cambridge. A taxonomy of stochastic hybrid models has been proposed in Deliverable 1.2 [13] with the goal of identifying the essential features that make the models appropriate for their use in ATM. Moreover, connections have been established between classes of stochastic hybrid systems and different safety critical ATM situations.

The second objective of Task 7.2 addresses the issue of observability and observer design for hybrid systems. The cornerstones of the literature on observability have been the Luenberger observer introduced in the context of linear deterministic systems [26], and the Kalman filter in the context of linear stochastic systems [25]. In the case of hybrid systems, the literature is still rather scant. We review in this Deliverable what is available in the case of hybrid systems. Note that observability conditions for hybrid systems give sufficient conditions for observability of particular classes of stochastic hybrid models where continuous disturbances are either not present or measurable.

The results on observability of hybrid systems cannot but be based on results available for discrete and continuous systems "adapted" to the hybrid case. Assuming the reader is well accustomed with the classical results on continuous systems, we review what is available for Discrete Event Dynamic Systems (DEDS) before analyzing what is available in the literature. Finally we present some new results on observability and observer design that have been obtained under the partial sponsorship of HYBRIDGE.

The contributions of this deliverable can then be summarized as follows:

- A critical review of the literature on the observability of hybrid systems. One of the outcomes of the review is that the very concept of observability is much more complex in the hybrid case than in the case of linear and non linear systems. In the review, we note that a systematic procedure for the construction of a hybrid observer is still largely missing.
- Some new results on the observability of a particular class of hybrid systems and a theory for hybrid observers.

The report is organized as follows. In Chapter 2 we formally define hybrid systems. In Chapter 3 we review the most important results concerning the observability of discrete event dynamic systems. In Chapter 4, we present the available concepts of observability in the context of hybrid systems. In Chapter 5 we propose a new definition of observability for linear hybrid systems. In Chapter 6, we illustrate a methodology for the design of hybrid observer from [5] and [4]. In Chapter 7, we offer some concluding remarks.

Chapter 2

Formal Definitions of Hybrid Systems

This chapter is devoted to the introduction of hybrid system formalism and to the formulation of the observability problem for hybrid systems. In Section 2.1, we present a general definition of hybrid systems that includes a large class of hybrid systems, such as jump linear systems [37, 38], piecewise affine system [7], switched linear systems [36] and switching linear systems [11], [17], [18]. Then, we introduce some assumptions of "regularity" in the behaviour of hybrid systems, which will be useful in the next development. In Section 2.2, we review briefly some stochastic hybrid models presented in the literature and show connections between these stochastic hybrid models and hybrid systems.

2.1 Hybrid Systems

The notion of a hybrid system that has been used in the control community is centered around a particular composition of discrete and continuous dynamics. In particular, a hybrid system has continuous evolution and occasional jumps. The jumps correspond to the change of state in an automaton whose transitions are caused by controllable or uncontrollable external events or by the continuous evolution. A continuous evolution is associated to each discrete state, described by differential equations, which may have different structure for each discrete state, and an initial state that is determined whenever a transition into the discrete state of the automaton is taken. While this informal description seems rather simple, the precise definition of the evolution of the system is quite complex. In the sequel, we describe formally a hybrid system. With this model, we intended to mediate between generality and understandability of its properties. Its bases are in the classic paper [27].

Definition 1 *(Hybrid Systems) A continuous time (resp. discrete time) hybrid system is a tuple* $\mathcal{H} = (\mathbf{Q}, \mathbf{P}, \mathbf{U}_D, X, U, V, Y, Init, \mathbf{S}_C, \mathcal{S}, E, \gamma, Inv, R, G)$, *(resp.* $\mathcal{H} = (\mathbf{Q}, \mathbf{P}, \mathbf{U}_D, X, U, V, Y, Init, \mathbf{S}_D, \mathcal{S}, E, \gamma, Inv, R, G)$*) where:*

- $\mathbf{Q} = \{q_i, i \in J\}$ *is the set of discrete states,* $J \subset \mathbb{N}$*;*
- $\mathbf{P} = \{p_i, i \in J\} \cup \{\epsilon\}$ *is the set of discrete outputs;* ϵ *is the "null" output;*
- $\mathbf{U}_D = \mathbf{U}_{D_{EXT}} \cup \mathbf{U}_{D_{CONTR}} \cup \{\epsilon_e\}$ *is the set of discrete inputs;* ϵ_e *is the "null" event;*
- X, U, V, Y *are subsets of finite dimensional vector spaces and are respectively the continuous state, input, disturbance and output space. Given* $x \in X$ *and* $q \in \mathbf{Q}$, $y = h(x, q) \in Y$, *where* $h : X \times \mathbf{Q} \to Y$. *We denote by* $\mathcal{U}_C$ *the set of measurable functions in the class of control functions for continuous signals* $u : T \to U$ *and by* $\mathcal{U}_d$ *the class of disturbance functions* $\delta : T \to V$, *where* T *denotes the set of reals* $\mathbb{R}$ *(resp. the set of integers* $\mathbb{Z}$*).*
- $Init \subset \mathbf{Q} \times X$ *denotes the set of hybrid initial states;*
- $\mathbf{S}_C$ *is a subclass of continuous time dynamical systems (resp.* $\mathbf{S}_D$ *is a subclass of discrete time dynamical systems).*

  - $S_i \in \mathbf{S}_C$ *is defined by the equation:*

  $$\dot{x}(t) = f_i(x(t), u(t), \delta(t)) \qquad i \in J$$

  where $t \in \mathbb{R}$, $x(t) \in X$ *and* f_i *is a function such that,* $\forall u \in \mathcal{U}_C$, $\forall \delta \in \mathcal{U}_d$, *the solution* $x(t)$ *exists and is unique for all* $t \in \mathbb{R}$ *.*

  (resp. - $S_h \in \mathbf{S}_D$ *is defined by the equation:*

  $$x(t+1) = f_i(x(t), u(t), \delta(t)) \qquad i \in J$$

  where $t \in \mathbb{Z}$, $x(t) \in X$ *and* f_i *is a vector field).*
- $\mathcal{S} : \mathbf{Q} \to \mathbf{S}_C$ *(resp.* $\mathcal{S} : \mathbf{Q} \to \mathbf{S}_D$*) is a mapping associating to each discrete state a continuous time (resp. a discrete time) dynamical system;*
- $E \subset \mathbf{Q} \times \mathbf{U}_D \times \mathbf{Q}$ *is a collection of discrete transitions;*
- $\gamma : E \to \mathbf{P}$ *is a mapping that associates a discrete output to each discrete transition;*

- $Inv : \mathbf{Q} \to 2^{X \times \mathbf{U}_D \times U \times V}$ *is a mapping called* invariant*;*
- $R : E \times X \times U \times V \rightarrow 2^X$ *is the reset mapping;*
- $G : E \to 2^{X \times U \times V}$ *is a mapping called* guard*.*

$Inv(.)$ may be used to constrain the continuous state, the continuous inputs and the continuous disturbances.

Discrete transitions, as defined by the set E, can be of different types:

- if $\sigma \in \mathbf{U}_{D_{EXT}}$, the transition is forced by a discrete disturbance and is called a **switching transition**;
- if $\sigma \in \mathbf{U}_{D_{CONTR}}$, the transition is determined by a controllable input event and is called a **controllable transition**.
- if an invariance condition is not satisfied, a transition called **invariance transition** occurs.

Switching and invariance transitions are both **uncontrollable transitions.**

Following [27], we introduce the concept of *hybrid time basis* for the temporal evolution of the system.

Definition 2 *(Time Basis) A hybrid time basis τ is an infinite or finite sequence of sets I_j, $j \in \mathbb{N}$ satisfying the following conditions:*

- $I_j = \left\{ t \in T : t_j \leq t \leq t'_j \right\}$; *unless τ is a finite sequence and I_L is the last set of the sequence, in which case it may be of the form $I_L = \{t \in T : t \geq t_L\}$ or of the form $I_L = [t_L, t'_L)$ with $t'_L < \infty$;*
- *For all j, $t_j \leq t'_j$ and for $j > 0$, $t_j = t'_{j-1}$;*

Let $\mathcal{T}$ be the set of all hybrid time bases. Given $\begin{pmatrix} \widehat{q} \\ x_0 \end{pmatrix} \in Init$ such that x_0 is in the projection of the mapping $Inv\left(\widehat{q}\right)$ onto the continuous state x, we define now an *execution of a hybrid system*, which describes its evolution in time.

Definition 3 *(Hybrid System Execution) An execution χ of a hybrid system $\mathcal{H}$, with initial state $\begin{pmatrix} \widehat{q} \\ x_0 \end{pmatrix} \in Init$, is a collection $\chi = (\widehat{q}, x_0, \tau, \sigma, q, p, u, \delta, \xi, \eta)$ with $\tau \in \mathcal{T}$, $\sigma : \tau \to \mathbf{U}_D$, $q : \tau \to \mathbf{Q}$, $p : \tau \to \mathbf{P}$, $u \in \mathcal{U}_C$, $\delta \in \mathcal{U}_d$, $\xi : T \times \mathbb{N} \to X$, $\eta : T \times \mathbb{N} \to Y$, satisfying:*

1. *Discrete evolution: the functions q and p satisfy the conditions*

$$q\left(I_0\right)=\widehat{q}; \quad q\left(I_{j+1}\right): e_j \in E, \quad e_j=\left(q\left(I_j\right), \sigma\left(I_{j+1}\right), q\left(I_{j+1}\right)\right);$$

$$p\left(I_0\right)=\gamma(\widehat{q}); \quad p\left(I_{j+1}\right)=\gamma\left(e_j\right);$$

2. *Continuous evolution: the functions ξ and η satisfy the conditions*

   - $\xi\left(t_0, 0\right)=x_0$
     $\xi\left(t_{j+1}, j+1\right) \in R\left(e_j, \xi\left(t_j^{\prime}, j\right), u\left(t_j^{\prime}\right), \delta\left(t_j^{\prime}\right)\right)$
     $\xi(t, j)=x(t)$,
     $\eta(t, j)=h\left(x(t), q(I_j)\right), \quad \forall t \in I_j$
     where $x(t)$ is the solution at time t of the dynamical system $S_h=\mathcal{S}\left(q\left(I_j\right)\right)$, with initial condition $x\left(t_j\right)=\xi\left(t_j, j\right)$, given some control input function $u \in \mathcal{U}_C$ and some disturbance function $\delta \in \mathcal{U}_d$,
   - *if $t_j<t_j^{\prime}$, then $\left(\xi(t, j), \sigma\left(I_j\right), u(t), \delta(t)\right) \in Inv\left(q\left(I_j\right)\right) \quad \forall t \in\left[t_j, t_j^{\prime}\right]$*
   - *if τ is a finite sequence and $t_j^{\prime} \neq t_L^{\prime}$, then $\left(\xi\left(t_j^{\prime}, j\right), u\left(t_j^{\prime}\right), \delta\left(t_j^{\prime}\right)\right) \in G\left(e_j\right)$.*

A classification of hybrid system executions has been introduced in [27]:

Definition 4 *A hybrid system execution is said to be*

- *trivial if $\tau=\{I_0\}$ and $t_0=t_0^{\prime}$;*
- *finite if τ is a finite sequence ending in a right closed interval;*
- *infinite if τ is an infinite sequence or $\sum_{j=0}^{card(\tau)-1} t_j^{\prime}-t_j=\infty$;*
- *Zeno, if τ is infinite but $\sum_{j=0}^{\infty} t_j^{\prime}-t_j<\infty$.*

If a discrete disturbance and a discrete control act simultaneously and an invariance condition is no longer satisfied, the evolution in time of the hybrid system is not well-defined. To make it so in this rather peculiar situation, priorities among transitions have to be established. Given the semantics of the transitions, it is natural to assign the switching transition the highest priority and the invariance transition higher priority than the controllable transition one. In this case, if any combination of the three events happens at the same time, only the one with the highest priority is considered, while the others are ignored. We therefore introduce the following assumption:

Assumption 1.1 The switching transition has the highest priority and the invariance transition has higher priority than the controllable transition.

Given a hybrid system $\mathcal{H}$ and any execution $\chi = (\widehat{q}, x_0, \tau, \sigma, q, p, u, \delta, \xi, \eta)$ of $\mathcal{H}$, we refer to the times sequence $\{t'_k\}_{k=0}^{card(\tau)-1}$ as **switching times** of χ. Moreover given an execution $\chi = (\widehat{q}, x_0, \tau, \sigma, q, p, u, \delta, \xi, \eta)$, we consider the following functions

$$\begin{aligned} y_c &: T \times \mathbb{N} \rightarrow Y \\ y_d &: T \times \mathbb{N} \rightarrow \mathbf{P} \end{aligned}$$

where $y_c(t, j) = \eta(t, j)$, $y_d(t_0, 0) = \varepsilon$ and for any $j = 1, 2, ..., card(\tau) - 1$,

$$y_d(t, j) = \begin{cases} p(I_j) & \text{if } t = t'_j, \\ \varepsilon & \text{if } t \in (t_j, t'_j). \end{cases}$$

Let $\mathcal{Y}_c$ and $\mathcal{Y}_d$ be the classes of admissible functions y_c and y_d, respectively. Given an execution $\chi = (\widehat{q}, x_0, \tau, \sigma, q, p, u, \delta, \xi, \eta)$ and $\mathbf{t} > 0$ such that $\mathbf{t} < t'_L$, being $L = card(\tau)$, set $\mathbf{j} = \max\left\{j \in \mathbb{N} : t'_j \leq \mathbf{t}\right\}$. The pair $(y_c(t, j), y_d(t, j))|_{t \in [t_0, \mathbf{t}],\, j \in \{1,2,...,\mathbf{j}\}}$ is said to be the **observed output** at time $\mathbf{t}$ related to the execution χ of the hybrid system $\mathcal{H}$.

Hybrid systems, as described in Definition 1 are a very general class of dynamical systems. To analyze structural properties of this class, such as stabilizability, reachability and observability, some "regularity" assumptions have been introduced in the literature. We require the existence of a "*minimum dwell time*" [29] before which no discrete disturbance and/or discrete input causes a discrete transition, and of a "*maximum dwell time*" [18] within which a discrete disturbance and/or discrete input is guaranteed to cause a transition.

Assumption 1.2 (Minimum and maximum dwell time) Given the hybrid system $\mathcal{H}$, there exist $\delta_m, \delta_M > 0$ such that $0 < \delta_m \leq t'_j - t_j \leq \delta_M$, $\forall j \in [0, card(\tau) - 1]$, for any execution χ of $\mathcal{H}$. The value δ_M can be finite or infinite.

Assumption 1.2 implies that all executions of $\mathcal{H}$ are non-Zeno.

2.2 Relations between Stochastic Hybrid Models and Hybrid Systems

In this section we underline some connections between hybrid systems according to Definition 1 given in Section 2.1 and some stochastic hybrid models available in the literature. First we present some particular classes of stochastic hybrid models. Then we compare the descriptive power of hybrid systems versus stochastic hybrid models.

Relatively few classes of stochastic hybrid processes are available in the literature: we briefly review Piecewise Deterministic Markov Processes (PDMPs) of [16], Switched Diffusion Processes (SDPs) of [21] and Stochastic Hybrid Systems (SHSs) of [24]. The most important difference among these models is where randomness is introduced. Some models allow diffusions to model continuous evolution (e.g. [21],[24]), while others do not (e.g. [16]). Some allow invariance transitions (e.g. [24]), while others allow switching transitions to take place at random times (e.g. using a generalized Poisson process [21]), while others again allow both (e.g. [16]).

In the following a qualitative description of PDMPs, SDPs and SHSs is offered; for a formal definition we refer to Deliverable 1.2 [13] and [32].

- **Piecewise Deterministic Markov Processes** (PDMPs) have a hybrid state space, with both continuous and discrete states. Randomness appears only in the discrete transitions. Between two consecutive transitions, the continuous state evolves according to a nonlinear ordinary differential equation. Transitions occur either when the state hits a state space boundary (invariance transitions), or in the interior of the state space (switching transitions) according to a generalized Poisson process. Whenever a transition occurs, the hybrid state is reset instantaneously according to a probability distribution that depends on the hybrid state before the transition. For a formal definition we refer to [12], [32] and [13].
- **Switching Diffusion Processes** (SDPs) have a hybrid state space, with both continuous and discrete states. The continuous state evolves according to a stochastic differential equation (SDE), while the discrete state is a controlled Markov chain (whose transitions are switching transitions). Both the dynamics of the SDE and the transition matrix of the Markov chain depend on the hybrid state. The continuous hybrid state evolves without jumps, i.e. the evolution of the continuous state can be assumed to be a continuous function of time. For a formal definition of SDPs we refer to [21], [32] and [13].
- **Stochastic Hybrid Systems** (SHSs) have a hybrid state space, with both continuous and discrete states. The continuous state obeys a SDE that depends on the hybrid state. Transitions occur when the continuous state hits the boundary of the state space (invariance transitions). Whenever a transition occurs, the hybrid state is reset instantly to a new value. The value of the discrete state after the transition is determined deterministically by the hybrid state before the transition. The new value of the continuous state, on the other hand, is governed by a probability law that depends on the last hybrid state. For a formal definition of SHSs we refer to [24], [32] and [13].

We now establish some formal relations between stochastic hybrid models previously described and hybrid systems. Given a hybrid system $\mathcal{H}$ or a stochastic hybrid model $\mathcal{H}$, we denote with $\mathcal{E}(\mathcal{H})$ the class of all executions generated by $\mathcal{H}$. It can be easily verified that PDMPs, SDPs and SHSs all fall within the hybrid systems of Definition 1, according to the Proposition below.

Proposition 1 *Given a PDMP H^{PDMP}, there exists a hybrid system $\mathcal{H}_D$ as in Definition 1, such that $\mathcal{E}(H^{PDMP}) = \mathcal{E}(\mathcal{H}_D)$, a.s. Given a SDP H^{SDP}, there exists a hybrid system $\mathcal{H}_D$ as in Definition 1, such that $\mathcal{E}(H^{SDP}) = \mathcal{E}(\mathcal{H}_D)$, a.s. Given a SHS H^{SHS}, there exists a hybrid system $\mathcal{H}_D$ as in Definition 1, such that $\mathcal{E}(H^{SHS}) = \mathcal{E}(\mathcal{H}_D)$, a.s.*

Proposition above states that the functional spaces of realizations $\mathcal{E}(\mathcal{H})$ generated by the previously described stochastic hybrid models $\mathcal{H}$, coincides with the functional spaces of executions generated by suitable hybrid systems according to Definition 1: in this sense the class of hybrid systems as in Definition 1 includes the previously considered stochastic hybrid models.

In the following we offer some informal guidelines describing how Proposition 1 works. More precisely we show how to define a suitable hybrid system, according to Definition 1 whose generated executions functional space coincide with the one of a given PDMP.

With reference to the notations introduced in Deliverable 1.2. [13] and [32], let us consider a PDMP $H^{PDMP} = ((Q, d, Inv), f, Init, \lambda, R)$. We assume that $d(q) = d$, for any $q \in Q$, since the general case may be proved by appropriately redefining the hybrid state space of hybrid systems of Definition 1. Define the following hybrid system $\mathcal{H} = (Q, P, \mathbf{U}_D, X, U, V, Y, Init', \mathbf{S}_C, \mathcal{S}, E, \gamma, Inv, R', G)$ where:

- $P = \{\varepsilon\}$;
- $\mathbf{U}_{D_{EXT}} = \{\sigma\}$; $\mathbf{U}_{D_{CONTR}} = \emptyset$;
- $X = \mathbb{R}^d$; $U = \emptyset$; $V = \emptyset$; $Y = X$; $y(q, x) = x$, for any $q \in Q$, $x \in \mathbb{R}^n$;
- $Init'$ is defined as follows: for any $A \subset Init' \cap \mathcal{B}(Init)$, A is such that $Init(A) > 0$;
- $\mathbf{S}_C$ is such that $S_i \in \mathbf{S}_C$ is defined by the ordinary differential equation $\dot{x}(t) = f(q_i, x(t))$ for any $i \in J$;
- E is defined as follows: $(q_i, \sigma, q_j) \in E$ if there exist Borel sets $A_i \subset Inv(q_i)$ and $A_j \subset Inv(q_j)$ such that $R(\{q_j\} \times A_j, \{q_i\} \times A_i) > 0$;
- $\gamma(q) = \varepsilon$ for any $q \in Q$;

- For any $(q_i, \sigma, q_j) \in E, x \in \mathbb{R}^d, R'(((q_i, \sigma, q_j), x, u, \delta)$ is defined as follows: for any $A \subset R'(((q_i, \sigma, q_j), x, u, \delta) \cap \mathcal{B}(Inv(q_j)), A$ is such that $R(\{q_j\} \times A, \{q_i\} \times \{x\}) > 0$;
- $G((q_i, \sigma, q_j)) = \{x \in \partial Inv(q_i) : R(\{q_j\} \times A, \{(q_i, x)\}) > 0$ for some Borel set $A \subset Inv(q_j)\}$;

It is easy to check that any realization of H^{PDMP} is also generated by $\mathcal{H}$ and viceversa: hence the class of executions generated by $\mathcal{H}_D$ coincide with the class of executions generated by H^{PDMP}.

The relation between the other stochastic hybrid models, i.e. SDPs and SHSs, and hybrid systems can be addressed following the approach above.

Results in Proposition 1 could be in principle of help when characterizing observability of stochastic hybrid models whose continuous disturbances are either not present or measurable.

Chapter 3

Observability of Discrete Event Dynamic Systems

DEDSs have been the topic of intense study during the past few years and many results are available in the literature about their stability properties and their observability (e.g. [15, 30, 31, 33, 34]). DEDS are characterized by a set of discrete states, a set of events enabling transitions from one discrete state to another one and by discrete outputs.

Two kinds of DEDS are of importance: the first one, known as *Moore machine* is characterized by discrete outputs associated to each discrete state, the second one, known as *Mealy machine* is characterized by discrete outputs associated to each discrete transitions. Mealy and Moore Machine define regular languages and were proven to be equivalent from the expressive power point of view. Mealy machines are traditionally considered easier to capture designers intent (even though they do not enjoy some important properties such as composability which are true for Moore machines). For this reason we focus on the approach developed in [30]. Other approaches may be used in order to appropriate define DEDS as the classical paper of Cieslak et al. in 1988 [15] or the one of Sampath et al. in 1996 [34]; we focus on the approach by Ozveren [30] since it will be applied in the next sections to the context of dynamical observers design for hybrid systems.

In [30], the model is called "*nondeterministic finite-state automaton with intermittent event observations*". The peculiarity of the model proposed by Ozveren et al. in [30] and [31] lies in the introduction of the so called "null" output occurring when the corresponding transition cannot be observed.

In the following sections, we review the formal definition of DEDS, a definition of observability and some technical results characterizing observability of DEDS. The following developments are based on [5], [2], [3] and [4].

Definition 5 *A Discrete Event Dynamic System (DEDS) is a tuple* $\mathcal{D} = (Q, U_D, P, \varphi, \phi, \gamma)$ *such that*

- $Q = \{q_i, i \in J\}$ *is the finite set of discrete states;* $J = \{1, 2, ..., N\}$, $N \in \mathbb{N}$*;*
- U_D *is the finite set of events;*
- $P = \{p_i, i \in J\} \cup \{\epsilon\}$ *is the finite set of discrete output;* ϵ *is the "null" output;*
- $\varphi : Q \times U_D \to 2^Q$ *is the transition function and specifies the set of possible states following a particular event;*
- $\phi : Q \to 2^{U_D}$ *is a map that specifies the possible events at each state;*
- $\gamma : Q \times U_D \times Q \to P$ *is the output function.*

The evolution of DEDSs are characterized by the following dynamics:

$$q(k+1) \in \varphi(q(k), \sigma(k+1)) \tag{3.1}$$

$$\sigma(k+1) \in \phi(q(k)) \tag{3.2}$$

$$p(k+1) = \gamma(q(k), \sigma(k+1), q(k+1)) \tag{3.3}$$

where $k \in \mathbb{N}$, $q(0) \in Q$ is the initial discrete state, $q(k) \in Q$ and $p(k) \in P$ are, respectively, the state and the output after the k-th input event $\sigma(k) \in U_D$.

Remark 1 *Note that DEDSs are a special subclass of hybrid systems according to Definition 1. Given a hybrid system* $\mathcal{H} = (\mathbf{Q}, \mathbf{P}, \mathbf{U}_D, X, U, V, Y, \mathbf{S}_c, S, E, \gamma, Inv, R, G)$*, the tuple* $\mathcal{D}_\mathcal{H} = (\mathbf{Q}, \mathbf{P}, \mathbf{U}_D, \gamma, E)$ *associated to* $\mathcal{H}$ *can be viewed as a DEDS.*

Given a DEDS $\mathcal{D} = (Q, U_D, P, \varphi, \phi, \gamma)$, sometimes it is useful to distinguish the case where transitions are characterized by a complete graph or not according to the following assumption:

Assumption 3.1 For any fixed $q_0 \in Q$ and for any $q \in Q$ there exists $\sigma \in \Sigma$ such that $q \in \varphi(q_0, \sigma)$ and $\sigma \in \phi(q_0)$.

This assumption will be used when characterizing observability of hybrid systems.

Given a DEDS $\mathcal{D} = (Q, U_D, P, \varphi, \phi, \gamma)$, we say that $\mathcal{D}$ is **alive** if for any discrete state q_0 there exists a transition taking the discrete state to another state, i.e. for any fixed $q_0 \in Q$ there exists at least $q \in Q$ and $\sigma \in \Sigma$ such that $q \in \varphi(q_0, \sigma)$. It is simple to check that if $\phi(q) \neq \emptyset$ for any $q \in Q$, then $\mathcal{D}$ is alive.

We can now give the definition of observability for DEDSs as done in [5].

Definition 6 *[5] An alive DEDS $\mathcal{D}$ is said to be* current–state observable *if there exists a positive integer k such that for every $i \geq k$ and for any unknown initial state $q(0) \in Q$, the state $q(i)$ can be determined from the output sequence $p(1), \ldots, p(i)$ for every possible input sequence $\sigma(1), \ldots, \sigma(i)$.*

Notice that Definition 6 is more demanding than the one introduced by Ozveren and Willsky in [30]. In particular, they consider partial observation problems defined on DEDSs with some unobservable transitions, for which the identification of the state is requested at points separated by at most a fixed number of input events.

A straightforward design of an observer producing identification of the state $q(k)$ after each output $p(k)$ can be done by computing the subset $\tilde{q}(k)$ of possible states $q(k)$ that the system $\mathcal{D}$ could have entered when the last event $\sigma(k)$ occurred. The observer $\mathcal{O}$ is a DEDS itself

$$\mathcal{O} = (Q_{\mathcal{O}}, U_{D_{\mathcal{O}}}, P_{\mathcal{O}}, \varphi_{\mathcal{O}}, \phi_{\mathcal{O}}, \gamma_{\mathcal{O}})$$

where $Q_{\mathcal{O}} \subseteq 2^Q$, $U_{D_{\mathcal{O}}} = P$, $P_{\mathcal{O}} = Q_{\mathcal{O}}$, $\gamma_{\mathcal{O}} = \varphi_{\mathcal{O}}$ and $\phi_{\mathcal{O}}$ will be defined in the following. Its dynamics are described by

$$\tilde{q}(k+1) = \varphi_{\mathcal{O}}\left(\tilde{q}(k), p(k+1)\right) \tag{3.4}$$

$$p(k+1) \in \phi_{\mathcal{O}}\left(\tilde{q}(k)\right) \tag{3.5}$$

$$\tilde{p}(k+1) = \varphi_{\mathcal{O}}\left(\tilde{q}(k), p(k+1)\right) = \tilde{q}(k+1) \tag{3.6}$$

where the input of the observer is the output $p(k)$ of $\mathcal{D}$ and $\tilde{q}(k) = \tilde{p}(k) \in Q_{\mathcal{O}}$ is the observer state (and output) and correspond to the subset of possible states $q(k)$ that the system $\mathcal{D}$ entered when the last output $p(k)$ was observed.

The observer transition function $\varphi_{\mathcal{O}}$ and the admissible event function $\phi_{\mathcal{O}}$ can be constructed by inspection of the given DEDS $\mathcal{D}$ following the algorithm for the computation of the *current-state observation tree* as described in [14]. The construction starts from the initial state $\tilde{q}(0)$: since the initial state of $\mathcal{D}$ is unknown, then $\tilde{q}(0) = Q$. When the first input event $p(1)$ is received, the observer makes a transition to the state $\tilde{q}$ corresponding to the set

$$\tilde{q} = \{q \in Q \mid \exists s \in \tilde{q}(0) \text{ and } \sigma \in \phi(s): \; q \in \varphi(s, \sigma), \text{ with } p(1) \in \gamma(s, \sigma, q)\}$$

that depends on the value of $p(1)$. In fact, the number of observer states at the second level depends on the number of possible distinct events $p(1)$. By iterating this step, one can easily construct the third level of the tree whose nodes correspond to the sets of possible states that $\mathcal{D}$ entered after the second event. Since this procedure produces at most $2^N - 1$ observer states, then the construction of the observer necessarily ends.

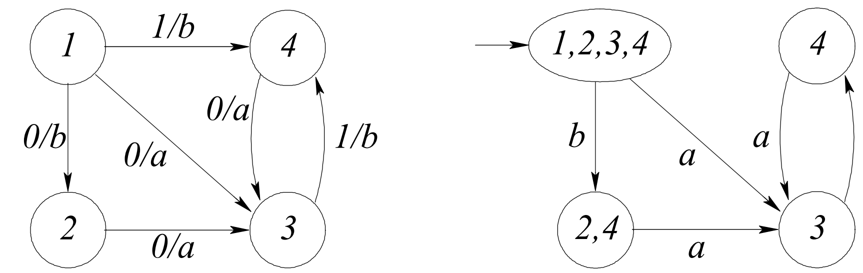

Figure 3.1: A simple DEDS $\mathcal{D}_1$ (left) and its observer $\mathcal{O}_1$ (right).

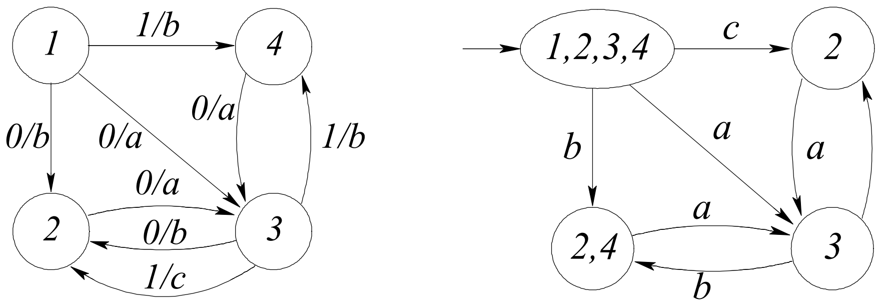

Figure 3.2: DEDS $\mathcal{D}_2$ (left) and its observer $\mathcal{O}_2$ (right).

Consider for example the DEDS $\mathcal{D}_1$ in Figure 3.1 for which $Q = \{1, 2, 3, 4\}$, $U_D = \{0, 1\}$ and $P = \{a, b\}$. The observer $\mathcal{O}_1$ of this DEDS has four states, i.e. $Q_\mathcal{O} = \{\{1, 2, 3, 4\}, \{2, 4\}, \{3\}, \{4\}\}$ (see Figure 3.1). Current–state observability can be considered by examining $\mathcal{O}$ by itself.

We now characterize current state observability for alive DEDSs. The following theorem has its origins in a result of [30].

Theorem 2 *[5] Let $\mathcal{D} = (Q, \Sigma, \Psi, \varphi, \Phi, \eta)$ be an alive DEDS. Then $\mathcal{D}$ is current–state observable if and only if for the corresponding observer $\mathcal{O}$ defined as in (3.4–3.6) there exists a nonempty set $E_\mathcal{O} \subset Q_\mathcal{O}$ of singleton states of $\mathcal{O}$ such that:*

(i) all cycles of $\mathcal{O}$ are contained in $E_\mathcal{O}$;[1]

(ii) the set $E_\mathcal{O}$ is $\varphi_\mathcal{O}$-invariant, i.e. $\displaystyle\bigcup_{\tilde{q} \in E_\mathcal{O}} \bigcup_{p \in \phi_\mathcal{O}(\tilde{q})} \varphi_\mathcal{O}(\tilde{q}, p) \subset E_\mathcal{O}$.

The algorithm reported in Figure 3.3 is based on Theorem 2 and can be used to check if a given alive DEDS $\mathcal{D}$ is current–state observable. The following notation is used in the description

[1] This condition corresponds to that of prestability of $\mathcal{O}$ with respect to the subset $E_\mathcal{O}$, as introduced in [31].

of the algorithm: $\mathcal{O}$ is the observer of the DEDS $\mathcal{D}$; $S_\mathcal{O} \subset Q_\mathcal{O}$ denotes the set of singleton states of $\mathcal{O}$ and $C_\mathcal{O} \subset Q_\mathcal{O}$ the set of states composing the cycles of $\mathcal{O}$. Moreover, $\text{Pre}(S)$ stands for the set of states of $Q_\mathcal{O}$ that can enter S in one step, i.e.

$$\text{Pre}(S) = \{\tilde{q} \in Q_\mathcal{O} | \exists p \in \phi_\mathcal{O}(\tilde{q}), \varphi_\mathcal{O}(\tilde{q}, p) \in S\}$$

BEGIN
- IF $(S_\mathcal{O} = \emptyset) \vee (C_\mathcal{O} \not\subseteq S_\mathcal{O})$ THEN
  - $\mathcal{D}$ is not current–state observable
  - RETURN
- END IF
- $E_\mathcal{O} = S_\mathcal{O}$
- WHILE $\left(\text{Pre}(\overline{E}_\mathcal{O}) \cap E_\mathcal{O} \neq \emptyset\right)$
  - $E_\mathcal{O} = E_\mathcal{O} \setminus \text{Pre}(\overline{E}_\mathcal{O})$
- END WHILE
- IF $C_\mathcal{O} \subseteq E_\mathcal{O}$ THEN
  - $\mathcal{D}$ is current–state observable
- ELSE
  - $\mathcal{D}$ is not current–state observable
- END IF

END

Figure 3.3: Algorithm to check if a given alive DEDS $\mathcal{D}$ is current–state observable

The algorithm computes the maximal set of singletons that is $\varphi_\mathcal{O}$–invariant and tests whether it contains all the cycles of the observer DEDS.

The following examples illustrate how Theorem 2 and the previous algorithm work. Consider the DEDS $\mathcal{D}_1$ in Figure 3.1. For the corresponding observer $\mathcal{O}_1$, shown in Figure 3.1, there is only one cycle, which is composed of all the singleton states of $\mathcal{O}_1$, i.e. $\{3\}$ and $\{4\}$. Moreover, $E_\mathcal{O} = S_\mathcal{O} = C_\mathcal{O} = \{\{3\}, \{4\}\}$ is $\varphi_\mathcal{O}$–invariant so that $\mathcal{D}_1$ is current–state observable. In fact, at most after two transitions one can identify the current state from the admissible outputs a and b.

Consider next the DEDS $\mathcal{D}_2$ and its observer $\mathcal{O}_2$ in Figure 3.2. The observer has four states: $Q_\mathcal{O} = \{\{1, 2, 3, 4\}, \{2, 4\}, \{2\}, \{3\}\}$. Since the cycle $\{\{2, 4\}, \{3\}\}$ of $\mathcal{O}_2$ is not entirely composed of singleton states, i.e. $C_\mathcal{O} \not\subseteq S_\mathcal{O}$, then $\mathcal{D}_2$ is not current–state observable. In fact, when $\mathcal{D}_2$ is in state 3, it is impossible to identify which state is entered when the output b is received. On the other hand, if the output c is emitted by the DEDS $\mathcal{D}_2$, then its observer $\mathcal{O}_2$ can properly conclude that $\mathcal{D}_2$ is in state 2.

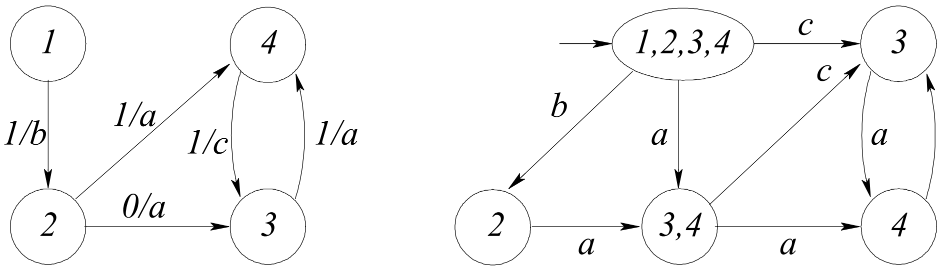

Figure 3.4: DEDS $\mathcal{D}_3$ (left) and its observer $\mathcal{O}_3$ (right).

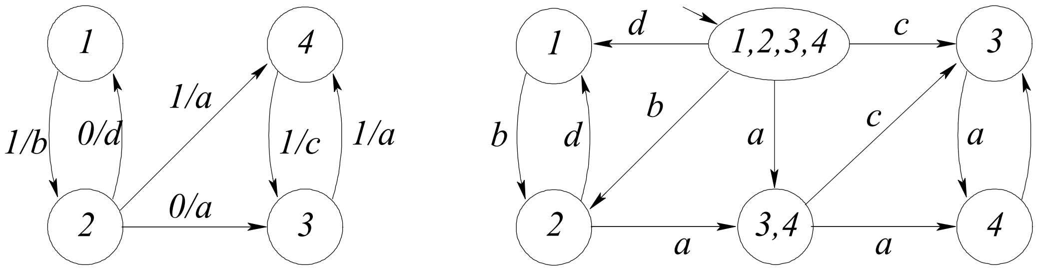

Figure 3.5: DEDS $\mathcal{D}_4$ (left) and its observer $\mathcal{O}_4$ (right).

The set $E_{\mathcal{O}}$ may be strictly contained in the set $S_{\mathcal{O}}$ of singleton states of $\mathcal{O}$. Consider for example the DEDS $\mathcal{D}_3$ and its observer $\mathcal{O}_3$ in Figure 3.4. The system $\mathcal{D}_3$ is current–state observable, as one can easily verify noting that, at most after three transitions the current state can be identified from the admissible outputs a and c. However, $E_{\mathcal{O}} = \{\{3\},\{4\}\} \subset S_{\mathcal{O}} = \{\{2\},\{3\},\{4\}\}$.

As a final example consider the DEDS $\mathcal{D}_4$ and its observer $\mathcal{O}_4$ in Figure 3.5. In this case $S_{\mathcal{O}} = C_{\mathcal{O}} = \{\{1\},\{2\},\{3\},\{4\}\}$, but the system is not current–state observable since $S_{\mathcal{O}}$ is not $\varphi_{\mathcal{O}}$–invariant. In fact, $E_{\mathcal{O}} = \{\{3\},\{4\}\}$ does not contain the set $C_{\mathcal{O}}$.

Remark 2 *A "richer" output information may help in the state identification process. For example, when also the input sequence $\sigma(k)$ is measurable, the DEDS $\mathcal{D}_2$ in Figure 3.2 becomes the DEDS $\mathcal{D}_5$ depicted in Figure 3.6, which is current–state observable as it can be verified in terms of its observer $\mathcal{O}_5$.*

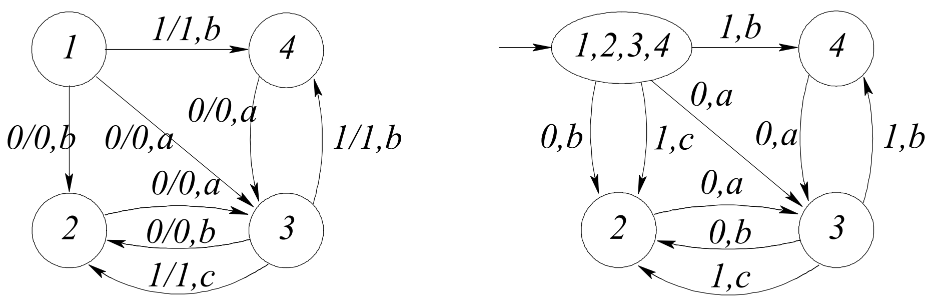

Figure 3.6: DEDS $\mathcal{D}_5$ (left) and its observer $\mathcal{O}_5$ (right).

Chapter 4

Observability of Jump Linear Systems, Piecewise Affine Systems and Switched Linear Systems

This chapter is devoted to review some definitions of observability in the context of hybrid systems that are available in the literature. The hybrid system models under consideration are jump linear systems, piecewise affine systems and switched linear systems. While in jump linear systems discrete transitions are switching transitions, in piecewise affine systems they are invariance transitions and in switched linear systems they are controllable transitions.

In particular, we survey the main results for jump linear systems due to Vidal et al. in [37], for piecewise affine systems due to Bemporad et al. in [7] and for switched linear systems due to Sun et al. in [36].

The commonality among the work of [37], [7] and [36] is that the hybrid systems considered have associated DEDS satisfying Assumption 3.1 and no discrete outputs.

When the associated DEDS does not satisfy Assumption 3.1 and/or discrete outputs are available in the hybrid system model, their knowledge gives more information for observing and recovering the hybrid state evolution. Hence, the results presented in this chapter can be considered as sufficient conditions characterizing observability of hybrid systems whose DEDS topology structure and/or discrete outputs are available. Of course, a theory that takes into account the topological structure of DEDS and the discrete outputs would yield tighter results.

4.1 Observability of Jump Linear Systems

In this section, we describe the concept of observability proposed by Vidal et al. in [37] and [38].

Jump linear systems are characterized by a hybrid state involving a continuous state and a discrete state and no control input. The evolution of the continuous state is governed by a linear ordinary differential equation while the discrete state evolution is governed by an external and unknown discrete disturbance acting on the hybrid system (or equivalently, discrete transitions are only switching transitions). We describe continuous time jump linear systems according to definitions given in [38], but by using the formalism given in Definition 1:

Definition 7 *A Jump Linear System (JLS)* $\mathcal{H}$ *is a hybrid system* $\mathcal{H} = (\mathbf{Q}, \mathbf{P}, \mathbf{U}_D, X, U, V, Y, Init, \mathbf{S}_C, S, E, \gamma, Inv, R, G)$ *where:*

1. $q_i = i, \forall i \in \mathbf{Q}$;
2. $\mathbf{P} = \{\epsilon\}$;
3. $U_{D_{CONTR}} = \emptyset$;
4. $X = \mathbb{R}^n$; $U = \emptyset$; $Y = \mathbb{R}^l$; $h(q_i, x) = C_i x$, $C_i \in \mathbb{R}^{l \times n}$, *for any* $i \in J$ *and* $x \in \mathbb{R}^n$; $\mathcal{U}_c = \emptyset$; $\mathcal{U}_d = \emptyset$;
5. $Init = Q \times X$;
6. $\mathbf{S}_c$ *is a subclass of linear, continuous time dynamical systems, and* $S_i \in \mathbf{S}_c$ *is defined by the equation* $\dot{x}(t) = A_i x(t)$, $A_i \in \mathbb{R}^{n \times n}$; $i \in J$;
7. E *is such that the associated DEDS* $\mathcal{D}_{\mathcal{H}} = (\mathbf{Q}, \mathbf{P}, \mathbf{U}_D, \gamma, E)$ *satisfies Assumption 3.1;*
8. $\gamma(e) = \epsilon$, *for any discrete transition* $e \in E$;
9. $Inv(q) = \mathbb{R}^n$ *for any discrete state* $q \in \mathbf{Q}$;
10. $R(., x, ., .) = x$ *for any continuous state* $x \in \mathbb{R}^n$;
11. $G(e) = \mathbb{R}^n$ *for any transition* $e \in E$.

With reference to Definition 7, condition 1 is introduced for the sake of notational simplicity; conditions 2 and 8 imply that there are no discrete outputs; condition 3 implies that the JLSs discrete transitions are not controllable transitions; condition 4 implies that there are no continuous inputs and no continuous disturbances acting on the system and that the continuous output function h is linear in x and depends on the current discrete state q_i; condition 6 implies that the continuous time dynamical systems involved in JLSs are linear in the continuous state x and depend on the current discrete state q_i; conditions 9 and 11 trivialize the rule of maps Inv and G, thus implying that JLSs have no invariance transitions; finally condition 10 implies that the reset mapping is the identity reset function, thus implying no jumps on the continuous state whenever a switching transition occurs.

In [38], the authors restrict their attention to non-Zeno JLSs, by assuming the existence of a minimum dwell time. Then, Assumption 1.2 holds with $\delta_m > 0$ and $\delta_M \leq \infty$.

Given a JLS execution χ, introduce the following triples $(\xi_0, \tau, \mathbf{q}) \in X \times (\cup_{\tau \in \mathcal{T}} \{\tau\} \times \mathbf{Q}^{card(\tau)})$ representing respectively the continuous initial state, the hybrid time basis and the sequence of discrete states 'visited' during execution χ. Note that the pair $(\tau, \mathbf{q})$ defines uniquely the discrete state evolution of a given execution χ.

The hybrid evolution reconstruction is equivalent to the reconstruction of the continuous initial state and of the evolution of the discrete state in time for all JLS executions. This fact led the authors to propose a notion of observability based on the concept of **indistinguishability of triples** $(\xi_0, \tau, \mathbf{q})$ generalizing classical indistinguishability concepts given for linear dynamical systems.

Remark 3 *It is simple to prove that, given a JLS with minimum dwell time* $\delta_m > 0$*, any triple* $(\xi_0, \tau, \mathbf{q}) \in X \times (\cup_{\tau \in \mathcal{T}} \{\tau\} \times \mathbf{Q}^{card(\tau)})$ *identifies uniquely an execution and viceversa. This is because an execution of JLSs is uniquely determined by the initial continuous state* ξ_0*, by the switching times, uniquely associated to* τ *and by the sequence of discrete states* $\mathbf{q}$ *'visited' during the execution.*

We now introduce the definition of observable JLSs by means of the concept of indistinguishable executions, rather than of indistinguishable triples $(\xi_0, \tau, \mathbf{q})$ as in the equivalent formulation of [38],.

Definition 8 *(Indistinguishability) Given a JLS* $\mathcal{H}$ *and* $\Delta > 0$*, two executions* χ_1 *and* χ_2 *of* $\mathcal{H}$ *are indistinguishable on the time interval* $[t_0, t_0 + \Delta]$ *if the corresponding continuous outputs are equal in the time interval* $[t_0, t_0 + \Delta]$*. We denote the class of executions which are indistinguishable from an execution* χ *as* $\mathcal{I}(\chi)$*.*

Definition 9 *(Observability) Given a JLS* $\mathcal{H}$ *and* $\Delta > 0$*, an execution* χ *of* $\mathcal{H}$ *is observable on the time interval* $[t_0, t_0 + \Delta]$ *if* $\mathcal{I}(\chi) = \{\chi\}$*. When any admissible execution is observable, we say that the JLS* $\mathcal{H}$ *is observable.*

Vidal et al. derive sufficient and necessary conditions to characterize the observability of JLSs according to the previous definition. To present the main result of [38] it is necessary to introduce the following notations:

$$\mathcal{O}_j(i) = \begin{bmatrix} C_i \\ C_i A_i \\ \vdots \\ C_i A_i^{j-1} \end{bmatrix}, \quad \mathcal{Y}_j(t_0) = \mathcal{O}_j(\widehat{q_0})\xi_0, \quad \nu = \max_{k \neq k'} \{\nu(k,k')\}, \quad \forall j \in \mathbb{N}, \forall k, k' \in J,$$

where $\nu(k,k')$ is the *joint observability index* of systems $\mathcal{S}(q_k)$ and $\mathcal{S}(q_{k'})$ and is defined as "*the minimum integer j such that the rank of the finite-dimensional joint observability matrix $O_j(k,k') = \begin{bmatrix} \mathcal{O}_j(k) & \mathcal{O}_j(k') \end{bmatrix}$ stops growing*" [38]. Moreover, given a matrix M, the matrix M^+ denotes the pseudo-inverse matrix of M defined as $M^+ = \left(M^T M\right)^{-1} M^T$.

Using Remark 3 and under Assumption 1.2, it is simple to state that a JLS is observable if and only if it is possible, on the basis of the knowledge of the observed output, to recover:

- the hybrid initial state;
- the first switching time.

The conditions outlined in [38] characterize the hybrid initial state and the first switching time recovery.

According to [38], if Assumption 1.2 holds with $\delta_m > 0$ and $\delta_M = +\infty$, it is possible to recover the hybrid initial state if and only if

$$\forall k, k' \in J, k \neq k', rank\left(\mathcal{O}_\nu(k,k')\right) = 2n. \tag{4.1}$$

Note that condition (4.1) is equivalent to the following two conditions:

$$\forall k, k' \in J, k \neq k',$$

$$rank\left(\mathcal{O}_\nu(k,k')\right) = rank\left(\mathcal{O}_\nu(k)\right) + rank\left(\mathcal{O}_\nu(k')\right); \tag{4.2}$$

$$rank\left(\mathcal{O}_\nu(k)\right) = n. \tag{4.3}$$

Condition (4.2) implies that the intersection of unobservability subspaces related to each subsystem composing a JLS is trivial, i.e., is the origin. Hence, it is essential in discrete state evolution reconstruction (for details see [38]). Condition (4.3) implies that each subsystem composing the JLS is observable. Hence, it is essential in the continuous initial state reconstruction.

According to [38], the first switching time for JLS satisfying Assumption 1.2 with $\delta_m > 0$ and $\delta_M = +\infty$ is observable if and only if

$$\forall k, k' \in J, rank\left(\mathcal{O}_\nu(k) - \mathcal{O}_\nu(k')\right) = n. \tag{4.4}$$

Note that if condition (4.4) is satisfied, whenever the first switching transition occurs ($t = t_1$), a discontinuity in the continuous output and/or its derivatives appears at time t_1. Viceversa if condition (4.4) is violated for some $k = q_0$ and $k' = q_0'$, then there exists a continuous state $\xi' \in \mathbb{R}^n$ such that $\mathcal{O}_\nu(q_0)\xi' = \mathcal{O}_\nu(q_0')\xi'$. Let $\xi_0 \in \mathbb{R}^n$, $t_1 > \delta_m$ be such that $\xi' = e^{A_{q_0}(t_1 - t_0)}\xi_0$. Let us consider two executions χ_1, χ_2 with initial hybrid state $\begin{pmatrix} \xi_0 \\ q_0 \end{pmatrix}$ and such that χ_1 has no switching transitions from the initial discrete state q_0 and χ_2 is such that a discrete transition from discrete state q_0 to q_0' at time t_1 holds. Then, since $\mathcal{O}_\nu(q_0)\xi' = \mathcal{O}_\nu(q_0')\xi'$ the switching time t_1 cannot be recovered.

Finally note that condition (4.1) implies condition (4.4). We can now present the main results of [38].

Theorem 3 *[38] Let $\mathcal{H}$ be a JLS satisfying Assumption 1.2 and $\Delta > 0$. Then $\mathcal{H}$ is observable on time interval $[t_0, t_0 + \Delta]$ if and only if condition (4.1) holds. Furthermore, the state trajectory can be uniquely recovered as:*

$$\begin{aligned} q(I_0) &= \left\{k : rank\left(\begin{bmatrix} \mathcal{O}_\nu(k) & \mathcal{Y}_\nu(t_0) \end{bmatrix}\right) = n\right\}, \\ \xi_0 &= \mathcal{O}_\nu(q(I_0))^+\mathcal{Y}_\nu(t_0), \\ t_i &= \min\left\{t > t_{i-1} : \mathcal{Y}_\nu(t^-) \neq \mathcal{Y}_\nu(t^+)\right\}, \\ q(I_i) &= \left\{k : rank\left(\begin{bmatrix} \mathcal{O}_\nu(k) & \mathcal{Y}_\nu(t_i) \end{bmatrix}\right) = n\right\}. \end{aligned} \tag{4.5}$$

The main contribution of this approach is that it is possible to characterize completely the observability properties for the class of JLSs.

The necessary and sufficient conditions of Theorem 3 for characterizing observability of jump linear systems become sufficient conditions for characterizing observability of hybrid systems whose transitions depend on the continuous state or equivalently whose discrete transitions are only invariance transitions.

If we require that

Condition A: *The JLS executions* $\chi = (\widehat{q}, \xi_0, \tau, \sigma, q, p, u, \delta, \xi, \eta)$ are such that $card(\tau) = \infty$

then, the hypotheses of Theorem 3 can be appropriately weakened. In order to fulfill Condition A, we assume the existence of a maximal dwell time according to Assumption 1.2 with $\delta_m > 0$ and $\delta_M < \infty$. Note that under Assumption 1.2 with $\delta_M < \infty$, a switching transition occurs at most every δ_M time units of permanence into each discrete state. In fact when characterizing observability for JLSs satisfying Assumption 1.2 with $\delta_M < \infty$, requiring observability of each subsystem composing the JLS is not necessary: indeed an example was presented in [38] where a JLS satisfying Assumption 1.2 with $\delta_M < \infty$, composed by two unobservable linear systems, is observable. For this reason, condition (4.1) (already present in Theorem 3), which is equivalent to conditions (4.2) and (4.3), can be weakened to condition (4.2), if Assumption 1.2 with $\delta_M < \infty$ holds.

We can then present a result, basically equivalent to Theorem 2 of [38], but with weaker assumptions.

Theorem 4 *Let $\mathcal{H}$ be a JLS satisfying Assumption 1.2 with $\delta_M < \infty$. The following results hold:*

1. *(Observability of the switching times) If condition (4.4) holds then the switching times can be uniquely recovered as the first time instances at which the continuous output is not $\mathcal{C}^\infty$ as a function of time, i.e.*

$$t_i = \min\left\{t > t_{i-1} : \mathcal{Y}_\nu(t^-) \neq \mathcal{Y}_\nu(t^+)\right\}.$$

We denote by j the number of switches in the time interval $[t_0, t_0 + \Delta]$.

2. *(Observability of the discrete state evolution) If in addition condition (4.2) holds, then the discrete state evolution can be uniquely recovered as follows:*

2.1) For the switching times t_i such that $\mathcal{Y}_\nu(t_i) \neq 0$, obtain the discrete state as

$$q(I_i) = \left\{k : rank\left(\left[\begin{array}{cc} \mathcal{O}_\nu(k) & \mathcal{Y}_\nu(t_i) \end{array}\right]\right) = rank\left(\mathcal{O}_\nu(k)\right)\right\}.$$

2.2) For the switching times t_i such that $\mathcal{Y}_\nu(t_i) = 0$,

- Compute

$$q(I_{i+1}) = \left\{k : rank\left(\left[\begin{array}{cc} \mathcal{O}_\nu(k) & \mathcal{Y}_\nu(t_{i+1}) \end{array}\right]\right) = rank\left(\mathcal{O}_\nu(k)\right)\right\};$$

- Compute

$$q(I_i) = \left\{k : rank\left(\left[\begin{array}{cc} \mathcal{O}_\nu(k) & 0 \\ \mathcal{O}_\nu(q(I_{i+1})) & \mathcal{Y}_\nu(t_{i+1}) \end{array}\right]\right) = rank\left(\left[\begin{array}{c} \mathcal{O}_\nu(k) \\ \mathcal{O}_\nu(q(I_{i+1})) \end{array}\right]\right)\right\}.$$

3. *(Observability of the initial continuous state): Under conditions (4.2) and (4.4), the initial value of the continuous state can be uniquely recovered as*

$$\xi_0 = \begin{bmatrix} \mathcal{O}_\nu(q(I_0)) \\ \mathcal{O}_\nu(q(I_1))e^{A_{q(I_0)}(t_1-t_0)} \\ \vdots \\ \mathcal{O}_\nu(q(I_j))e^{A_{q(I_{j-1})}(t_j-t_{j-1})} \cdots e^{A_{q_0}(t_1-t_0)} \end{bmatrix}^+ \begin{bmatrix} \mathcal{Y}_\nu(t_0) \\ \mathcal{Y}_\nu(t_1) \\ \vdots \\ \mathcal{Y}_\nu(t_j) \end{bmatrix}.$$

4.2 Observability of Piecewise Affine Systems

Piecewise Affine Systems are a special class of constrained hybrid systems; they are characterized by a hybrid state involving a continuous state and a discrete state. The evolution of the continuous state is governed by an affine dynamical system while the discrete state evolution is governed by invariance transitions. A formal definition of piecewise affine systems can be easily derived as a special case of general hybrid systems given in Definition 1 as follows.

Definition 10 *A PieceWise Affine system (PWA)* $\mathcal{H}$ *is a hybrid system* $\mathcal{H} = (\mathbf{Q}, \mathbf{P}, \mathbf{U}_D, \gamma, X, U, V, Y, Init\mathbf{S}_C, S, E, Inv, R, G)$ *where:*

1. $\mathbf{P} = \{\epsilon\}$;
2. $U_{D_{EXT}} = \emptyset$; $U_{D_{CONTR}} = \emptyset$;
3. $X = \mathbb{R}^n$; $U = \mathbb{R}^m$; $Y = \mathbb{R}^l$; $h(q_i, x) = C_i x + g_i$, $C_i \in \mathbb{R}^{l\times n}$, $g_i \in \mathbb{R}^l$, *for any* $i \in J$ *and* $x \in \mathbb{R}^n$; $\mathcal{U}_d = \emptyset$;
4. $Init = Q \times X$;
5. $\mathbf{S}_c$ *is a subclass of affine, discrete time dynamical systems, and* $S_i \in \mathbf{S}_c$ *is defined by the equation* $x(t+1) = A_i x(t) + B_i u(t) + f_i$, $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $f_i \in \mathbb{R}^n$; $i \in J$;
6. E *is such that the associated DEDS* $\mathcal{D}_{\mathcal{H}} = (\mathbf{Q}, \mathbf{P}, \mathbf{U}_D, \gamma, E)$ *satisfies Assumption 3.1;*
7. $\gamma(e) = \epsilon$, *for any discrete transition* $e \in E$;
8. $Inv(q) \subset X \times U$ *such that* $Inv(q) \cap Inv(q') = \emptyset$ *for any* $q, q' \in \mathbf{Q}, q \neq q'$;
9. $R(., x, ., .) = x$ *for any continuous state* $x \in \mathbb{R}^n$;
10. $G(e) = Inv(q)$ *for any transition* $e = (., ., q) \in E$.

Conditions 1 and 7, imply that there are no discrete outputs; condition 2 implies that the discrete transitions of PWA are nor switching transitions neither controllable transitions; condition 3 implies that there are no continuous disturbances acting on the system and that the continuous output function h is affine in x and depends on the current discrete state q_i; condition 4 implies that the continuous time dynamical systems involved in PWAs are affine in the continuous state x and in the continuous input u and depends on the current discrete state q_i; conditions 8 and 10 implicitly say that the pair (x, u) of continuous state x and continuous input u has to belong to the *constraining set* $\mathcal{Z} = \cup_{q \in \mathbf{Q}} Inv(q)$ and that the sequence of sets $\{Inv(q)\}_{q \in \mathbf{Q}}$ is a partition of the constraining set $\mathcal{Z}$; condition 9 implies that the reset mapping is the identity reset function, thus implying no jumps on the continuous state whenever an invariance transition occurs.

In [6], the authors introduced the class of Mixed Logical Dynamical (MLD) systems where logic based transitions are embedded into the model.

The general MLD form is:

$$x(t+1) = Ax(t) + B_1 u(t) + B_2 \delta(t) + B_3 z(t) \tag{4.6}$$

$$y(t) = Cx(t) + D_1 u(t) + D_2 \delta(t) + D_3 z(t) \tag{4.7}$$

$$E_2 \delta(t) + E_3 z(t) \leq E_1 u(t) + E_4 x(t) + E_5 \tag{4.8}$$

where $x \in \mathbb{R}^{n_c} \times \{0,1\}^{n_l}$ are the continuous and binary states, $u \in \mathbb{R}^{m_c} \times \{0,1\}^{m_l}$ are the inputs, $y \in \mathbb{R}^{p_c} \times \{0,1\}^{p_l}$ are the outputs, $\delta \in \{0,1\}^{r_l}$ and $z \in \mathbb{R}^{r_c}$ are auxiliary variables. All constraints on the continuous state x, the continuous input u, the variables δ and z are formalized in inequality (4.8): notice that inequality (4.8) is not linear since δ is an integer variable. The introduction of auxiliary variables δ and z is motivated by the necessity to transform propositional logic in linear inequalities [7].

Relations between PWAs and MLDs are very strong: indeed a PWA whose invariance sets $Inv(.)$ are polyhedral, can be written as MLD (by introducing logical δ_i variables such that $[\delta_i = 1] \longleftrightarrow \left[\begin{bmatrix} x \\ u \end{bmatrix} \in Inv(q_i) \right]$ and by imposing the *exclusive or* condition $\bigoplus_{i=1}^{card(\mathbf{Q})} [\delta_i = 1]$). Conversely a result was presented in [7] stating that a MLD is a PWA. The definition of observability for MLD (or equivalently for PWA), called **incremental observability** is centered on the possibility to reconstruct the hybrid state on the basis of the observed outputs. A MLD is said to be incrementally observable if any pair of continuous initial states and continuous inputs yields "sufficiently different" continuous outputs after a finite number of steps. The authors require that different outputs be obtained for **any** input sequence, because the hybrid observer is considered to be part of a control loop where the controller uses the results of the observer. A formal definition of the proposed concept of observable MLD is given below.

Definition 11 *Let $\mathcal{H}$ be MLD. Let $\mathcal{Z} \subset \mathbb{R}^{n_c} \times \{0,1\}^{m_l}$ be the set of initial continuous states and $\mathcal{U} \subset \mathbb{R}^{m_c} \times \{0,1\}^{m_l}$ be the set of continuous inputs. $\mathcal{H}$ is incrementally observable in Δ steps on $\mathcal{Z}$ uniformly with respect to $\mathcal{U}$ if there exist two norms $\|\cdot\|_a$ (on $\mathbb{R}^{n_c+n_l}$) and $\|\cdot\|_b$ (on $\mathbb{R}^{p_c+p_l}$) and a positive scalar w such that for any $x_1, x_2 \in \mathcal{Z}$ and for any input sequence $\{u(t)\}_{t=0}^{\Delta-1} \subset \mathcal{U}$:*

$$\sum_{t=0}^{\Delta-1} \|y(t,x_1,u) - y(t,x_2,u)\|_b \geq w\,\|x_1 - x_2\|_a$$

where $y(t,x,u)$ denotes the continuous output evolution at time t starting from the initial condition $x(0) = x$ and driven by the input $u(t)$.

Note that if a PWA is incrementally observable for some $\Delta > 0$ then it is incrementally observable for any $\Delta' > \Delta$.

In [7], the authors point out that there is no relation between incremental observability and the observability properties of each subsystem, i.e., a system consisting of unobservable subsystems can be incrementally observable and a system consisting of observable subsystems may not be incrementally observable. The following examples were used to make this point.

Example 1 *[7]* ***An Incremental Observable PWA system whose subsystems are unobservable.*** *Consider a two-dimensional PWA system composed of two linear systems:*

$$S(q_1): \begin{cases} x(k+1) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} x(k) \\ y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) \end{cases} ; \quad S(q_2): \begin{cases} x(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(k) \\ y(k) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(k) \end{cases}$$

where $Inv(q_1) = S_1 \cup S_3$ and $Inv(q_2) = S_2 \cup S_4$ and

$$\begin{aligned} S_1 &= \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 : x_1 > 0, x_1 > x_2 \right\}; \\ S_2 &= \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 : x_2 > 0, x_1 \leq x_2 \right\}; \\ S_3 &= \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 : x_1 \leq 0, x_1 > x_2 \right\}; \\ S_4 &= \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 : x_2 \leq 0, x_1 \leq x_2 \right\};. \end{aligned}$$

Note that each subsystem is unobservable. Let $\mathcal{Z}$ be a bounded subset of $S_1 \cup S_2$. If $x(0) \in S_1$ we have $y(0) = x_1(0)$ and the first component of the continuous initial state is observed. There *exists* $\Delta > 0$ *such that* $x_1(\Delta) \in S_2$ *thus implying that the state trajectory enters in* S_2*: the second component of the initial state can then be recovered using the knowledge of the continuous output* $y(.)$ *and of* Δ*. If* $x(0) \in S_2$*, the same reasoning applies. Then, the system is incrementally observable in* Δ *steps. If* $\mathcal{Z}$ *is a bounded subset of* $S_3 \cup S_4$*, the state trajectory never hits the guard. Hence, there is no transition and, since the subsystems at hand are unobservable, it is not possible to recover the initial continuous state. Therefore, the PWA system is not incrementally observable if the set of initial states is any bounded subset of* $S_3 \cup S_4$*.*

Example 2 *[7]* ***An Unobservable PWA system whose subsystems are observable.*** *Consider a two-dimensional PWA system composed of the two linear systems:*

$$S(q_1):\begin{cases} x(k+1) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(k) \\ y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) \end{cases} \quad ; \quad S(q_2):\begin{cases} x(k+1) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(k) \\ y(k) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(k) \end{cases}$$

where $Inv(q_1)$ *and* $Inv(q_2)$ *are as in the previous example. Note that each subsystem is observable. Let* $\mathcal{Z}$ *be a bounded subset of* $S_3 \cup S_4$*. Setting a continuous initial state* $x(0)$ *in* S_3*, we have* $y(0) = x_2(0)$ *and* $y(t) = 0, \forall t > 0$*. Moreover the state evolution for* $t > 0$ *is* $x(t) = [\ x_1(0) \quad 0\ ]^T$ *if* t *is even,* $x(t) = [\ 0 \quad x_1(0)\ ]^T$ *if* t *is odd, and* $x_1(0) < 0$*. Since the same reasoning can be applied in the case where the initial continuous state lies in* S_4*, we conclude that the PWA is not incrementally observable.*

These examples show that if a discrete transition from the unobservable subsystems occurs in finite time, then we may have incremental observability for the overall systems.

Theorem 1 in [7] implies that testing for incremental observability of a MLD can be done by solving a mixed-integer linear program (MILP). However, the computational complexity of the test becomes intractable as the time horizon Δ becomes large.

4.3 Observability of Switched Linear Systems

In this section we describe the concept of observability proposed by Sun et al. in [36].

Switched Linear Systems (SLS) are characterized by a hybrid state involving a continuous state and a discrete state. The continuous state evolution is governed by a linear dynamical system while the discrete state evolution is governed by a discrete input acting on the hybrid system (or equivalently discrete transitions are only controllable transitions). We describe continuous time switched linear systems by using the formalism of hybrid system given in Definition 1.

Definition 12 *A Switched Linear System (SLS)* $\mathcal{H}$ *is a hybrid system* $\mathcal{H} = (\mathbf{Q}, \mathbf{P}, \mathbf{U}_D, \gamma, X, U, V, Y, Init, \mathbf{S}_C, S, E, Inv, R, G)$ *where:*

1. $q_i = i, \forall i \in \mathbf{Q}$;
2. $\mathbf{P} = \{\epsilon\}$;
3. $U_{D_{EXT}} = \emptyset$;
4. $X = \mathbb{R}^n$; $U = \mathbb{R}^m$; $Y = \mathbb{R}^l$; $h(q_i, x) = C_i x$, $C_i \in \mathbb{R}^{l \times n}$, *for any* $i \in J$ *and* $x \in \mathbb{R}^n$; $\mathcal{U}_d = \emptyset$;
5. $Init = Q \times X$;
6. $\mathbf{S}_c$ *is a subclass of linear, continuous time dynamical systems, and* $S_i \in \mathbf{S}_c$ *is defined by the equation* $\dot{x}(t) = A_i x(t) + B_i u(t)$, $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times m}$, $i \in J$;
7. E *is such that the associated DEDS* $\mathcal{D}_{\mathcal{H}} = (\mathbf{Q}, \mathbf{P}, \mathbf{U}_D, \gamma, E)$ *satisfies Assumption 3.1;*
8. $\gamma(e) = \epsilon$, *for any discrete transition* $e \in E$;
9. $Inv(q) = \mathbb{R}^n$ *for any discrete state* $q \in \mathbf{Q}$;
10. $R(., x, ., .) = x$ *for any continuous state* $x \in \mathbb{R}^n$;
11. $G(e) = \mathbb{R}^n$ *for any transition* $e \in E$.

With reference to Definition 12, condition 1 is introduced for the sake of notational simplicity; conditions 2 and 7, imply that there are no discrete outputs; condition 3 implies that the discrete transitions of JLSs are not switching transitions; condition 4 implies that there are no continuous disturbances acting on the system and that the continuous output function h is linear in x and depends on the current discrete state q_i; condition 5 implies that the continuous time dynamical subsystems of JLSs are linear in the continuous state x and on the control law u and depend on the current discrete state q_i; conditions 8 and 10 imply that SLSs have no invariance transitions; finally condition 9 implies that the reset mapping is the identity reset function, thus implying no jumps on the continuous state whenever a controllable transition occurs.

In [36] a complete characterization of the structural properties (reachability, controllability and observability) of SLSs is developed. In this report, we focus on observability.

The question addressed in [36] is: under what conditions is it possible to observe and reconstruct the continuous initial state from the measure of the continuous outputs and continuous inputs, by designing a suitable discrete control strategy?

Definition 13 *(Observability) A SLS $\mathcal{H}$ is observable if there exist a time $T > 0$ and a discrete input sequence such that the initial continuous state ξ_0 can be determined from the knowledge of the continuous output and continuous input in time interval* $[0, T]$.

Note that requiring recovery of the continuous initial state is equivalent to reconstruct the hybrid state evolution for SLS since discrete transitions are controllable transitions. Then, if the discrete control sequence is known, the discrete evolution can be reconstructed. Finally the hybrid state evolution of SLS can be recovered from the knowledge of discrete state evolution and of the continuous initial state.

Note that by definition of SLSs and observability, a sufficient condition for a SLS to be observable is that each subsystem be observable: this is true because SLS discrete transitions are controllable transitions. However, there are as before situations where the subsystems are non observable but the overall system is.

Sun et al. derive sufficient and necessary conditions to characterize the observability of SLSs according to the previous definition. In order to give the main result of [36] it is necessary to introduce the following notations: $\mathcal{R}(M)$ is the range of a matrix M; given matrices $M_1, M_2 \in \mathbb{R}^{n\times n}$ and a subspace $\mathbf{S} \subset \mathbb{R}^n$, define

$$\begin{aligned} \Gamma_{M_1}\mathbf{S} &= \mathbf{S} + M_1\mathbf{S} + ... + M_1^{n-1}\mathbf{S} \\ \Gamma_{M_1}\Gamma_{M_2}\mathbf{S} &= \Gamma_{M_1}\left(\Gamma_{M_2}\mathbf{S}\right) \end{aligned}$$

We are ready now to introduce the main result of [36].

Theorem 5 *A SLS $\mathcal{H}$ is observable if and only if $\mathcal{O} = \mathbb{R}^n$ where $\mathcal{O} = \sum_{j=0}^{\infty} \mathcal{O}_j$ and*

$$\mathcal{O}_1 = \textstyle\sum_{q\in Q} \mathcal{R}(C_q^T); \quad \mathcal{O}_{j+1} = \sum_{q\in Q} \Gamma_{A_q^T}\mathcal{O}_j.$$

This result is based on the following duality property: a SLS is observable if and only if its dual is reachable (see [36]). The authors of [36] use their reachability results to derive Theorem 5.

The conditions of Theorem 5 involve the calculation of the subspaces $\mathcal{O}_j$ sequence, which may be prohibitive in terms of computational complexity. The authors gave an algorithm to synthesize a control based on two steps: in the first one, a periodic discrete control sequence is derived; in the second one, the continuous control is derived. Leveraging the duality properties, they show how it is possible to design a control law that involves discrete inputs only to make a SLS observable.

Chapter 5

A New Definition of Observability for Linear Hybrid Systems

Linear Hybrid Systems are hybrid systems whose subsystems are linear dynamical systems and whose reset map is given by means of a linear function of the continuous state x and depends on the current discrete state q. Discrete transitions are either switching, controllable or invariance transitions. In this chapter, we introduce the subclass of Switching linear Systems, i.e., Linear Hybrid Systems whose discrete transitions are only switching transitions. Characterizing observability for Switching linear Systems is equivalent to characterizing observability of Linear Hybrid Systems in view of Assumption 1.1 in Chapter 2.1.

Our new definition of observability is based on reconstructability of the hybrid state evolution. In this definition of observability the continuous input has to be chosen so that the continuous and the discrete state are both recoverable. In this chapter, we show that the continuous control laws enabling hybrid state reconstructability are almost all piecewise continuous functions taking value in a given input set U. This result is very important if a hybrid observer is to be included in a feedback control scheme. This approach can be seen as a generalization of the classic work on observability by Sontag et al. in [35] for observability of nonlinear systems.

This chapter contains:

- A complete characterization of observability for Switching linear Systems under Assumption 1.2 with $\delta_M = \infty$. This characterization is based on structural properties of the subsystems composing the switching system and on the topological properties of the DEDSs associated to the hybrid system under consideration.

- Weaker conditions characterizing observability of switching systems satisfying Assumption

1.2 with $\delta_M < \infty$.

5.1 Switching linear Systems

In this section, we introduce the class of Switching linear Systems (SlS) [8], [9], [10], a subclass of hybrid systems. The evolution of the continuous state is governed by a linear dynamical system while the discrete state evolution is governed by an external and unknown discrete disturbance acting on the hybrid system (or equivalently, discrete transitions are only switching transitions). Whenever a switching transition occurs, the continuous state is reset instantly to a new value which is given deterministically by a function that is linear in the continuous state x and depends on the last switching transition.

Characterizing observability for Switching linear Systems is important for the following reasons:

1. Since switching transitions are due to external and unknown disturbance acting on the plant, results characterizing observability for Switching linear Systems, can be seen as sufficient results characterizing observability for hybrid systems with invariance transitions (as in the case of piecewise affine systems) or with controllable transitions (as in the case of switched linear systems);

2. Since in this report a priority order is given on concurrent transitions (see Assumption 1.1), characterizing observability of Switching linear Systems is equivalent to characterizing observability of linear hybrid systems whose discrete transitions can be either switching, invariance or controllable transitions.

In this section, we also consider the case where the DEDS associated to the Switching linear System do not satisfy Assumption 3.1. Using the topological structure of the DEDS is useful to derive conditions characterizing observability that are less conservative with respect to the ones presented in the previous chapter.

We also consider here both continuous and discrete outputs. Continuous outputs are associated to the continuous evolution of the Switching linear Systems and are assumed to be linear in the state x and to depend on the current discrete state q. Discrete outputs are events generated every time a transition occurs. In this last case, we consider also partial observability of discrete events by introducing the null output: this is done according to the definition introduced in Chapter 3.

As before, the formal definition of Switching linear Systems, is cast in the terms specified in Definition 1

Definition 14 *A Switching linear System (SlS) is a hybrid system* $\mathcal{H} = (\mathbf{Q}, \mathbf{P}, \mathbf{U}_D, \gamma, X, U, V, Y, Init, \mathbf{S}_C, S, E, Inv, R, G)$ *where:*

1. $U_{D_{CONTR}} = \emptyset$*;*

2. $X = \mathbb{R}^n$*;* $U = \mathbb{R}^m$*;* $Y = \mathbb{R}^l$*;* $V = \mathbb{R}^d$*;* $h(q_i, x) = C_i x$*,* $C_i \in \mathbb{R}^{l\times n}$*, for any* $i \in J$ *and* $x \in \mathbb{R}^n$*;* $\mathcal{U}_d = \emptyset$*;*

3. $Init = Q \times X$*;*

4. $\mathbf{S}_c$ *is a subclass of linear, continuous time dynamical systems, and* $S_i \in \mathbf{S}_c$ *is defined by the equation* $\dot{x}(t) = A_i x(t) + B_i u(t)$*,* $A_i \in \mathbb{R}^{n\times n}, B_i \in \mathbb{R}^{n\times m}$*;* $i \in J$*;*

5. $Inv(q) = \mathbb{R}^n \times U_D \times \mathbb{R}^m \times \mathbb{R}^d$ *for any discrete state* $q \in \mathbf{Q}$*;*

6. $R(e, x, ., .) = M_e x$ *for any continuous state* $x \in \mathbb{R}^n$ *and any discrete transition* $e \in E$*, where* $M_e \in \mathbb{R}^{n\times n}, \forall e \in E$*;*

7. $G(e) = \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d$ *for any transition* $e \in E$*.*

With reference to Definition 14, condition 1 implies that the SlSs discrete transitions are neither controllable transitions nor invariance transitions; condition 2 implies that there are no continuous disturbances acting on the system and that the continuous output function h is linear in x and depends on the current discrete state q_i; condition 4 implies that the continuous–time dynamical systems involved in SlSs are linear dynamical systems whose matrices depend on the current discrete state q_i; conditions 5 and 7 imply no invariance transitions on SlSs; finally condition 6 implies that the reset mapping is linear in the continuous state x and depends on the last switching transition.

From now on, we refer to a SlS as a tuple $\mathcal{S} = \big(\mathbf{Q}, \mathbf{P}, \mathbf{U}_D, \gamma, \mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^l, \ \mathbf{S}_c, S, E, R)$ for the sake of notational simplicity. We assume the existence of a minimum dwell time δ_m and a maximum dwell time δ_M according to Assumption 1.2.

5.2 Observability of Switching linear Systems

In the following paragraphs, we build the case for a new definition of observability [18].

The definition of observability for JLSs [38] seems rather strong if applied to SlSs since it considers only the free response to reconstruct the state. In fact, consider a SlS $\mathcal{S} = (\mathbf{Q}, \mathbf{P}, \mathbf{U}_D, \gamma, \mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^l, \mathbf{S}_c, S, E, R\big)$ and let X_0 be a set of initial states such that, for any $x_0 \in X_0$, the systems in $\mathbf{S}_c$ have the same free continuous output. However, assume that all the systems in $\mathbf{S}_c$ are observable and that there exists an input $u \in \mathcal{U}_c$ and $\Delta \in (0, \delta_m)$ such that

$$\int_0^{\Delta} \|y_i(s) - y_j(s)\| \, ds > 0 \quad \forall i, j \in J$$

Then, even though $\mathcal{S}$ is not observable in the sense of [38], at time $t_j + \Delta$ the discrete state $q(I_j)$ can be determined, $\forall j \in [0, card(\tau) - 1]$, and the continuous state $\xi(t, I_j)$ can be reconstructed $\forall t \in \left(t_j + \Delta, t'_j\right]$, $\forall j \in [0, card(\tau) - 1]$.

The forced response of the system is used in the definition of observability of PWA [7]. This definition of incremental observability can be trivially extended to the class of continuous time SlS that we are considering. To better analyze the consequences of this definition, consider a SlS $\mathcal{S} = \left(\mathbf{Q}, \mathbf{P}, \mathbf{U}_D, \gamma, \mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^l, \mathbf{S}_c, S, E, R\right)$ with minimum dwell time $\delta_m > 0$. Assume that all dynamic systems in $\mathbf{S}_c$ are controllable, $\mathbb{R}^l = \mathbb{R}^n$, and suppose that the matrices C_i are nonsingular and are such that $\rho(C_i - C_j) = n$, $\forall i, j \in J$, $i \neq j$. In that case, $\forall x \in \mathbb{R}^n - \{0\}$, $C_i x \neq C_j x$. Therefore, for any pair of initial states $\begin{pmatrix} \xi \\ q_i \end{pmatrix}$ and $\begin{pmatrix} \xi \\ q_j \end{pmatrix} \in X_0 \subset \mathbb{R}^n - \{0\} \times \mathbf{Q}$, $i \neq j$, and for any input function, the output functions of the SlS $\mathcal{S}$ do not coincide, for any execution of $\mathcal{S}$. Hence, $\mathcal{S}$ is incrementally observable for any set of initial states $X_0 \subset \mathbb{R}^n - \{0\} \times \mathbf{Q}$. However, the discrete state evolution of $\mathcal{S}$ cannot be reconstructed for *any* input function. In fact, since the systems in $\mathbf{S}_c$ are controllable, for all x belonging to any subset of $\mathbb{R}^n - \{0\}$ and for any $\widehat{t} \in (t_0, t_0 + \delta_m)$ there exists an input function such that $\xi(t, I_0) = 0$, $\forall t \geq \widehat{t}$. As a consequence, it is not always possible to reconstruct the discrete state evolution, even if the state $q(I_0)$ were known. This shows that, for switching systems, the incremental observability notion, based on a distinguishability property that holds for any input, does not guarantee state reconstruction.

Consider now a definition of observability based on distinguishability of initial states from the observed output, for a *suitable* input function. The following example shows that this notion has problems too. In fact, consider the SlS $\mathcal{S} = \left(\mathbf{Q}, \mathbf{P}, \mathbf{U}_D, \gamma, \mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^l, \mathbf{S}_c, S, E, R\right)$ with $\delta_m > 0$ and $\delta_M < \infty$, where

$$\begin{aligned}
&\mathbf{Q} = \{q_1, q_2, q_3, q_4\} \\
&\mathbf{P} = \{p_1, p_2, p_3\} \\
&\mathbf{U}_D = \{\sigma\} \\
&E = \{(q_3, \sigma, q_1), (q_4, \sigma, q_1), (q_1, \sigma, q_2), (q_2, \sigma, q_1)\} \\
&\gamma((q_1, \sigma, q_2)) = p_1, \gamma((q_2, \sigma, q_1)) = p_2, \\
&\gamma((q_3, \sigma, q_1)) = \gamma((q_4, \sigma, q_1)) = p_3,
\end{aligned}$$

the systems S_1 and S_2 are observable, the dynamical matrices describing systems S_3 and S_4, are such that $A_3 = A_4$, $B_3 = B_4 = 0$ in observability canonical form, and the reset function is the identity. Any pair of initial states $\begin{pmatrix} 0 \\ q_3 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ q_4 \end{pmatrix}$ is indistinguishable, since for any input function $u \in \mathcal{U}_c$, exactly the same output functions are observed. However, after the first switching, the discrete state evolution can be uniquely determined from the discrete output and the continuous state evolution can be recovered for any continuous input function, since S_1 and S_2 are observable.

Hence, we conclude from this discussion that the notion of observability for hybrid systems should be based on state reconstructability, rather than on state indistinguishability. Following the approach developed in [3], where a discrete and a continuous observer were combined to estimate the state of hybrid plants, we propose the following definition of observability.

Definition 15 *[18] A SlS $\mathcal{S} = (\mathbf{Q}, \mathbf{P}, \mathbf{U}_D, \gamma, \mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p, \mathbf{S}_c, S, E, R)$ is **observable** if there exist a function $\varphi : \mathcal{Y}_c \times \mathcal{Y}_d \times \mathcal{U}_c \to \mathbb{R}^n \times \mathbf{Q}$, an integer $\mathbf{j} \geq 0$ and a real $\Delta \in (0, \delta_m)$ such that:*

$$\forall \begin{pmatrix} \xi_0 \\ \widehat{q} \end{pmatrix} \in \mathbb{R}^n \times \mathbf{Q}, \forall \tau \in \mathcal{T}, \ \forall \sigma \ \textit{admissible w.r.t.} \ \widehat{q}$$

there exists an execution $\chi = (\widehat{q}, \xi_0, \tau, \sigma, q, p, u, \xi, \eta)$ such that

$$\varphi \left(y_c|_{[t_0,t]}, y_d|_{[t_0,t]}, u|_{[t_0,t]} \right) = \begin{pmatrix} \xi(t,j) \\ q(I_j) \end{pmatrix}, \forall t \in \left(t_j + \Delta, t'_j\right], \forall j \in [\mathbf{j}, card(\tau) - 1].$$

Note that a continuous time SlS is said to be observable according to Definition 15 if and only if there exists a continuous control law $u(.)$, depending on the observed output at time $t = t_0$, such that the hybrid state evolution can be recovered from the observed output. Note also that this definition, when applied to dynamical linear systems, yields the classical definition of observability.

5.3 Characterizing Observability of Switching linear Systems

The results presented in this section are based on [18].

Introduce the following notation: given a Switching linear System $\mathcal{S} = (\mathbf{Q}, \mathbf{P}, \mathbf{U}_D, \gamma, \mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^l, \mathbf{S}_c, S, E, R)$ we define $F_i(t) = C_i e^{A_i t} B_i, \forall t \geq 0$ for any $q_i \in \mathbf{Q}$; moreover for any $q \in Q$, we define $succ(q) = \{q' \in Q : (q, . , q') \in E\}$.

We can now give some results characterizing observability of Switching linear Systems under Assumption 1.2 with $\delta_m > 0$ and $\delta_M = \infty$. It is simple to check that a SlS with $\delta_m > 0$ and

$\delta_M = \infty$, is observable if and only if it is possible to recover, on the basis of the knowledge of the observed output, the hybrid initial state and the first switching time.

The following result completely characterizes the recoverability of the hybrid initial state of a SlS with $\delta_m > 0$ and $\delta_M = \infty$.

Proposition 6 *Given a SlS* $\mathcal{S} = (\mathbf{Q}, \mathbf{P}, \mathbf{U}_D, \gamma, \mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p, \mathbf{S}_c, S, E, R)$ *satisfying Assumption* 1.2 *with* $\delta_m > 0$ *and* $\delta_M = \infty$*. Then the hybrid initial state is recoverable if and only if*

- $S(q_i)$ *is observable for any* $q_i \in \mathbf{Q}$*;*
- *for any* q_i, q_j*,* $q_i \neq q_j$*,* $F_i(.) \neq F_j(.)$*.*

The following result states a sufficient condition for recovering the first switching time.

Proposition 7 *Given a SlS* $\mathcal{S} =(\mathbf{Q}, \mathbf{P}, \mathbf{U}_D, \gamma, \mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p, \mathbf{S}_c, S, E, R)$ *satisfying Assumption* 1.2 *with* $\delta_m > 0$ *and* $\delta_M = \infty$*. Then the first switching time* t_1 *is recoverable if for any* $e = (q_i, ., q_j) \in E$ *such that* $\gamma(e) = \varepsilon$*,* $F_i(.) \neq F_j(.)$*.*

By combining results above we have the following Theorems.

Theorem 8 *Given a SlS* $\mathcal{S} = (\mathbf{Q}, \mathbf{P}, \mathbf{U}_D, \gamma, \mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p, \mathbf{S}_c, S, E, R)$ *satisfying Assumption* 1.2 *with* $\delta_m > 0$ *and* $\delta_M = \infty$*. Then* $\mathcal{S}$ *is observable if and only if:*

- $\mathcal{S}(q_i)$ *is observable for any* $q_i \in Q$*,*
- *For any* $q_i, q_j \in Q$*,* $q_i \neq q_j$*,* $F_i(.) \neq F_j(.)$*.*

The result above is independent from the topological structure of the DEDS associated to the SlS. If the initial discrete state is assumed to be known, then less conservative results may be obtained on the basis of the topological properties of the DEDS associated to the SlS.

Theorem 9 *Given a SlS* $\mathcal{S} = (\mathbf{Q}, \mathbf{P}, \mathbf{U}_D, \gamma, \mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p, \mathbf{S}_c, S, E, R)$ *satisfying Assumption* 1.2 *with* $\delta_M = \infty$*. Suppose that the discrete initial state is known. Then* $\mathcal{S}$ *is observable if:*

- $\mathcal{S}(q_i)$ *is observable for any* $q_i \in Q$*,*
- *for any* $q_i \in Q$*, for any* $q_j, q_k \in succ(q_i), q_j \neq q_k$ *such that* $\gamma((q_i, ., q_j)) = \gamma((q_i, ., q_k))$*,* $F_j(.) \neq F_k(.)$*;*
- *for any* $e = (q_i, ., q_j) \in E$ *such that* $\gamma(e) = \varepsilon$*,* $F_i(.) \neq F_j(.)$*.*

Here, we considered the case of $\delta_m > 0$ and $\delta_M = \infty$. We assume now the existence of a finite maximum dwell time $\delta_M < \infty$. The above property combined with the topological properties of the DEDS associated to the SlS under consideration yields less conservative results.

Given a SlS $\mathcal{S}$, whose DEDS $\mathcal{D}_\mathcal{S}$ is alive and current state observable with $K = \mathbf{k}$, we denote with $Q_\mathbf{k} \subset \mathbf{Q}$ the set of all discrete states that may be reached after the $\mathbf{k}$-th event.

Theorem 10 *Given a SlS* $\mathcal{S} = (\mathbf{Q}, \mathbf{P}, \mathbf{U}_D, \gamma, \mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p, \mathbf{S}_c, S, E, R)$ *with* $\delta_m > 0$ *and* $\delta_M < \infty$. *Then* $\mathcal{S}$ *is observable if:*

- *The DEDS* $\mathcal{D}_\mathcal{S}$ *is alive;*
- *The DEDS* $\mathcal{D}_\mathcal{S}$ *is current state observable with* $K = \mathbf{k}$*;*
- $\mathcal{S}(q_i)$ *is observable for any* $q \in Q_\mathbf{k}$*,*
- *For any* $q_i, q_j \in Q_\mathbf{k}$, $q_i \neq q_j$, $F_i(.) \neq F_j(.)$.

Chapter 6

Observers Design for Hybrid Systems

In this chapter, we describe a methodology for designing a hybrid observer that recovers the hybrid state evolution of a hybrid system on the basis of its observed output. These results are due to Balluchi et al. in [5], [2], [3] and [4]. [1]

The hybrid system under consideration is linear in the continuous dynamics and the reset map is an affine function of the continuous state before the transition and of the discrete states involved in the transition. Discrete transitions may be either switching, controllable or invariance transitions. This model is formulated in the general hybrid system framework of Definition 1:

Definition 16 *The model under consideration is a hybrid system* $\mathcal{H} = (\mathbf{Q}, \mathbf{P}, \mathbf{U}_D, X, U, V, Y, Init, \mathbf{S}_C, S, E, \gamma, Inv, R, G)$ *where:*

1. $X \subset \mathbb{R}^n$*;* $U \subset \mathbb{R}^m$*;* $Y \subset \mathbb{R}^l$*;* $V \subset \mathbb{R}^n$*;* $h(q_i, x) = C_i x$*,* $C_i \in \mathbb{R}^{l\times n}$*, for any* $i \in J$ *and* $x \in \mathbb{R}^n$*;* U_d *is the class of bounded function taking values in* V*;*

2. $Init = Q \times X$*;*

3. $\mathbf{S}_c$ *is a subclass of linear, continuous time dynamical systems, and* $S_i \in \mathbf{S}_c$ *is defined by the equation* $\dot{x}(t) = A_i x(t) + B_i u(t) + \delta(t)$*,* $A_i \in \mathbb{R}^{n\times n}, B_i \in \mathbb{R}^{n\times m}$*;* $i \in J$*;*

4. $R(e, x, ., .) = M_e x + m_e$ *for any continuous state* $x \in \mathbb{R}^n$ *and any discrete transition* $e \in E$*;*

[1] Recently, Hwang et al. in a yet unpublished report [23] applied these results to a particular class of stochastic hybrid systems. In our next deliverable, we will review these results together with our own extensions.

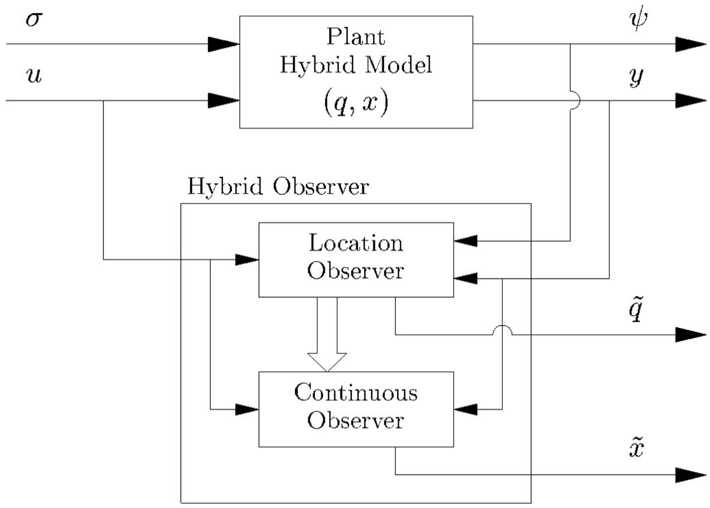

Figure 6.1: Observer structure: location observer and continuous observer.

Note that these hybrid systems are very general.

The hybrid observer $\mathcal{H}_O$ is a hybrid system itself: its task is providing an estimate $\tilde{q}(k)$ and an estimate $\tilde{x}(t)$ for the current location $q(k)$ and continuous state $x(t)$ of the hybrid plant. Its inputs are the continuous input and output, u and y_c, and the discrete output y_d.

The observers have to satisfy the following stability property:

Definition 17 *Given a hybrid system $\mathcal{H}$ as in Definition 16, a hybrid observer $\mathcal{H}_O$ is said to be* exponentially ultimately bounded *if there exists a positive integer K and constants $c \geq 1$, $\mu > 0$ and $b \geq 0$ such that*

$$\tilde{q}(k) = q(k) \qquad \forall k \geq K, \tag{6.1}$$

$$\|\tilde{x}(t) - x(t)\| \leq c\|\tilde{x}(t_K) - x(t_K)\| \; e^{-\mu t} + b \qquad \forall t > t_K. \tag{6.2}$$

for every initial hybrid state $(q(0), x(0)) \in Q \times X$, every continuous input $u(\tau)$ with $\tau \in [0, t]$, every possible input sequence $\sigma(1), \ldots, \sigma(k)$ and output sequence $p(1), \ldots, p(k)$. In (6.2), μ is the rate of convergence and b is the ultimate bound.

If $b = 0$, the observer is said to be exponentially convergent.

Given a hybrid plant with state $\begin{pmatrix} x \\ q \end{pmatrix}$, the structure of the proposed hybrid observer $\mathcal{H}_O$ is illustrated in Figure 6.1. The *location observer* describes the evolution of the discrete location of $\mathcal{H}_O$ while the *continuous observer* governs the evolution of the continuous state of $\mathcal{H}_O$.

The *location observer* receives as input the continuous input $u(t)$ and the observed output $\left(y_c|_{[t_0,\mathbf{t}]}, y_d|_{[t_0,\mathbf{t}]} \right)$. Its task is to provide the estimate $\tilde{q}(k)$ of the discrete location $q(k)$ of the hybrid plant at the current time. Based on the discrete evolution of the location observer, the *continuous observer* constructs an estimate $\tilde{x}(t)$ of the plant continuous state that converges exponentially to $x(t)$. The continuous plant input $u(t)$ and output $y_c(t)$ are used by the continuous observer for this purpose. Moreover in the following it will be assumed that the continuous disturbance $\delta(t)$ is measurable.

6.1 Observers for Current-location Observable Plant

We consider a subclass of hybrid plants $\mathcal{H}_P$ for which exponentially convergent observers can be obtained by using the results on DEDS observability described in Chapter 3 and the results of switching systems stability introduced in [22] and [29].

We assume that Assumption 1.2 holds with $\delta_m > 0$ and $\delta_M < \infty$, and that the DEDS associated to the hybrid system under consideration is current–state observable.

In the sequel we denote by $Succ(q_i)$ the set of plant locations reachable in one step from q_i, that is

$$Succ(q_i) = \{q \mid q \in Q : (q_i, ., q) \in E\} .$$

For current–location observable hybrid plants, the current location of the system can be reconstructed instantaneously by *using the discrete output information only*, no matter what the evolution of the continuous state variables is. The structure of the hybrid observer $\mathcal{H}_O$ reduces in this case to that illustrated in Figure 6.2.

Given a hybrid plant $\mathcal{H}_P$, the location observer, capturing the discrete dynamics of the hybrid observer, is described by the DEDS (3.4–3.6) obtained by computing the discrete observer for the DEDS $\mathcal{D}_{\mathcal{H}_P}$ associated to the hybrid plant $\mathcal{H}_P$. Current–state observability of $\mathcal{D}_{\mathcal{H}_P}$ ensures that the location estimate sequence $\tilde{q}(k)$ equals the plant location sequence $q(k)$ after a finite number of steps from the initial time. In fact, by Theorem 2, the location observer is guaranteed to enter after a finite number of steps a nonempty $\varphi_{\mathcal{O}}$-invariant subset $E_{\mathcal{O}} \subset Q_{\mathcal{O}}$ of singleton locations.

The continuous observer, describing the evolution of the estimate $\tilde{x}(t)$ of the plant continuous state $x(t)$, is obtained as follows:

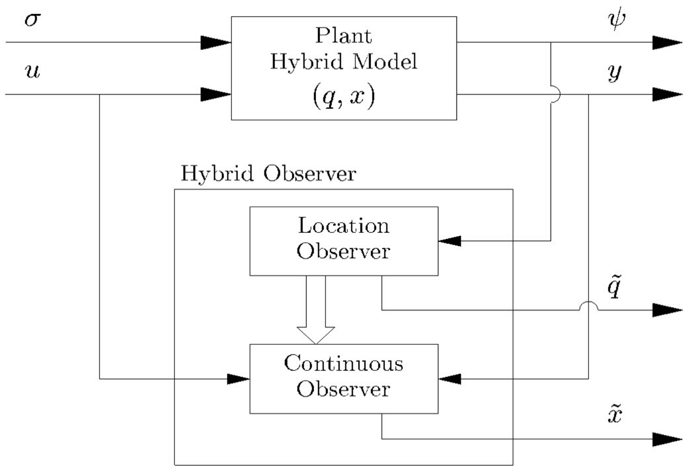

Figure 6.2: Observer structure for current-location observable hybrid plants.

1. to each location observer state $\tilde{q}$ is associated the continuous dynamics

$$\begin{cases} \dot{\tilde{x}}(t) = 0 & \text{if } \tilde{q} \in Q_{\mathcal{O}} \setminus E_{\mathcal{O}} \\ \dot{\tilde{x}}(t) = (A_i - G_i C_i)\,\tilde{x}(t) + B_i u(t) + G_i y(t) & \text{if } \tilde{q} = \{q_i\} \in E_{\mathcal{O}} \end{cases} \tag{6.3}$$

   where A_i, B_i, C_i are as in Definition 16, and the observer gain matrix $G_i \in \mathbb{R}^{n \times p}$ is the design parameter used to set the velocity of convergence in each location $\tilde{q} \in E_{\mathcal{O}}$.

2. to each location observer transition $\{q_i\} \to \{q_j\}$, with $\{q_i\} \in E_{\mathcal{O}}$ and $q_j \in Succ(q_i)$, is associated the reset

$$\tilde{x}(t_k) = \tilde{x}(t_k^+) = M_e \tilde{x}(t_k^-) + m_e \tag{6.4}$$

   where t_k denotes a switching time and M_e and m_e are as in Definition 16, being $e = (q_i, \sigma, q_j) \in E$ for some $\sigma \in U_D$.

Since DEDSs associated to hybrid systems are current–state observable, there exists a finite K such that $\tilde{q}(k) = q(k)$ for any $k \geq K$. The time t_K from which the plant location is properly identified corresponds to the time at which the location observer enters the $\varphi_{\mathcal{O}}$-invariant subset $E_{\mathcal{O}}$ of singleton locations.

By (6.3) and (6.4), for $t \leq t_K$ the continuous observer state $\tilde{x}$ is locked at the initial value $\tilde{x}(0)$, since till time t_K the location observer was not able to identify the current plant location q.

For $t \geq t_K$ a Luenberger's observer (see the survey work [26]) is implemented. The dynamic parameters of the observer are A_i, B_i, C_i and G_i and they switch synchronously with the hybrid plant switchings. Further, by (6.4), the continuous observer applies the same plant resets.

Introducing the observation error $\zeta = \tilde{x} - x$, by (6.3) and the dynamics of the continuous state x and continuous output y, for $t \geq t_K$ we have

$$\dot{\zeta}(t) = (A_i - G_i C_i)\,\zeta(t) - \delta(t) \tag{6.5}$$

with resets at switching times t_k

$$\zeta(t_k) = \zeta(t_k^+) = M_e\,\zeta(t_k^-). \tag{6.6}$$

As pointed out in [1], the stabilization of this continuous observer is more complex than the stabilization of each single dynamics in (6.3). In particular, exponential convergence of the hybrid observer is guaranteed by the following theorem that easily follows by [22] and by the assumption of current–location observability:

Theorem 11 *Given a hybrid system as in Definition 16 whose DEDS $\mathcal{D}_{\mathcal{H}_P}$ is current state observable, if no continuous state resets are present, i.e. $M_e = I, m_e = 0, \forall e \in E$, and there exist gain matrices G_i such that the Lie algebra $\{A_i - G_i C_i : i = 1, ..., N\}_{LA}$ is solvable, then the hybrid observer $\mathcal{H}_O$ is exponentially convergent.*

Since the family of matrices that generate solvable Lie algebras form a nowhere dense set, then the observer resulting from the application of the above theorem may not exhibit a robust behavior. Moreover, to the best of our knowledge, there is no methodology for the selection of matrices G_i achieving solvable Lie algebra $\{A_i - G_i C_i : i = 1, ..., N\}_{LA}$.

Easier–to–achieve sufficient conditions for exponential convergence of the hybrid observer can be obtained for hybrid plants $\mathcal{H}_P$ that exhibit a minimum dwell time δ_m between location switchings, according to Assumption 1.2. To this end let us introduce some notation: for a given square matrix A, let $\alpha(A)$ denote its spectral abscissa and let $k(A) = \|T\|\,\|T^{-1}\|$ denote the condition number with respect to inversion of the matrix T such that $T^{-1}AT$ is in the Jordan canonical form (note that T is not unique).

The next theorem gives sufficient conditions for a hybrid observer to be exponentially convergent:

Theorem 12 *Given a hybrid system $\mathcal{H}_P$ as in Definition 16, satisfying Assumption* 1.2. *with minimum dwell time $\delta_m > 0$, maximum dwell time $\delta_M < \infty$ and whose DEDS $\mathcal{D}_{\mathcal{H}_P}$ is current state observable, if for each $\{q_i\} \in E_{\mathcal{O}}$ there exists a gain matrix G_i such that:*

1. $A_i - G_iC_i$ *has distinct eigenvalues, and*

2. $\alpha(A_i - G_iC_i) + \frac{\max\{0, \log[M_i\, k(A_i - G_iC_i)]\}}{\delta_m} \leq -\mu < 0$

where $M_i = \max_{q_j \in Succ(q_i)} \|M_{(i,.,j)}\|$*, then the hybrid observer* $\mathcal{H}_O$ *is exponentially ultimately bounded – exponentially convergent in the absence of disturbance – with rate of convergence* μ*.*

Existence of observer gain matrices G_i satisfying the conditions of this theorem is guaranteed for sufficiently large dwell times δ_m and sufficiently small rates of convergence μ.

6.2 Observers for General Plants

When the evolutions of the discrete inputs and outputs of the hybrid plant are not sufficient to estimate the current location, the continuous plant inputs and outputs can be used to obtain some additional information that may be useful for the identification of the plant current location. In this case, the structure of the hybrid observer is as shown in Figure 6.1. In the following a methodology for selecting where the continuous information should be supplied and how to process it, is described. The processing of the continuous signals of the plant gives reliable discrete information only after some delay with respect to plant location switchings.

When the hybrid plant is not current–location observable, then, to estimate the current location of the plant, it is natural to try to leverage the information available from the continuous evolution of the plant. A transition of the hybrid plant can be detected by observing the corresponding change of the continuous dynamics. Continuous dynamics changes can be identified by comparing the evolution of the continuous inputs and outputs of the hybrid plant with the evolutions that correspond to the dynamics associated to the locations to be identified. In this way additional discrete signals, to be used as extra inputs to the DEDS observer, are produced. These signals are referred to as *signatures*.

6.2.1 Signatures

In order to explain the role of the signatures, consider for example a hybrid plant whose discrete behavior is represented by the system $\mathcal{D}_2$ shown in Figure 3.2, which is not current–location observable due to the cycle $\{\{2,4\},\{3\}\}$ in the observer $\mathcal{O}_2$. To distinguish between location 2 and location 4, assume that two extra discrete outputs are available, referred to as signature r_2 and signature r_4, which detect if the continuous state x is subject to the dynamics associated to either the location 2 or the location 4, respectively. Assume moreover that the signatures are

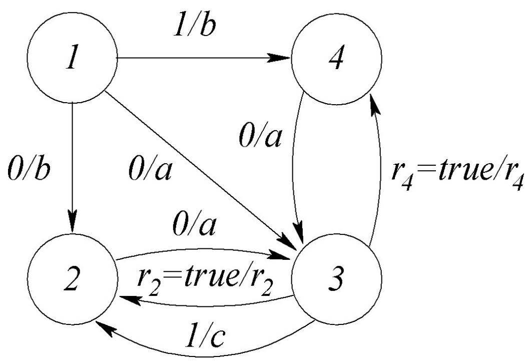

Figure 6.3:

produced possibly with some delay with respect to hybrid plant transitions to location 2 or 4, but before any subsequent transition of the hybrid plant.

A representation of the DEDS associated to the hybrid plant *composed* with the generator of the signatures r_2, r_4 can be obtained by modifying the original DEDS $\mathcal{D}_2$ as follows:

- the transition from location 3 to location 2, triggered by 0 with output b, is replaced with a transition from 3 to 2 triggered by $r_2 = true$ and with associated output label r_2;
- the transition from location 3 to location 4, triggered by 1 with output b, is replaced with a transition from 3 to 4 triggered by $r_4 = true$ and with associated output label r_4.

The new transitions may occur with a delay with respect to the original hybrid plant transitions (triggered by either 0 or 1). However, under the assumption that signatures r_2 and r_4 become true before the next transition of the hybrid plant, the sequence of plant locations is preserved.

The DEDS describing the discrete evolution of the hybrid model obtained by composing the original plant $\mathcal{D}_2$ and the signatures generator is the DEDS $\mathcal{D}_6$ shown in Figure 6.3. As it can be easily checked, the introduction of the signatures makes $\mathcal{D}_6$ current–state observable. The observer $\mathcal{O}_6$ of $\mathcal{D}_6$, obtained applying the synthesis described in Section 6.1, is shown in Figure 6.4. The observer $\mathcal{O}_6$ properly identifies the original hybrid plant location sequence, after at most 2 steps. From that time, the current location of $\mathcal{D}_2$ is identified instantaneously when the observer $\mathcal{O}_6$ uses the plant outputs a, b, c, and with some delay when it uses the outputs r_2 and r_4.

The previous example shows that, if the hybrid plant is not current–location observable, one may introduce a number of signatures detecting some of the different continuous dynamics of

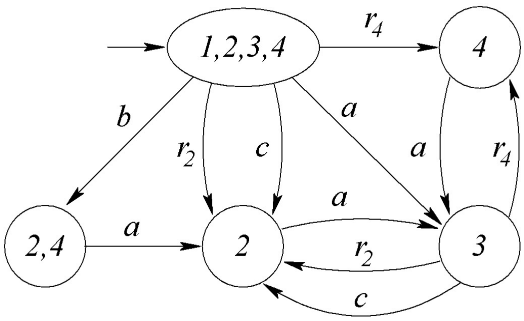

Figure 6.4:

the plant to achieve current–location observability for the combination of the hybrid plant and the signature generator. In what follows, we describe how this can be achieved in the general case.

Consider a generic hybrid plant $\mathcal{H}$ and its associated DEDS $\mathcal{D}_{\mathcal{H}}$. If the plant $\mathcal{H}$ is not current–location observable, then the observer $\mathcal{O}$ of $\mathcal{D}_{\mathcal{H}}$ does not satisfy conditions of Theorem 2. This means that the smallest $\varphi_{\mathcal{O}}$–invariant set containing all the cycles of the observer, say $F_{\mathcal{O}}$, is composed not only of singleton states. Assume that for each $\tilde{q} \in F_{\mathcal{O}} \setminus S_{\mathcal{O}}$ the continuous dynamics (A_i, B_i, C_i) associated to every $q_i \in \tilde{q}$ can be distinguished from the observation of the continuous output. Then one can introduce a signature r_i for each location $q_i \in \tilde{q}$. The DEDS describing the discrete evolution of the composition of the original hybrid plant and the signatures generator is obtained by modifying the original plant DEDS $\mathcal{D}_{\mathcal{H}}$ as follows. For each $\tilde{q} \in F_{\mathcal{O}} \setminus S_{\mathcal{O}}$ consider all the observer states $\hat{q} \in Succ(\tilde{q}) \cap F_{\mathcal{O}}$ (note that $\hat{q}$ may not be a singleton state) and replace in $\mathcal{D}_{\mathcal{H}}$ all the existing transitions from $q_j \in \hat{q}$ to $q_i \in \tilde{q}$ producing the output

$$p \in \phi_{\mathcal{O}}(\hat{q}) \text{ such that } \varphi_{\mathcal{O}}(\hat{q}, \psi) = \tilde{q}$$

with a transition from q_j to q_i triggered by $r_i = true$ and with associated output label r_i.

When all the states $\tilde{q} \in F_{\mathcal{O}} \setminus S_{\mathcal{O}}$ have been considered, the DEDS $\mathcal{D}_H^{ext}$ that describes the discrete evolution of the hybrid model obtained by composing the original plant and the signatures generator is current–state observable by construction. In this case we will say that the original hybrid plant $\mathcal{H}$ is *current–location observable via signatures*. Since the processing of the continuous input/output evolutions of the plant cannot be performed instantaneously, then the use of residual signals introduces a delay between the hybrid plant transitions and the location observer ones. Nevertheless, under the assumptions that

- the signature corresponding to the current location of the hybrid plant becomes true before

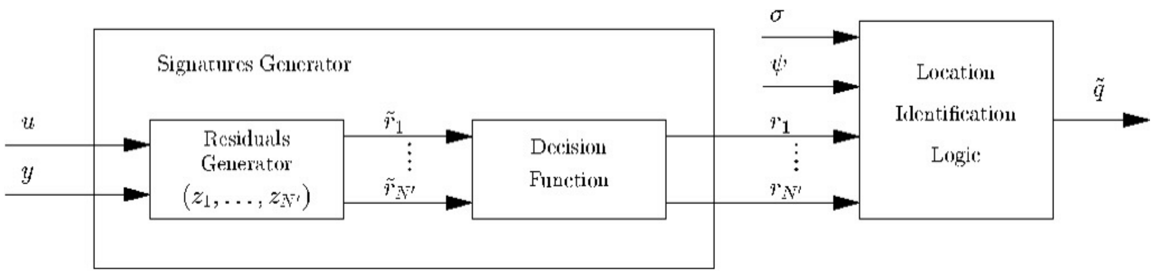

Figure 6.5: Location observer structure.

the next transition of the hybrid plant;

- all the other signatures (if any) associated to the outgoing arcs of the current observer location remain false;

the sequence of plant locations is correctly identified. The verification of these two assumptions is discussed in the next subsection where the design of the signatures generator is presented.

The complete scheme of the location observer is shown in Figure 6.5. The *signatures generator* is described below. The *location identification logic* is a discrete observer synthesized as described in Section 6.1.

6.2.2 Signatures generator

The task of the signature generator is similar to that of a fault detection and identification algorithm (see [28] for a tutorial). Indeed, the signatures generator has to decide whether or not the continuous system is obeying to a particular dynamics in a set of known ones. Assuming that the location observer has properly recognized that the hybrid plant is in location q_i, i.e. $\tilde{q} = \{q_i\}$, then the location observer should detect a fault from the evolution of $u(t)$ and $y(t)$ when the hybrid plant changes the location to some $q_j \neq q_i$ and should identify the new location q_j. The time delay in the location change detection and identification is critical to the convergence of the overall hybrid observer. We denote by Δ an upper bound for such delay.

Since, when a change of location occurs, the continuous dynamics of the plant suddenly change, then the fault detection algorithms of interest are those designed for abrupt faults [20]. The general scheme is composed of three cascade blocks: the *residuals generator*, the *decision function*, and the fault decision logic, renamed here *location identification logic*, see Figure 6.5. The *signatures generator* is the pair residuals generator–decision function. Assume that, in order to achieve current–location observability for the hybrid plant the signature generator has to detect N' different continuous dynamics given by points 1. and 2. in Definition 16, associated to a subset of states $Q' \subseteq Q$. The simplest and most reliable approach for our application is to use a bank of N' Luenberger observers (see [20]), one for each plant dynamics in Q', as residual generators:

$$\dot{z}_j(t) = (A_j - L_j C_j) z_j(t) + B_j u(t) + L_j y(t) \tag{6.7}$$

$$\tilde{r}_j(t) = C_j z_j(t) - y(t) \tag{6.8}$$

where L_j are design parameters. The N' residual signals $\tilde{r}_j$ are used to identify the continuous dynamics the plant is obeying to. Indeed, no–vanishing residuals $\tilde{r}_j(t)$ correspond to $j \neq i$. The decision function outputs N' binary signals as follows:

$$r_j = \begin{cases} \textit{true} & \text{if } \|\tilde{r}_j(t)\| \leq \varepsilon \\ \textit{false} & \text{if } \|\tilde{r}_j(t)\| > \varepsilon \end{cases} \qquad \text{for } j = 1, \ldots, N' \tag{6.9}$$

where the threshold ε is a design parameter.

As previously pointed out, to ensure correct identification of the sequence of plant locations, it is necessary that when the hybrid plant makes a transition to a location q_i (for which the use of a signature r_i is needed), the i–th signature r_i becomes *true* before the next transition of the hybrid plant. To achieve this result we impose all the signatures to become *true* within a bounded time Δ, where Δ is chosen smaller than the minimum dwell time $\delta_m > 0$ of the hybrid plant.

Proposition 13 *For a given $\Delta > 0$, $\varepsilon > 0$ and a given upper bound Z_0 on $\|x - z_i\|$, if the estimator gains L_i in (6.7) are chosen such that $A_i - L_i C_i$ have distinct eigenvalues and*

$$\frac{\alpha(A_i - L_i C_i)}{k(A_i - L_i C_i)} < -\frac{\sqrt{n}\|C_i\|\, W}{\varepsilon} \tag{6.10}$$

$$-\frac{1}{\alpha(A_i - L_i C_i)} \log \frac{k(A_i - L_i C_i)\, \|C_i\|\, Z_0}{\varepsilon + \sqrt{n}\|C_i\| \frac{k(A_i - L_i C_i)}{\alpha(A_i - L_i C_i)} W} \leq \Delta \tag{6.11}$$

then r_i becomes true before a time Δ elapses after a change in the plant dynamics parameters to the values (A_i, B_i, C_i), and it remains true till the next transition of the hybrid plant.

6.2.3 Continuous observer design

The continuous observer has to produce an estimate $\tilde{x}$ of the continuous state x of the plant on the basis of the current location q identified by the location observer.

The design of the location observer described in Section 6.2.2 is based on the DEDS $\mathcal{D}_H^{ext}$ that is obtained by appropriately modifying the DEDS $\mathcal{D}_H$ associated to the hybrid system $\mathcal{H}$. Hence, in the location observer the dependence of the hybrid plant transitions on x and u is neglected (see the expression of $\bar{\phi}$ in Definition 17). As a matter of fact, in some cases, the knowledge of this dependence can be used to improve the convergence of the observer continuous state $\tilde{x}$ to the plant continuous state x as described below.

Instantaneous continuous state identification. Since the plant discrete transitions may depend on the value of plant continuous state x through the guards modelled by mappings $G(\cdot)$, then instantaneous detection of a discrete transition of the hybrid plant can provide some information on the value of the plant continuous state x at transition time.

Suppose that, at time t_k, the hybrid plant makes a transition from the location $q(k-1) = q_i$ to the location $q(k) = q_j$. Assume that this transition is due to an event $\overline{\sigma}$ whose enabling/auto-generating condition depends on the values of the state x and the input u, i.e. $(x(t_k^-), u(t_k^-), \delta(t_k^-)) \in G((q_i, \overline{\sigma}, q_j))$.

Assume also that the event $\overline{\sigma}$ is instantaneously identified by the location observer. To this end, note that:

- Current–location observability (via signatures) guarantees that, when $k \geq K$, the new location q_j is identified.

- Instantaneous detection can only be obtained if the new location q_j is identified without the use of signatures, that is using only the hybrid plant discrete output $p(I_k)$ and location q_i.

- The event $\overline{\sigma}$ is identified iff the set valued output function η is invertible, i.e. $\sigma = \overline{\sigma}$ is the unique solution of $p(I_k) = \gamma((q_i, \sigma, q_j))$.

Then, the exact value of the plant continuous state at time t_k^- can be determined if the following system of equation has a unique solution for x:

$$\begin{array}{lcl} C_i\, x & = & y(t_k^-) \\ (x, u(t_k^-), \delta(t_k^-)) & \in & G((q_i, \overline{\sigma}, q_j)). \end{array} \tag{6.12}$$

This allows the continuous observer to jump to the current value of the plant continuous state, zeroing instantaneously the observation error. Consider, for example, the case in which a transition from location q_i to location q_j occurs when the continuous state hits the guard $D_{ij}x + E_{ij} = 0$ with $D_{ij} \in \mathbb{R}^{s\times n}$ and $E_{ij} \in \mathbb{R}^s$. Then, assuming that such transition is detected instantaneously at time t_k, the following holds:

$$\begin{bmatrix} D_{ij} \\ C_i \end{bmatrix} x = \begin{bmatrix} -E_{ij} \\ y(t_k^-) \end{bmatrix} \tag{6.13}$$

If this equation has a unique solution $x = x(t_k^-)$, then the plant's continuous state is known at time t_k^- and the observation error is zeroed.

We will denote by $\mathbf{T}$ the set of the location observer transitions $\{q_i\} \to \{q_j\}$ for which the observation error can be instantaneously zeroed.

Continuous observer scheme.

The continuous observer that describes the evolution of the estimate $\tilde{x}(t)$ of the plant continuous state $x(t)$, is basically the same introduced in Section 6.1. We denote by $E_{\mathcal{O}}$ the nonempty $\varphi_{\mathcal{O}}$–invariant subset of singleton states of the observer DEDS of the DEDS describing the discrete evolution of the composition between the hybrid plant and the signature generator, as illustrated in the previous section. Recall that, for the case at hand, observer location transitions may be not synchronous with hybrid plant discrete transitions. More precisely, the continuous observer is as follows:

1. to each location observer state $\tilde{q}$ is associated the continuous dynamics (6.3);
2. to each location observer transition $\{q_i\} \to \{q_j\} \notin \mathbf{T}$, with $\{q_i\} \in E_{\mathcal{O}}$ and $q_j \in Succ(q_i)$, is associated the reset

   $$\tilde{x}(\hat{t}_k) = \tilde{x}(\hat{t}_k^+) = M_e \tilde{x}(\hat{t}_k^-) + m_e \tag{6.14}$$

   where $\hat{t}_k$ denotes the k-th location observation transition time, and M_e, m_e are as in Definition 16;
3. to each location transition $\{q_i\} \to \{q_j\} \in \mathbf{T}$, is associated the reset

   $$\tilde{x}(\hat{t}_k) = M_e \tilde{x}(\hat{t}_k^-) + m_e$$

   where $\hat{t}_k = t_k$, i.e. the k-th location observation transition is synchronous with the hybrid plant one, and the value $\tilde{x}(\hat{t}_k^-)$ is determined as described above.

Exponential convergence of the continuous observer is analyzed considering the complete hybrid system $\mathcal{H}_P \otimes \mathcal{H}_O$ obtained by composing the hybrid model $\mathcal{H}_P$ and the observer hybrid model $\mathcal{H}_O$ as defined above, from the time t_K at which the location observer enters the $\varphi_{\mathcal{O}}$-invariant subset $E_{\mathcal{O}}$ of singleton locations.

The discrete states of the overall hybrid system $\mathcal{H}_P \otimes \mathcal{H}_O$ are of type $(q_i, \{q_j\})$, the former corresponding to plant locations $q_i \in Q$ and the latter corresponding to observer locations $\{q_j\} \in Q_{\mathcal{O}}$. The continuous dynamics of the overall hybrid system $\mathcal{H}_P \otimes \mathcal{H}_O$ that govern the evolution of the composed state (x, ζ) for locations in the subset $Q \times E_{\mathcal{O}}$ are as follows:

1. to each location $(q, \tilde{q})$ in the subset $Q \times E_{\mathcal{O}}$ are associated the continuous dynamics

$$\dot{x}(t) = A_i\, x(t) + B_i\, u(t) + w(t) \qquad \text{if } q = q_i \tag{6.15}$$

   and

$$\left\{ \begin{array}{lcll} \dot{\zeta}(t) & = & F_i\, \zeta(t) - w(t) & \text{if } \tilde{q} = \{q_i\} \\ \dot{\zeta}(t) & = & F_j\, \zeta(t) + v_{ji}(t) - w(t) & \text{if } \tilde{q} \neq \{q_i\} \end{array} \right. \tag{6.16}$$

   where $F_j = A_j - G_j C_j$ and $v_{ji}(t) = [(A_j - A_i) - G_j(C_j - C_i)]\, x(t) + (B_j - B_i)u(t)$.

2. to each discrete transition between locations in the subset $Q \times E_{\mathcal{O}}$ the following resets are applied:

   (a) for transitions $(q_j, \{q_\ell\}) \rightarrow (q_i, \{q_\ell\})$, with $q_i \neq q_j$, occurring at times $t_k \neq \hat{t}_k$, the continuous state (x, ζ) is subject to the reset

$$x(t_k) = x(t_k^+) = M_e\, x(t_k^-) + m_e \tag{6.17}$$

$$\zeta(t_k) = \zeta(t_k^+) = \zeta(t_k^-) - m_e + [I - M_e]\, x(t_k^-). \tag{6.18}$$

   (b) for transitions $(q_\ell, \{q_j\}) \rightarrow (q_\ell, \{q_i\})$, with $\{q_i\} \neq \{q_j\}$, occurring at times $\hat{t}_k \neq t_k$, only the component ζ of the continuous state is reset according to

$$\zeta(\hat{t}_k) = \zeta(\hat{t}_k^+) = M_e\, \zeta(\hat{t}_k^-) + m_e - [I - M_e]\, x(\hat{t}_k). \tag{6.19}$$

   (c) for transitions $(q_j, \{q_j\}) \rightarrow (q_i, \{q_i\})$, with $\{q_i\} \neq \{q_j\}$, occurring at times $t_k = \hat{t}_k$, the continuous state (x, ζ) is subject to the resets (6.17) and

$$\zeta(t_k) = \zeta(t_k^+) = M_e\, \zeta(\hat{t}_k^-)$$

   if $\{q_j\} \rightarrow \{q_i\} \notin \mathbf{T}$, or

$$\zeta(t_k) = 0$$

   otherwise.

Notice that the description above is complete in the sense that if the hybrid plant $\mathcal{H}_P$ and the hybrid observer $\mathcal{H}_O$ make a transition synchronously at some time $t_k = \hat{t}_k$, then necessarily the transition is between plant locations correctly identified by the hybrid observer.

Dynamics (6.15–6.16) are readily obtained from dynamics of the continuous state in Definition 17 and (6.3). While resets (6.18) and (6.19) are given by the reset map as in Definition 17 and (6.14). The following theorem gives sufficient conditions for exponential ultimate boundedness of the observation error for hybrid systems current–location observable via signatures.

Theorem 14 *Given a hybrid system $\mathcal{H}_P$ as in Definition 16 that is current–location observable via signatures, with minimum dwell time δ_m and such that matrices A_i in $\mathcal{H}_P$ have distinct eigenvalues for each i such that $\{q_i\} \in E_{\mathcal{O}}$, if for each $\{q_i\} \in E_{\mathcal{O}}$ there exists a gain matrix G_i such that*

1. *$A_i - G_iC_i$ has distinct eigenvalues;*
2. *the location observer identifies a change in the hybrid system location within time Δ with $0 \leq \Delta \leq \delta_m$;*
3. *$\alpha(A_i - G_iC_i) + \dfrac{\max\{0, \log[r_i^1\, k(A_i - G_iC_i)]}{\delta_m - \Delta} \leq -\mu < 0$*

where $R_i = \max\limits_{q_j \in Reach(q_i)} \|R_{ij}^1\|$, then the hybrid observer $\mathcal{H}_O$ is exponentially ultimately bounded with rate of convergence μ.

In the case of absence of disturbances and continuous state resets, any desired value for the ultimate bound can be achieved by choosing Δ small enough. Otherwise, the ultimate bound cannot be lower than a minimum threshold value.

Chapter 7

Conclusions

In an ATM closed loop system with mixed computer–controlled and human–controlled subsystems, recovery from non-nominal situations implies the existence of an outer control loop that has to identify these situations and act accordingly to prevent non-nominal situations to evolve into accidents. observers can, then, have an important role in controlling error propagation. A fairly comprehensive model of an ATM should be based on stochastic hybrid systems, a class of hybrid systems for which few results are available.

In this report, we presented in a unified framework the results available in the literature on observability of hybrid systems as a first step in developing a theory of observability and algorithms for observer design that can be applied to error propagation control in ATM. We began our report by describing in formal terms hybrid systems to construct the framework where the future work on error propagation control will be carried out. Then we defined and reviewed the work on hybrid stochastic systems. Observability has come to the attention of the hybrid control community only recently. Hence, much work remains to be done to solidify the field. In our report, we reviewed the literature on observability and observers for hybrid systems as a first step in our quest for a general hybrid system observer. We then illustrated synthesis methods for hybrid observers.

Our future work is two-pronged: we will do research to develop theory and algorithms for observability of hybrid systems with particular attention to the stochastic case and we will address some examples in the ATM domain to show the applicability of our methods to solve problems of interest to the ATM community.

Acknowledgments: We wish to thank A. Balluchi, L. Benvenuti and A. Sangiovanni-Vincentelli for very helpful discussions on the content of this report.

Bibliography

[1] A. Alessandri and P. Coletta, Design of Luenberger Observers for a Class of Linear Systems, Systems: Computation and Control, Springer–Verlag, vol. 2034, LNCS, Berlin Heidelberg, pp. 7–18, 2001.

[2] A. Balluchi, L. Benvenuti, M.D. Di Benedetto, A.L. Sangiovanni- Vincentelli: A Observer for the Driveline Dynamics, ECC 2001, pp.618–623, Porto (Portugal), Sept. 4–7, 2001.

[3] A. Balluchi, L. Benvenuti, M.D. Di Benedetto, A.L., Sangiovanni-Vincentelli, Design of Observers for Hybrid Systems, In Lecture Notes in Computer Science 2289, C.J. Tomlin and M.R. Greensreet Eds., pp. 76-89, Springer-Verlag, 2002.

[4] A. Balluchi, L. Benvenuti, M.D. Di Benedetto, A.L., Sangiovanni-Vincentelli, Dynamical Observers for Hybrid Systems. Submitted for publication, October 2002.

[5] A. Balluchi and L. Benvenuti and A. L. Sangiovanni-Vincentelli, Observers for hybrid systems with continuous state resets, *Proc. 10th Mediterranean Conference on Control and Automation - MED2002*, Lisbon, Portugal, July 2002.

[6] A. Bemporad and D. Mignone and M. Morari, Moving horizon estimation for hybrid systems and fault detection, *Proceedings of the 1999 American Control Conference*, 4, pp.2471-2475, San Diego, CA, USA, 1999.

[7] A. Bemporad and G. and Ferrari-Trecate and M. Morari, Observability and controllability of piecewise affine and hybrid systems, *IEEE Transactions on Automatic Control,* vol. 45, 10, pp. 1864–1876, October, 2000.

[8] L. Berardi, E. De Santis, M.D. Di Benedetto: Control of switching constrained systems under state and input constraints. Invited paper, *European Control Conference '99*, August 31 – Sept. 3, Karlsruhe, Germany, 1999.

[9] L. Berardi, E. De Santis, M.D. Di Benedetto: A structural approach to the control of switching systems with an application to engine control. Invited paper, it 38th IEEE Conference on Decision and Control, Phoenix, AZ, Dec. 7–10, 1999.

[10] L. Berardi, E. De Santis, M.D. Di Benedetto: Invariant sets and control synthesis for switching systems with safety specifications. *Hybrid Systems: Computation and Control*, N.Lynch and B.H.Krogh, Eds., Lecture Notes in Computer Science, vol. 1790, pp. 59–72, Springer–Verlag, 2000.

[11] L. Berardi, E. De Santis, M.D. Di Benedetto, systems with safety specifications: procedures for the computation and approximations of controlled invariant sets, Department of Electrical Engineering, University of L'Aquila, *Research Report no. R.03-66*, Jan. 2003, http://www.diel.univaq.it/tr/web/web_search_tr.php (submitted).

[12] M.L. Bujorianu and J. Lygeros, Reachability questions in piecewise deterministic markov processes, Hybrid Systems: Computation and Control, Springer–Verlag, vol. 2623, LNCS, Berlin Heidelberg, pp. 126-140, 2003.

[13] M.L. Bujorianu, J. Lygeros, W. Glover and G. Pola, A Stochastic Hybrid System Modeling Framework, Deliverable 1.2, Project IST-2001-32460 HYBRIDGE, February 1, 2003, http://www.nlr.nl/public/hosted-sites/hybridge.

[14] P. E. Caines and R. Greiner and S. Wang, Dynamical logic observers for finite automata, *Proceedings of 27th Conference on Decision and Control*, pp. 226–233, Austin, TX, 1988.

[15] R. Cieslak, C. Desclaux, A. Fawaz, P. Varaja, Supervisory control of discrete event processes with partial informations, *IEEE Transactions on Automatic Control*, vol.33, pp. 249-260, 1988.

[16] M.H.A. Davis, Markov Processes and Optimization, Chapman & Hall, London, 1993.

[17] E. De Santis, M.D. Di Benedetto, M.D., G. Pola, Equivalence entre stabilisabilité et sécurité pour les systèmes à commutation. *CIFA 2002*, Conférence Internationale Francophone d'Automatique, Nantes, France, 8-10 July 2002.

[18] E. De Santis, M.D. Di Benedetto, M.D., G. Pola, On observability and detectability of continuous–time linear switching systems. Submitted to CDC 2003.

[19] M.D. Di Benedetto, G. Pola, Inventory of Error Evolution Control Problems in Air Traffic Management, Deliverable 7.1, Project IST-2001-32460 HYBRIDGE, November 4, 2002.

[20] P. Frank, Fault diagnosis in dynamic systems using analytical and knowledge–based redundancy — a survey and some new results. *Automatica*, 26(3):459–474, 1990.

[21] M.K. Ghosh, A. Araphostathis, S.I. Marcus, Ergodic control of switching diffusions, Siam Journal Control Optimization, vol. 35, 6, pp. 1952–1988.

[22] J. Hespanha and A. Morse, Stability of switched systems with average dwell-time. In Proceedings of the 38th IEEE Conference on Decision and Control, CDC1999, vol. 3, pp. 2655–2660, Phoenix, AZ, USA, December 1999.

[23] I. Hwang, H. Balakrishnan, C. Tomlin, Observability criteria and estimator design for stochastic linear hybrid systems. Preprint, October 2002.

[24] J. Hu, Lygeros, S. Sastry, Toward a theory of stochastic hybrid system. Hybrid Systems: Computation and Control, Springer–Verlag, Nancy Lynch and Bruce H. Krogh Eds, pp. 160-173 in LNCS, 2000.

[25] R.E. Kalman, A new approach to linear filtering and prediction problems, *Transactions of the ASME – Journal of Basic Engineering*, Vol. D, pp. 35-45, 1960.

[26] D.G. Luenberger, An introduction to observers, *IEEE Transactions on Automatic Control*, vol.16, 6, pp. 596-602, Dec, 1971.

[27] J. Lygeros, C. Tomlin, S. Sastry, Controllers for reachability specifications for hybrid systems, *Automatica*, Special Issue on Hybrid Systems, vol. 35, 1999.

[28] M.A. Massoumnia, G.C., Verghese, and A. Willsky. Failure detection and identification. *IEEE Transactions on Automatic Control*, 34(3):316–21, March 1989.

[29] A.S. Morse, Supervisory control of families of linear set-point controllers- part 1: exact matching. *IEEE Trans. on Automatic Control*, vol. 41, 10, pp. 1413-1431, October 1996.

[30] C. M.Ozveren and A. S. Willsky, Observability of discrete event dynamic systems, *IEEE Trans. on Automatic Control*, 35, 7, 797-806, July, 1990.

[31] C. M. Ozveren and A. S. Willsky and P. J. Antsaklis, Stability and stabilizability of discrete event dynamic systems, *Journal of the Association for Computing Machinery*, 38, 3, pp. 730-752, July, 1991.

[32] G. Pola, M.L. Bujorianu, J. Lygeros, M.D. Di Benedetto, Stochastic hybrid models: An Overview, *IFAC Conference on Analysis and Design of Hybrid Systems*, ADHS03, St. Malo, 16-18 June 2003.

[33] P. J. Ramadge, Observability of discrete event–systems, *Proceedings of 25th Conference on Decision and Control*, pp. 1108–1112, Athens, Greece, 1986.

[34] M. Sampath, R. Sengupta, S. Lafortune, K. Sinnamohideen, D. Teneketzis, Failure diagnosis using discrete-event models, *IEEE Trans. Cont. Syst. Tech.*, vol. 4,2, pp. 105-126, March 1996.

[35] E.D. Sontag, On the observability of polynomial systems, I: finite–time problems, *SIAM J. Control and Optimization*, vol. 17, 1, pp. 139–151, 1979.

[36] Z. Sun, S.S. Ge and T.H. Lee, Controllability and reachability criteria for switched linear systems, *Automatica*, Vol. 38, pp. 775-786, 2002.

[37] R. Vidal and A. Chiuso and S. Soatto, Observability and identificability of jump linear systems, *Proceedings of 41st Conference on Decision and Control*, CDC2002, Las Vegas, 2002.

[38] R. Vidal and A. Chiuso and S. Soatto and S. Sastry, Observability of linear hybrid systems, Hybrid Systems: Computation and Control, Springer–Verlag, vol. 2623, LNCS, Berlin Heidelberg, pp. 526-539, 2003.