HYBRIDGE

Distributed Control and Stochastic Analysis of Hybrid Systems
Supporting Safety Critical Real-Time Systems Design

WP2: Stochastic hybrid systems based modelling of accident risk

Stochastic analysis background of accident risk assessment for Air Traffic Management

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Abstract

Due to the increasing dependency of human society on worldwide networks, the management of safety critical activities tends to become more complex, and Air Traffic Management forms a typical example of this phenomenon. Air Traffic Management (ATM) involves interactions between multiple human operators, procedures and technical systems, all of which are highly distributed. This yields that providing safety is more than making sure that each of the ATM elements functions properly safe; it is the complex interaction between them that determines safety. The assessment of isolated indicators falls short in covering the complex interactions between procedures, human operators and technical systems in safety-critical non-nominal situations. To improve this situation, this paper develops a hybrid state space modelling approach towards the assessment of an ATM operational concept on mid-air collision risk.

CONTENTS

1. Introduction

Due to the increasing dependency of human society on worldwide networks, the management of safety critical activities tends to become more complex and Air Traffic Management forms a typical example of this phenomenon. One of the main objectives of Air Traffic Management (ATM) is to guarantee that flight trajectories stay well separated. Throughout the years, this has led to the development of a balanced set of spacing and separation criteria for various kinds of air traffic environments. A typical example of a spacing criterion is the minimal distance required between two parallel runways in order to use them independently of each other. Another typical example of a spacing criterion is minimum vertical distance required between two flight levels that are allocated to different aircraft. In addition to spacing criteria that apply to planned flight paths, there are separation criteria that apply to the actual flight paths. A typical separation criterion is the minimum horizontal distance to be maintained by air traffic control between two flights that have been allocated to the same flight level. Currently, this is 5 Nm (9.26 km) in controlled En-route airspace, and 3 Nm (5.56 km) in the terminal airspace around airports. Other examples of separation criteria are the minimal time lag required between aircraft that respectively fly behind each other on the same assigned route and flight level, or that land behind each other at the same runway.

During the last fifty years, many spacing and separation criteria have been developed, often co-ordinated at international level through the International Civil Aviation Organisation (ICAO). For the development of these criteria, both the demand for air traffic and the technological status in ATM have been taken into account. In view of the increasing air traffic demand and the ongoing technological and organisational development in ATM, there is an ongoing need to continue with the development of the set of safe spacing and safe separation criteria. A typical example of such a development is the reduction of minimum vertical spacing criterion between two flight levels in upper airspace from 2000 to 1000 ft (e.g. since January 2002 in Europe and planned for January 2005 in USA). In theory, this potentially doubled the upper airspace air traffic capacity; in practice, this started a process to implement organisational changes that step by step allow the most effective realisation of potential capacity improvements.

ATM is the result of complex interactions between human operators, procedures (including spacing and separation criteria), and technical systems (hardware and software) all of which are highly distributed. Providing safety is more than making sure that each of these elements function properly and safely. The complex interactions between the various elements of ATM significantly determine safety. Therefore, it is imperative to understand the safety impact of these interactions, particularly in relation to non-nominal situations. Traditional ATM design approaches tend first to design advanced ATM that provides sufficient capacity, and next to extend the design with safety features. The advantage of this approach is that ATM developments can be organised around the clusters of individual elements, i.e., the communication cluster, the navigation cluster, the surveillance cluster, the automation tools cluster, the human machine interfaces (HMIs), the advanced procedures, etc. The disadvantage of this traditional approach is that it fails to address the impact of complex interactions on safety.

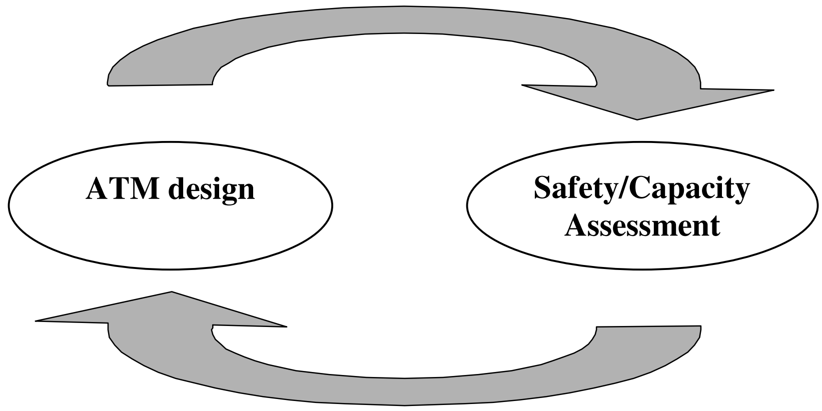

Figure 1 Safety feedback based ATM design.

A more effective approach is to design an ATM operational concept that is inherently safe at the capacity-level required. From this perspective, safety assessment should be one of the primary filters in ATM concept development. An early filtering of ATM design concepts on safety grounds can potentially avoid a costly development program that turns out to be ineffective, or an even more costly implementation program that fails. Although understanding this idea is principally not very difficult, it can be brought into practice only when an ATM safety assessment approach is available that provides appropriate feedback to the ATM designers at an early stage of the concept development (Figure 1). This feedback should not only provide information on whether the design is safe enough, but it should also identify the safety-capacity bottlenecks. By now, consensus is building that appropriate ATM safety modelling approaches are needed to understand the mechanisms behind designing advanced ATM. It is also recognised that, once such an ATM safety modelling approach is available, a safety feedback based design approach of future ATM will become feasible (Haraldsdottir et al., 1997), (Odoni et al., 1997), (Eurocontrol, 1998).

For air traffic, the probability of a fatal mid air collision during a flight should be on the order of 10^{-8} per flight. To develop some feeling of the difficulty to assess such rare events, it is quite helpful to understand why the well-known fast-time air traffic simulators like Total Airspace and Airport Modeller (TAAM), National Airspace Systems Performance Analysis Capability (NASPAC) and Reorganised ATC Mathematical Simulator (RAMS) fall short for that purpose. One major limitation of these tools is that they are not really capable of modelling the aviation safety-critical combinations of non-nominal events; they often do not even model the single non-nominal events. Another major shortcoming is that an accident rate of, say, 10^{-8} per flight cannot in a practically reasonable way be reached through a straightforward simulation, because this would require a simulation of more than 10^9 aircraft flights including the interactions and non-nominal events. This problem is well illustrated by the ATM safety iceberg (Figure 2).

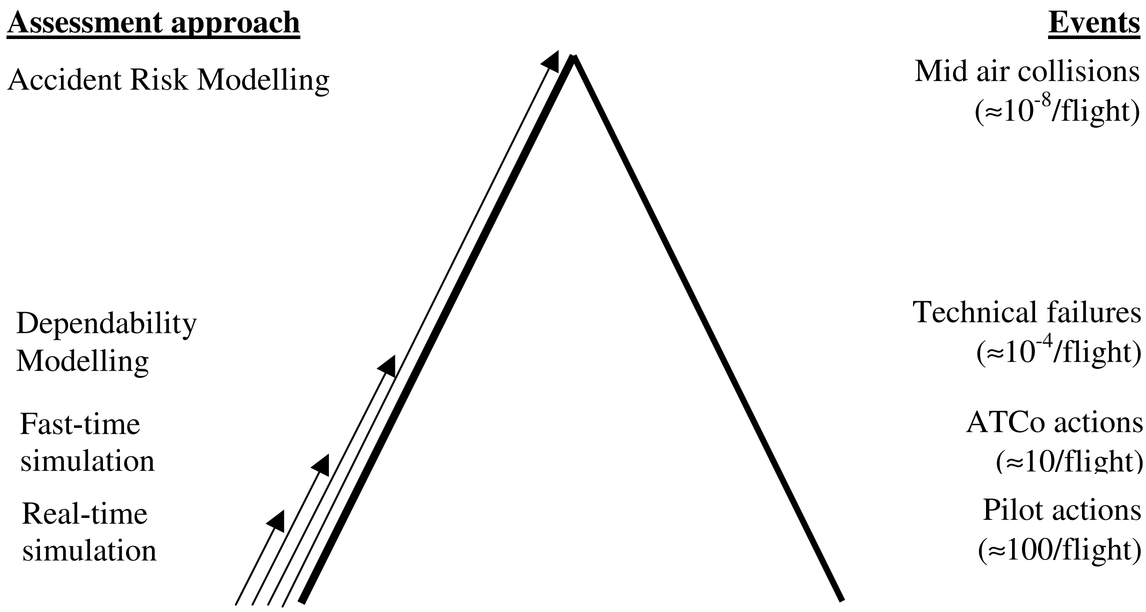

Figure 2 ATM safety iceberg.

To assess a catastrophic accident rate, one really needs to decompose the risk assessment problem into an effective hierarchy of simpler conditional assessment problems, in which simplicity means an appropriate combination of scope (e.g., volume of airspace) and depth (i.e., level of model detail) at each conditional assessment level. Indeed, fast-time air traffic simulation tools apply to assessments that address a broad scope in combination with a low level of non-nominal detail. In view of the limitations of fast-time simulation, for applications to air traffic management there is need of a mid-air collision risk assessment approach

In general, the accident risk assessment problem has been widely studied for other safety-critical operations, such as the nuclear and chemical industries, and for these applications, numerous techniques and tools have been developed. Established techniques vary from qualitative hazard identification methods such as preliminary hazard analysis (PHA), common cause analysis (CCA) and failure mode and effect analysis (FMEA), through static assessment techniques such as fault tree analysis (FTA) and event tree analysis (ETA), to dynamic assessment techniques such as Petri net and Markov chain modelling and dynamic event trees (e.g. Aldemir et al., 1994), and the incorporation of human reliability models (e.g. Kirwan, 1994). Each of these techniques has advantages and disadvantages, but these appear to be minor in comparison to what is required for modelling ATM-related risk. The key finding is that the established techniques fail to support a systematic approach toward modelling stochastic dynamical behaviour over time for complex interactions of highly distributed ATM (see Fig. 3). The established techniques would therefore force one to adopt a rather heuristic type of argumentation in trying to capture the complex interactions inherent to ATM.

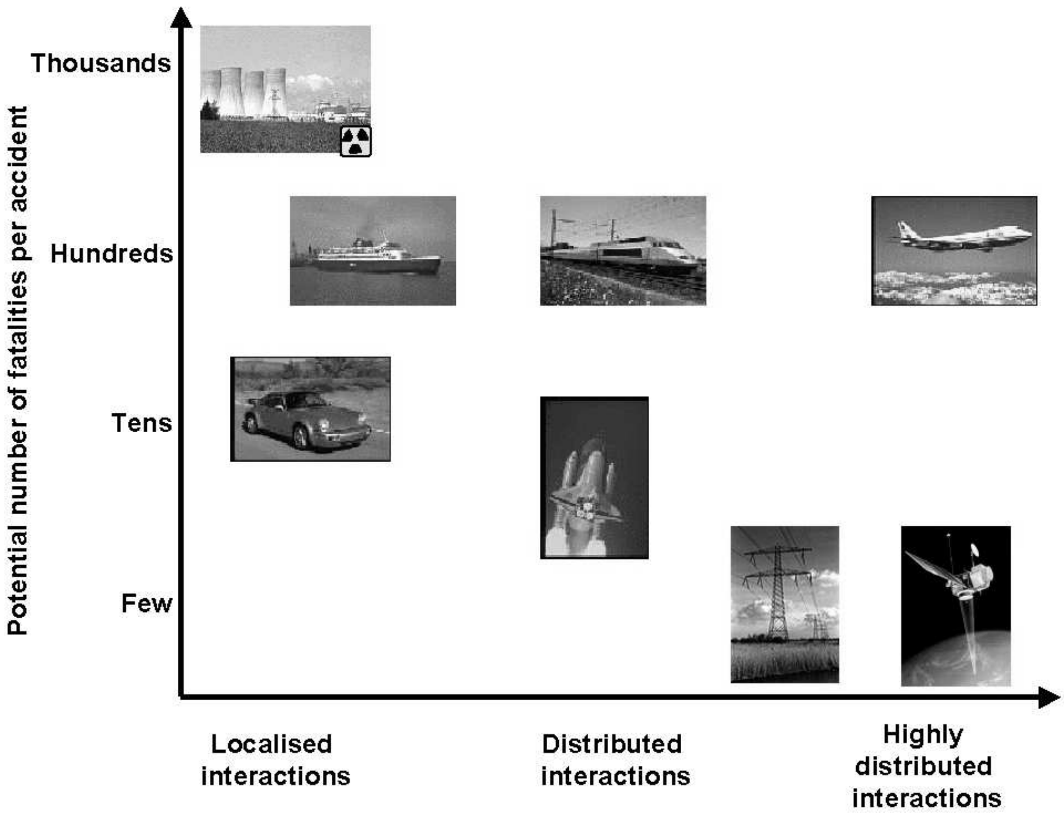

Figure 3 Potential fatalities and distribution level of ATM and other safety critical activities.

Hampered by the limitations of the established safety assessment techniques, significant extensions beyond this have also and successfully been developed for other safety critical industries, into the following two directions:

- Human cognition performance modelling, context dependent modelling (e.g. Hollnagel, 1993; Cacciabue, 1998). Within the advanced ATM development context, this direction is being studied and explored in a number of studies, e.g. Corker (2000), Daams et al. (2000), Blom et al. (2001b), Isaac & Pounds (2001), Shorrock & Kirwan (2002).
- Exploitation of hybrid state Markov process models. The particular processes studied are ordinary differential equations (ODE) with switching coefficients, such that the resulting hybrid state process is Markov. For risk evaluation of this class of hybrid state Markov processes several combinations of analytical and numerical techniques have been developed. A good recent overview is given by Labeau et al. (2000). For ATM applications, a hybrid state Markov process approach has been outlined in Blom et al. (2001a).

The aim of the current paper is to study the hybrid state Markov process framework towards the modelling of the risk of mid-air collision between two aircraft. In contrast to the limitation to ODE's by Labeau et al. (2000), we consider stochastic differential equations (SDE) with switching coefficients.

The report is organised as follows. Section 2 gives an overview of accident risk metrics and criteria in air traffic. Section 3 develops mid-air collision risk equations. Section 4 develops a Markov stopping-time based risk decomposition. Section 5 illustrates some results obtained through the approach of sections 3 and 4 for a realistic application. Section 6 draws conclusions.

2. Accident risk in air traffic

A key issue in the development of safe spacing and separation criteria for air traffic management is the perspective of what is safe and what not. Safety is a general notion that has specific meanings depending of the perspective taken. In general, there are three such perspectives:

- Dependability of a system and its use (e.g., of a computer program, an aircraft navigation system, a satellite-based communication system, etc.). In line with this, **Dependability** is defined as **that property of a computer system such that reliance can justifiably be placed on the services it delivers** (Laprie, 1995). The service delivered by a system is its behaviour as it is perceived by its user(s); a user is another entity (e.g. system or human) which interacts with the former. Dependability metrics are widely studied in literature (e.g. Randell et al., 1995) and are well developed to cover technical systems and their use in civil aviation. These dependability techniques do not cover procedural aspects such as spacing and separation criteria.
- Safety perception (e.g., by pilot, controller, passenger, human society, etc.). A spacing or separation criterion that is perceived as being unsafe will not easily be accepted by the pilots and controller involved. Hence, a positive perception about the safety of spacing and separation criteria for a particular operation in air traffic is a critical requirement. By its very nature, however, safety perception is a subjective notion and therefore not sufficient to approve safety-critical changes in ATM.
- Accident risk is a notion that is commonly in use for other safety-critical operations such as in the chemical and nuclear industries. Royal Society (1983, 1992) identifies several risk definitions, and chooses one of these as their preferred one. In line with this, **accident risk** is defined as **the probability that an accident occurs during a stated period of time**. Hence, accident risk is a frequency and depends on the accident types considered. In civil aviation, it also is common practice to express safety statistics and safety requirements for air traffic operations in terms of frequencies of various accident types. This will be explained in some more detail below for separation related accidents in air traffic.

In view of the safe spacing and safe separation assessment needs, the accident risk perspective has the best joint characteristics: 1) it implies the use of objective metrics; 2) it has proven its usability to safety-critical operations in other industries (e.g. nuclear, chemical); and 3) it is supported by ICAO, JAA and Eurocontrol. As such, in the sequel of this paper safety will be considered from an accident risk perspective.

Following (ICAO, Annex 13), an **accident** is defined as: "an occurrence associated with the operation of an aircraft, which takes place between the time any person boards the aircraft with the intention of flight until such time as all such persons have disembarked, in which:

a) a person is fatally or seriously injured as a result of being in the aircraft, or of direct contact with any part of the aircraft, including parts which have become detached from the aircraft, or of direct exposure to jet blast (except when the injuries are from natural causes, self-inflicted, or inflicted by other persons, or when the injuries are to stowaways hiding outside the areas normally available to the passenger and crew); or
b) the aircraft sustains damage or structural failure which adversely affects the structural strength, performance or flight characteristics of the aircraft, and would normally require major repair or replacement of the affected component (except for engine failure or damage, when the damage is limited to the engine, its cowlings or accessories; or for

damage limited to propellers, wing tips, antennas, tires, brakes, fairings, small dents or puncture holes in the aircraft skin); or

c) the aircraft is missing or is completely inaccessible."

In order to avoid ambiguity, [ICAO, Annex 13] also gives definitions of fatality and fatal accident. A **fatality** is defined as the death of a person resulting from injuries within thirty days of the date of the accident. A **fatal accident** is an accident with at least one fatality among the persons mentioned under a) above. Note that the ICAO definition counts one collision between two aircraft as two accidents. Also note that the ICAO definition largely excludes 3rd party damage, injuries and fatalities.

Separation related accident statistics

Van Es (2001) has performed a statistical analysis of accidents, fatal accidents and fatalities by Large Aeroplanes (certified takeoff weight is 5670 kg or more) in commercial aviation (but excluding flights with Russian-built and business jet aircraft) over the period 1980 through 1999, and with emphasis on separation-related accidents, i.e.

- Accident involved two or more commercial aviation aircraft, or
- Accident involved one aircraft and one or more ground vehicles, or
- Accident induced by the wake vortex of another aircraft, or
- Accident induced by a near-miss escape manoeuvre.

Over this 20-year period, the total number of accidents in the sample considered amounts 2340, of which 613 are fatal accidents with a total of 15,554 fatalities, while the estimated number of applicable flights amounts 420 million. This statistical data is shown in Table 1.

Table 1 Accident statistics of Large Aeroplane flights in commercial aviation

	Accidents	Fatal Accidents	Fatalities
1980-1999 period	2340	613	15,554
Average per year	117	30.7	777.7
Average per flight	**5.57 E-6**	**1.46 E-6**	**37.0 E-6**
Separation related	7.9%	3.75%	5.0%

The separation related share of accidents is 185 (7.9%), of fatal accidents it is 23 (3.75%) and of fatalities it is 783 (5.0%). Roughly, this means about one separation related fatal accident per year. Further characteristics of the separation related accidents are shown in Tables 2 and 3. It should be noticed that a collision between an aircraft in the sample and an aircraft not in the sample (e.g. a general aviation aircraft or a business jet) has been counted as one accident. Hence, the number of mid-air collisions cannot be obtained by dividing the number of mid-air accidents in the tables by two.

Table 2 Separation related accident statistics of Large Aeroplanes in commercial aviation

	Accidents	Fatal accidents	Fatalities
1980-1999	185	23 (12.4%)	783
Per year	9.25	1.15	39.15
Per flight	**44.0 E-8**	**5.5 E-8**	**1.86 E-6**
Airborne	9.5 E-8 (22%)	3.35 E-8 (61%)	1.47 E-6 (79%)
Non-airborne	34.5 E-8 (78%)	2.15 E-8 (39%)	0.39E-6 (21%)

Table 3 The distribution of separation-related accidents (light), fatal accidents (grey) and fatalities (black) over various accident types. Source: Van Es (2001; 2002)

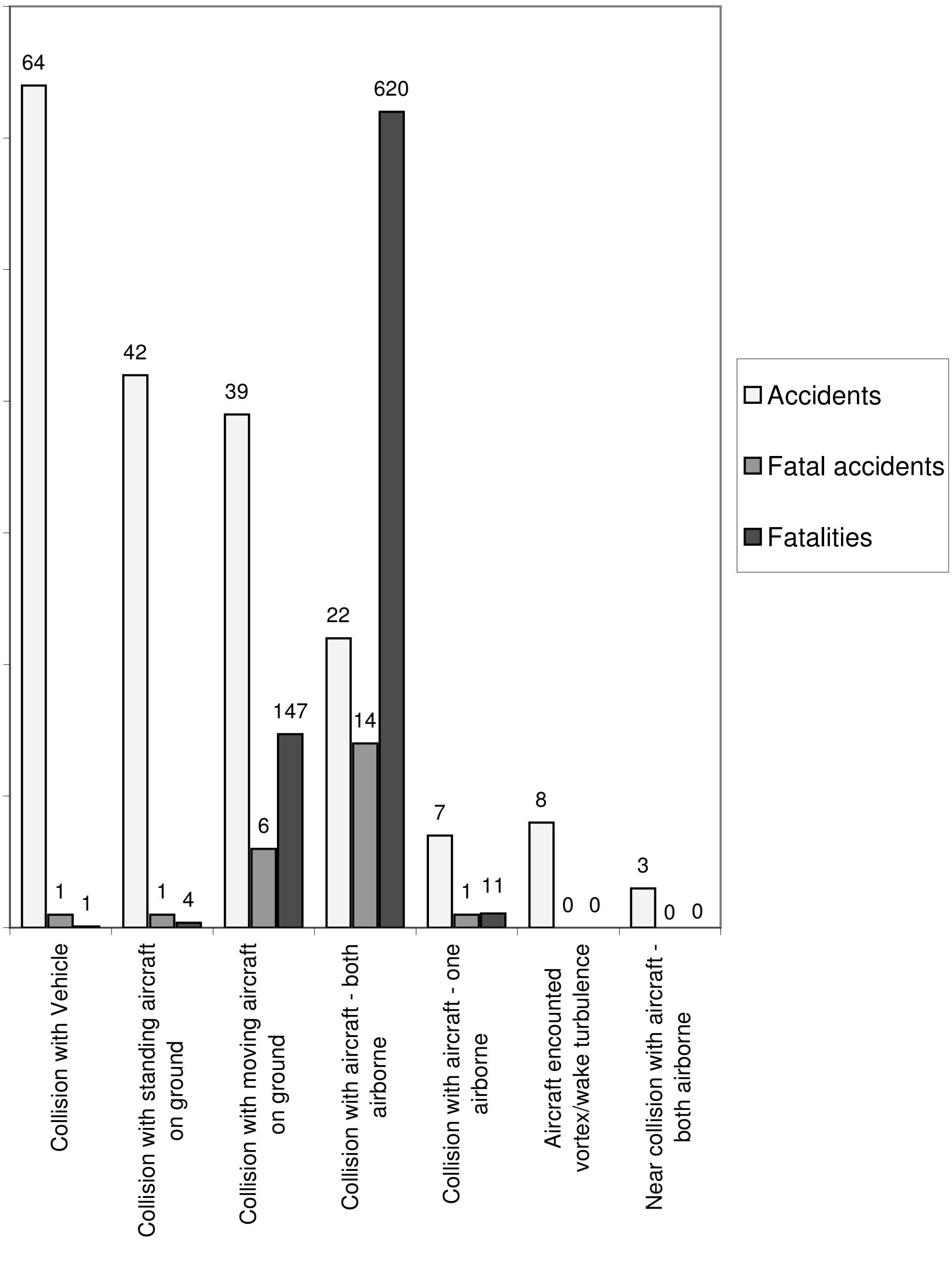

Table 2 shows that 79% of the separation related fatalities are due to mid-air collisions, although these constitute 22% only of all separation related accidents. The remaining 21% separation related fatalities are constituted by 78% of the separation-related accidents at the airport, and in particular between two aircraft. Table 3 shows that 11 out of 185 accidents, i.e. 6%, are not constituted by a collision but by last moment manoeuvring to avoid a collision or by hitting the wake vortex turbulence from another aircraft. Moreover, these non-collision accidents did not cause any fatality. Van Es (2001) has also shown that the number of separation related accidents per flight seems to be rather constant over different areas in the world (with a positive exception for the Australia/Pacific area), and rather stable over the years. However, one should be aware that the sample sizes often are too small to draw firm conclusions regarding this year and place invariance.

Finally, Table 4 shows the average number of fatalities per accident due to the various collision types in Table 2. This clearly shows that there are large differences in the consequences per type of collision. The average number of fatalities per accident varies from 0.016 for an accident due to collision with vehicle, to 28.2 for an accident due to a mid-air collision. Hence, if consequences are measured in number of fatalities then an accident due to a mid-air collision is a factor 1760 (= 28.2 /0.016) more severe than an accident due to a collision with a vehicle.

Table 4 Average number of fatalities per accident for various collision types

Type of collision determining the accident	Average # fatalities
Collision with aircraft – both airborne	28.2
Collision with moving aircraft on ground	3.8
Collision with aircraft – one airborne	1.57
Collision with standing aircraft on ground	0.095
Collision with vehicle	0.016

Accident risk criteria in air traffic

In commercial aviation it is common practice to set accident risk criteria that take into account available statistical information on frequency and severity of different accident types. The following risk criteria sources in particular have been identified, the first three of which address overall risks, while the others are separation related:

1. JAA (1994, 2000) specifies a requirement for **serious accidents (fatal or hull loss) during the airborne part of the flight due to aircraft system failure**. Currently, the requirement is maximally 1.0E-7 serious accident per flight hour, due to aircraft system failure.
2. ICAO (1993) specifies a requirement for **collisions with obstacles during Instrument Landing System (ILS) based final approach or missed approach**. Currently, the requirement is 1.0 E-7 collisions per ILS approach.
3. ICAO (1994) proposes a **hull loss risk** target for non-propeller Large Aeroplanes that is an order of magnitude better than current statistics. The proposed hull loss risk target is 1.5 E-7 per flight (from gate to gate), or 1.0 E-7 per flight hour.
4. ICAO (1998) specifies a maximum on the risk of **accidents due to collisions in en-route and oceanic airspace.** For the period 2000-2010, this level is 1.5 E-8 accidents per flight hour, due to mid-air collisions. The requirement is regularly updated (e.g. prior to January 2000 the allowed risk level was a factor 4 higher).
5. ICAO (2001) proposes a maximum on the risk of **serious accidents (fatal or hull loss) during the non-airborne phase of a flight (gate to gate).** The proposed level is 1.2 E-8 serious accidents per flight (gate-to-gate).
6. Eurocontrol (2000) specifies that a requirement for **ATM directly contributing to an accident during the airborne part of the flight** should take into account the annual growth of commercial traffic in Europe. Assuming a traffic growth of 6.7% per year, traffic increases a factor 1.91 from 2000 to 2010.

7. Eurocontrol (2001) specifies a requirement for **ATM directly contributing to an accident during the airborne part of the flight**. The current requirement is 1.55 E-8 of such accidents per flight hour or 2.31 E-8 of such accidents per flight.

The variation in required levels reflects significant differences in severity of the consequences over the accident classes, and also the significant differences in the timelines of the requirement. For example, the third requirement tends to outdate the first two requirements.

3. Mid-air collision risk equations

For oceanic air traffic, the civil aviation community has developed a mathematical model to estimate mid-air collision risk levels as a function of spacing (ICAO, 1988). This model is known as the Reich collision model (Reich, 1964). Following Hsu (1981), in mathematical terms this model assumes that the physical shape of each aircraft is a box, having a fixed x,y,z orientation, and the collision risk between two boxes is obtained by integrating the incrossing rate over the time period in which these boxes may be close to each other. Hence, mathematically it is better to speak of incrossing integral rather than collision risk. Unfortunately, this Reich model does not adequately cover busy continental situations with radar surveillance based tactical interventions by an air traffic controller. The aim of this and the next sections is to develop novel equations for this collision risk modelling and assessment problem.

Throughout this and the next sections, all stochastic processes are defined on a complete stochastic basis (Ω, $\mathcal{F}$, $\mathbb{F}$, P, $\mathbb{T}$) with (Ω, $\mathcal{F}$, P) a complete probability space, and $\mathbb{F}$ is an increasing sequence of sub-σ-algebra's on the positive time line $\mathbb{T}=\mathbb{R}_+$, i.e. $\mathbb{F} \triangleq \{\mathcal{J},(\mathcal{F}_t, t\in \mathbb{T}),\mathcal{F}\}$, $\mathcal{J}$ containing all P-null sets of $\mathcal{F}$ and $\mathcal{J}\subset\mathcal{F}_s\subset\mathcal{F}_t\subset\mathcal{F}$ for every $s < t$.

Consider an M-aircraft evolution model that is represented by stochastic differential equations[1] with switching coefficients, one for each of the M aircraft, i.e. for $i=1,\ldots,M$,

$$dx_t^i = f^i(x_t,\theta_t,t)dt + g^i(x_t,\theta_t,t)dw_t^i, \tag{1}$$

with $x_t \triangleq \mathrm{Col}\{x_t^1,\ldots,x_t^M\}$, $\theta_t \triangleq \mathrm{Col}\{\theta_t^1,\ldots,\theta_t^M\}$, $w_t \triangleq \mathrm{Col}\{w_t^1,\ldots,w_t^M\}$, $\{w_t^i\}$ an n-dimensional standard Brownian motion, x_t^i assumes values in $\mathbb{R}^n$ and θ_t^i a finite (N) state process such that $\{x_t,\theta_t\}$ is a semi-martingale and a strong Markov process[2]. The mappings f and g may represent planning and control strategies. Some elements of x_t^i form the 3D position of aircraft i,

$$y_t^i = Hx_t^i, \tag{1'}$$

with H a 3×n-matrix. To avoid Brownian motion behaviour in positions, we adopt the assumption

A.1 $Hg^i(x_t,\theta_t,t)=0$ for $i=1,\ldots,M$.

Under assumption **A.1**, we get for $i=1,\ldots,M$,

$dy_t^i = v_t^i dt$ with $v_t^i \triangleq Hf^i(x_t,\theta_t,t)$.

Next, with y_t^i and y_t^j representing the positions of the centres of aircraft pair (i,j), the relative 3D position is represented by the process $y_t^{ij} = y_t^i - y_t^j$, and the relative velocity is represented by the process $v_t^{ij} = v_t^i - v_t^j$. Hence

$$dy_t^{ij} = v_t^{ij}dt. \tag{2}$$

[1] Labeau et al. (2000) assume $g = 0$, i.e. no diffusion.

[2] Appendix A develops a more general semi-martingale hybrid state Markov process setting. The main result is Corollary 4.2, which allows that $\{x_t^i\}$ has jumps, which may anticipate a simultaneous switching of $\{\theta_t^i\}$. These hybrid jumps typically appear in air traffic models, e.g. switching of an aircraft control mode typically comes with a simultaneous jump in a control input.

A collision means that $\{y_t^{ij}\}$ enters a closed collision area D^{ij} around the origin; i.e. an area where aircraft *i* and *j* are not separated anymore. Under the assumption that the length of the aircraft equals the width of the aircraft, and that the volume of an aircraft is represented by a box the orientation of which does not change in time, then the size of D^{ij} is the sum of the size of two individual aircraft, i.e.

$$D^{ij} = D_1^{ij} \times D_2^{ij} \times D_3^{ij},$$

with $D_k^{ij} = [-m_k^{ij}, m_k^{ij}]$, $m_k^{ij} = \frac{1}{2}(s_k^i + s_k^j)$, s_1^i the length, s_2^i the width, s_3^i the height of aircraft *i* and $s_1^i = s_2^i$. If the relative position $\{y_t^{ij}\}$ enters D^{ij} at time τ, i.e. if $y_{\tau-\Delta}^{ij} \notin D^{ij}$ and $y_\tau^{ij} \in D^{ij}$ for $\Delta \downarrow 0$, then we say an incrossing event occurred. For equation (1) we assume that D^{ij} is transient (i.e. non-absorbing).

Next, we define for each (*i*, *j*) an indicator process $\{\chi_t^{ij}\}$ as follows:

$$\chi_t^{ij} = \begin{cases} 1 & \text{if } y_t^{ij} \in D^{ij}, \\ 0 & \text{else} \end{cases}$$

C.1 For any (*i*, *j*) the indicator process $\{\chi_t^{ij}\}$ has finite variation over any finite interval.

Lemma 1

Under assumption **C.1** the indicator process $\{\chi_t^{ij}\}$ admits on any finite interval a unique decomposition:

$$\chi_t^{ij} = \chi_{t_0}^{ij} + \chi_t^{ij+} - \chi_t^{ij-}, \tag{3}$$

with $\{\chi_{t_0}^{ij+} = \chi_{t_0}^{ij-} = 0\}$, while $\{\chi_t^{ij+}\}$ and $\{\chi_t^{ij-}\}$ are increasing processes such that,

$$\int_{t_0}^{t_1} \left| d\chi_s^{ij} \right| = \chi_{t_1}^{ij+} + \chi_{t_1}^{ij-}.$$

Proof: With $\{y_t^{ij}\}$ progressively measurable for all *t*, and D^{ij} a Borel set, the indicator process $\{\chi_t^{ij}\}$ is also progressively measurable for all *t*. Due to assumption **C.1** any realisation $\{\chi_t^{ij}(\omega)\}$ is a real-valued measurable function with finite variation for all *t*, which implies decomposition (3) (Wong and Hajek, 1985, p.218). Q.E.D.

Remark 1: Notice that $\{\chi_t^{ij+}\}$ and $\{\chi_t^{ij-}\}$ count the in-crossings and out-crossings respectively of $\{y_t^{ij}\}$ in D^{ij}.

Next, we define $I_{in}^{ij}(t_0, t_1)$ as the expected number of incrossings between the two aircraft considered during (t_0, t_1) , $(t_0 < t_1 < \infty)$, i.e.,

$$I_{in}^{ij}(t_0, t_1) \triangleq E\{\chi_{t_1}^{ij+} - \chi_{t_0}^{ij+}\}, \tag{4}$$

and define the collision probability $P_{col}^{ij}(t_0, t_1)$ by

$$P_{col}^{ij}(t_0, t_1) \triangleq \mathrm{P}\{\chi_{t_1}^{ij+} \neq \chi_{t_0}^{ij+}\}. \tag{5}$$

Remark 2: Equation (5) implies that the first incrossing on a given interval is the collision on that interval.

Furthermore, define τ_0 as the moment of the first incrossing after t_0, i.e. $\tau_0 \underline{\underline{\Delta}} \inf(t > t_0, \chi_t^{ij+} \neq \chi_{t-}^{ij+})$.

Theorem 1

Under assumption **C.1**, the collision risk $P_{col}^{ij}(t_0,t_1)$ defined in equation (5) satisfies:

$$P_{col}^{ij}(t_0,t_1) = \frac{I_{in}^{ij}(t_0,t_1)}{1 + \int_{t_0}^{t_1} I_{in}^{ij}(t,t_1 \mid \tau_0 = t) \cdot p_{\tau_0 \mid \tau_0 \leq t_1}(t)dt}.$$

Proof: From equation (4), we have that

$$\begin{aligned} I_{in}^{ij}(t_0,t_1) &= E\{\chi_{t_1}^{ij+} - \chi_{t_0}^{ij+}\} \\ &= P\{\chi_{t_1}^{ij+} = \chi_{t_0}^{ij+}\} \cdot E\{\chi_{t_1}^{ij+} - \chi_{t_0}^{ij+} \mid \chi_{t_1}^{ij+} = \chi_{t_0}^{ij+}\} + P\{\chi_{t_1}^{ij+} \neq \chi_{t_0}^{ij+}\} \cdot E\{\chi_{t_1}^{ij+} - \chi_{t_0}^{ij+} \mid \chi_{t_1}^{ij+} \neq \chi_{t_0}^{ij+}\} \\ &= P\{\chi_{t_1}^{ij+} = \chi_{t_0}^{ij+}\} \cdot 0 + P\{\chi_{t_1}^{ij+} \neq \chi_{t_0}^{ij+}\} \cdot E\{\chi_{t_1}^{ij+} - \chi_{t_0}^{ij+} \mid \chi_{t_1}^{ij+} \neq \chi_{t_0}^{ij+}\} \\ &= P_{col}^{ij}(t_0,t_1) \cdot E\{\chi_{t_1}^{ij+} - \chi_{t_0}^{ij+} \mid \tau_0 \leq t_1\} \\ &= P_{col}^{ij}(t_0,t_1) \cdot E\{\chi_{t_1}^{ij+} - \chi_{\tau_0}^{ij+} + \chi_{\tau_0}^{ij+} - \chi_{t_0}^{ij+} \mid \tau_0 \leq t_1\} \\ &= P_{col}^{ij}(t_0,t_1) \cdot (E\{\chi_{\tau_0}^{ij+} - \chi_{t_0}^{ij+} \mid \tau_0 \leq t_1\} + E\{\chi_{t_1}^{ij+} - \chi_{\tau_0}^{ij+} \mid \tau_0 \leq t_1\}). \end{aligned}$$

Since the process $\{y_t^{ij}\}$ is continuous and D^{ij} is closed, $\{\chi_t^{ij+}\}$ is cadlag, hence,

$$\begin{aligned} I_{in}^{ij}(t_0,t_1) &= P_{col}^{ij}(t_0,t_1) \cdot (1 + E\{\chi_{t_1}^{ij+} - \chi_{\tau_0}^{ij+} \mid \tau_0 \leq t_1\}) \\ &= P_{col}^{ij}(t_0,t_1) \cdot (1 + I_{in}^{ij}(\tau_0,t_1 \mid \tau_0 \leq t_1)). \end{aligned}$$

From this we get

$$P_{col}^{ij}(t_0,t_1) = \frac{I_{in}^{ij}(t_0,t_1)}{1 + I_{in}^{ij}(\tau_0,t_1 \mid \tau_0 \leq t_1)}.$$

Applying the law of total probability yields:

$$\begin{aligned} P_{col}^{ij}(t_0,t_1) &= \frac{I_{in}^{ij}(t_0,t_1)}{1 + \int_{t_0}^{t_1} I_{in}^{ij}(\tau_0,t_1 \mid \tau_0 = t, \tau_0 \leq t_1) \cdot p_{\tau_0 \mid \tau_0 \leq t_1}(t)dt} \\ &= \frac{I_{in}^{ij}(t_0,t_1)}{1 + \int_{t_0}^{t_1} I_{in}^{ij}(t,t_1 \mid \tau_0 = t) \cdot p_{\tau_0 \mid \tau_0 \leq t_1}(t)dt}. \end{aligned}$$

QED.

C.2 For all (i, j), $\Delta > 0$, $E\{(\chi_{t+\Delta}^{ij-} - \chi_t^{ij-})(\chi_{t+\Delta}^{ij+} - \chi_t^{ij+})\} = o(\Delta)$.

Theorem 2

Under assumptions **C.1** and **C.2**, equation (4) yields:

$$I_{in}^{ij}(t_0,t_1) = \int_{t_0}^{t_1} E\{d\chi_t^{ij+}\} = \int_{t_0}^{t_1} \phi^{ij}(t)dt \tag{6}$$

with $\phi^{ij}(t)$ the incrossing rate, which is defined, if the limit exists, as

$$\phi^{ij}(t) \underline{\underline{\Delta}} \lim_{\Delta \downarrow 0} \frac{P\{y_{t-\Delta}^{ij} \notin D^{ij}, y_t^{ij} \in D^{ij}\}}{\Delta}. \tag{7}$$

Proof: Define the incrossing rate $\phi^{ij}(t)$ as

$$\phi^{ij}(t) \triangleq \lim_{\Delta \downarrow 0} \Delta^{-1} E\{\chi^{ij+}_{t+\Delta} - \chi^{ij+}_{t}\} = \lim_{\Delta \downarrow 0} \Delta^{-1} [E\{\chi^{ij+}_{t+\Delta}\} - E\{\chi^{ij+}_{t}\}].$$

Taking the integral yields,

$$\int_{t_0}^{t_1} \phi^{ij}(t)dt = \int_{t_0}^{t_1} \lim_{\Delta \downarrow 0} \Delta^{-1} [E\{\chi^{ij+}_{t+\Delta}\} - E\{\chi^{ij+}_{t}\}]dt .$$

Due to assumption **C.1**, we can exchange limit and integral, which yields:

$$\begin{aligned}
\int_{t_0}^{t_1} \phi^{ij}(t)dt &= \lim_{\Delta \downarrow 0} \Delta^{-1} \int_{t_0}^{t_1} [E\{\chi^{ij+}_{t+\Delta}\} - E\{\chi^{ij+}_{t}\}]dt \\
&= \lim_{\Delta \downarrow 0} \Delta^{-1} \left[\int_{t_0}^{t_1} E\{\chi^{ij+}_{t+\Delta}\}dt - \int_{t_0}^{t_1} E\{\chi^{ij+}_{t}\}dt \right] \\
&= \lim_{\Delta \downarrow 0} \Delta^{-1} \left[\int_{t_0+\Delta}^{t_1+\Delta} E\{\chi^{ij+}_{t}\}dt - \int_{t_0}^{t_1} E\{\chi^{ij+}_{t}\}dt \right] \\
&= \lim_{\Delta \downarrow 0} \Delta^{-1} \left[\int_{t_1}^{t_1+\Delta} E\{\chi^{ij+}_{t}\}dt - \int_{t_0}^{t_0+\Delta} E\{\chi^{ij+}_{t}\}dt \right]
\end{aligned}$$

Since $\{\chi^{ij+}_{t}\}$ is cadlag, $E\{\chi^{ij+}_{t}\}$ is also cadlag and we get,

$$\begin{aligned}
\int_{t_0}^{t_1} \phi^{ij}(t)dt &= \lim_{\Delta \downarrow 0} \Delta^{-1} \left[\int_{t_1}^{t_1+\Delta} E\{\chi^{ij+}_{t}\}dt - \int_{t_0}^{t_0+\Delta} E\{\chi^{ij+}_{t}\}dt \right] \\
&= E\{\chi^{ij+}_{t_1}\} - E\{\chi^{ij+}_{t_0}\}.
\end{aligned}$$

To show the last equality, we can write the upper and lower bound for both integrals as:

$$\Delta \cdot E\{\chi^{ij+}_{t_1}\} \le \int_{t_1}^{t_1+\Delta} E\{\chi^{ij+}_{t}\}dt \le \Delta \cdot E\{\chi^{ij+}_{t_1+\Delta}\},$$

$$\Delta \cdot E\{\chi^{ij+}_{t_0}\} \le \int_{t_0}^{t_0+\Delta} E\{\chi^{ij+}_{t}\}dt \le \Delta \cdot E\{\chi^{ij+}_{t_0+\Delta}\}.$$

Using the cadlag property and taking the limits, yields,

$$\lim_{\Delta \downarrow 0} \Delta^{-1} \int_{t_1}^{t_1+\Delta} E\{\chi^{ij+}_{t}\}dt = E\{\chi^{ij+}_{t_1}\},$$

$$\lim_{\Delta \downarrow 0} \Delta^{-1} \int_{t_0}^{t_0+\Delta} E\{\chi^{ij+}_{t}\}dt = E\{\chi^{ij+}_{t_0}\}.$$

It remains to be shown that

$$\phi^{ij}(t) \triangleq \lim_{\Delta \downarrow 0} \frac{P\{y^{ij}_{t-\Delta} \notin D^{ij}, y^{ij}_{t} \in D^{ij}\}}{\Delta}.$$

We have,

$$\begin{aligned}
E\{(\chi^{ij-}_{t+\Delta} - \chi^{ij-}_{t})(\chi^{ij+}_{t+\Delta} - \chi^{ij+}_{t})\} &= \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} \alpha\beta \cdot P\{(\chi^{ij-}_{t+\Delta} - \chi^{ij-}_{t}) = \alpha, (\chi^{ij+}_{t+\Delta} - \chi^{ij+}_{t}) = \beta\} \\
&= \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} \alpha\beta \cdot \Big\{ P\{(\chi^{ij-}_{t+\Delta} - \chi^{ij-}_{t}) = \alpha, (\chi^{ij+}_{t+\Delta} - \chi^{ij+}_{t}) = \beta, \chi^{ij}_{t} = 0\} + \\
&\quad + P\{(\chi^{ij-}_{t+\Delta} - \chi^{ij-}_{t}) = \alpha, (\chi^{ij+}_{t+\Delta} - \chi^{ij+}_{t}) = \beta, \chi^{ij}_{t} = 1\} \Big\}
\end{aligned}$$

$$\geq \sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty}\Big\{\alpha\cdot P\{(\chi_{t+\Delta}^{ij-}-\chi_t^{ij-})=\alpha,(\chi_{t+\Delta}^{ij+}-\chi_t^{ij+})=\beta,\chi_t^{ij}=0\}+$$
$$+\beta\cdot P\{(\chi_{t+\Delta}^{ij-}-\chi_t^{ij-})=\alpha,(\chi_{t+\Delta}^{ij+}-\chi_t^{ij+})=\beta,\chi_t^{ij}=1\}\Big\}$$
$$=\sum_{\alpha=0}^{\infty}\alpha\cdot P\{(\chi_{t+\Delta}^{ij-}-\chi_t^{ij-})=\alpha,\chi_t^{ij}=0\}+\sum_{\beta=0}^{\infty}\beta\cdot P\{(\chi_{t+\Delta}^{ij+}-\chi_t^{ij+})=\beta,\chi_t^{ij}=1\}$$
$$=E\{(1-\chi_t^{ij})(\chi_{t+\Delta}^{ij-}-\chi_t^{ij-})\}+E\{\chi_t^{ij}(\chi_{t+\Delta}^{ij+}-\chi_t^{ij+})\}=\mathrm{o}(\Delta)$$

where the last equality is implied by assumption **C.2**.
Next, we get:

$$\begin{aligned}E\{\chi_{t+\Delta}^{ij+}-\chi_t^{ij+}\}&=E\{(1-\chi_t^{ij})(\chi_{t+\Delta}^{ij+}-\chi_t^{ij+})\}+E\{\chi_t^{ij}(\chi_{t+\Delta}^{ij+}-\chi_t^{ij+})\}\\&=E\{(1-\chi_t^{ij})(\chi_{t+\Delta}^{ij}-\chi_t^{ij})\}+E\{\chi_t^{ij}(\chi_{t+\Delta}^{ij+}-\chi_t^{ij+})\}+E\{(1-\chi_t^{ij})(\chi_{t+\Delta}^{ij-}-\chi_t^{ij-})\}\\&=E\{(1-\chi_t^{ij})(\chi_{t+\Delta}^{ij}-\chi_t^{ij})\}+\mathrm{o}(\Delta)\\&=P\{\chi_t^{ij}=0,\chi_{t+\Delta}^{ij}=1\}+\mathrm{o}(\Delta)\\&=P\{y_t^{ij}\notin D^{ij},y_{t+\Delta}^{ij}\in D^{ij}\}+\mathrm{o}(\Delta)\end{aligned}$$

The latter implies,

$$\begin{aligned}\phi^{ij}(t)&=\lim_{\Delta\downarrow 0}\Delta^{-1}E\{\chi_{t+\Delta}^{ij+}-\chi_t^{ij+}\}\\&=\lim_{\Delta\downarrow 0}\Delta^{-1}P\{y_t^{ij}\notin D^{ij},y_{t+\Delta}^{ij}\in D^{ij}\}\end{aligned}$$

Q.E.D.

Next, some assumptions are introduced under which $\phi^{ij}(t)$ is characterised. These assumptions are:

- **A.2** $P\{y_t^{ij}\in D^{ij},(y_t^{ij}-\Delta v_t^{ij})\notin D^{ij},y_{t-\Delta}^{ij}\in D^{ij}\}-P\{y_t^{ij}\in D^{ij},(y_t^{ij}-\Delta v_t^{ij})\in D^{ij},y_{t-\Delta}^{ij}\notin D^{ij}\}=o(\Delta)$
- **A.3** For any $k\in\{1,2,3\}$, there is a constant L_k such that for all t and for all $y_k\in[-m_k^{ij},m_k^{ij}]$: $E\{(v_{k,t}^{ij})^2\}\leq L_k$ and $E\{(v_{k,t}^{ij})^2\mid y_{k,t}^{ij}=y_k\}\leq L_k$.
- **A.4** A rather technical assumption on the joint density function of the pair (y_t^{ij},v_t^{ij}) (see Bakker & Blom, 1993).

<u>Theorem 3</u>
Under assumptions **A.1**, **A.2**, **A.3** and **A.4**, the incrossing rate $\phi^{ij}(t)$ defined in (7) satisfies:

$$\phi^{ij}(t)=\sum_{k=1}^{3}\int_{\underline{D}_k^{ij}}\left\{\int_0^{\infty}v\,p_{\underline{y}_{k,t}^{ij},y_{k,t}^{ij},v_{k,t}^{ij}}(\underline{y},-m_k^{ij},v)dv+\int_{-\infty}^{0}-v\,p_{\underline{y}_{k,t}^{ij},y_{k,t}^{ij},v_{k,t}^{ij}}(\underline{y},m_k^{ij},v)dv\right\}d\underline{y} \tag{8}$$

where

$$\begin{aligned}&\underline{D}_1^{ij}\triangleq D_2^{ij}\times D_3^{ij},\quad \underline{D}_2^{ij}\triangleq D_1^{ij}\times D_3^{ij},\quad \underline{D}_3^{ij}\triangleq D_1^{ij}\times D_2^{ij},\\&\underline{y}_{1,t}^{ij}\triangleq(y_{2,t}^{ij},y_{3,t}^{ij}),\quad \underline{y}_{2,t}^{ij}\triangleq(y_{1,t}^{ij},y_{3,t}^{ij}),\quad \underline{y}_{3,t}^{ij}\triangleq(y_{1,t}^{ij},y_{2,t}^{ij}).\end{aligned} \tag{9}$$

Proof: See Bakker & Blom (1993, Theorem 1).

Remark 3: Equations similar to (8) have been derived by Leadbetter (1966, 1973) and by Marcus (1977) for a one-dimensional process and by Belyaev (1968) for a multi-dimensional process.

Remark 4: In Blom & Bakker (2002), the incrossing rate is further characterised for Gaussian and Gaussian mixture shapes of $p_{y_t^{ij},v_t^{ij}}(\cdot)$.

4. Stopping time based decomposition

Theorem 3 shows that $\phi^{ij}(t)$ can be evaluated as a function of the probability density of the joint relative state (y_t^{ij}, v_t^{ij}). In general, a characterisation of this probability density is complex, especially since there are combinatorially many types of non-nominal events. In order to improve this situation, we introduce a stopping time based approach for decomposing the incrossing risk for a pair of aircraft. Following Section 3, the evolution of the M-aircraft situation is modelled as a Markov process $\{\xi_t\} = \{x_t, \theta_t\}$ in a hybrid state space $X = (\mathbb{R}^n \times \mathbb{M})^M$. From the theory of Markov processes, e.g. Davis (1993), it follows that for a time homogeneous Markov process the evolution of the density distribution $p_{\xi_t}(\cdot)$ of the joint process can be characterised by a Chapman-Kolmogorov equation:

$$P\{\xi_t \in A\} = \int_X P\{\xi_t \in A \mid \xi_0 = \xi\} P\{\xi_0 \in d\xi\},\ t \geq 0, \tag{10}$$

for any Borel set $A \subset X$.

Labeau et al. (2000) have made extensive studies of evaluating equation (10) when no Brownian motion is involved and when the state space is rather limited. In air traffic models, the state space is very large and Brownian motion plays an essential role. Hence, the approaches of Labeau et al. (2000) are not sufficient to evaluate accident risk in air traffic. In order to improve this situation, the stopping time approach suggested in Blom et al. (2001a) is described next.

The first step is to recognise that if the strong Markov property holds true for $\{\xi_t\}$, then equation (10) holds true for any stopping time τ as well:

$$P\{\xi_{\tau+t} \in A\} = \int_X P\{\xi_t \in A \mid \xi_0 = \xi\} P\{\xi_\tau \in d\xi\},\ t \geq 0 \tag{11}$$

which for example means that, more colloquially, Monte Carlo simulations of a strong Markov process may be restarted from an empirical distribution that has been obtained for any stopping time. Now for a stopping time $\tau^{ij} \in (t_0, t_1)$ that is smaller than the first incrossing moment τ_0^{ij} between aircraft pair (i, j) on (t_0, t_1), i.e. $t_0 < \tau^{ij} < \tau_0^{ij}$, equation (6) becomes

$$I_{in}^{ij}(t_0, t_1) = \int_{t_0}^{\tau^{ij}} E\{d\chi_t^{ij+}\} + \int_{\tau^{ij}}^{t_1} E\{d\chi_t^{ij+}\} = \int_{\tau^{ij}}^{t_1} \phi^{ij}(t)dt \tag{12}$$

Next, we introduce a conditioning on classes of non-nominal events. To do so, we define an event sequence classification process $\{\kappa_t^{ij}\}$ assuming values in a discrete set $\mathcal{K}$, and such that κ_t^{ij} is a function of θ_t, i.e. $\kappa_t^{ij} = \mathcal{K}^{ij}(\theta_t)$, with $\mathcal{K}^{ij}$ an application specific mapping of θ_t into $\mathcal{K}$. Hence, $\{\xi_t, \kappa_t^{ij}\}$ too is a strong Markov process. Then for any stopping time τ^{ij} for the aircraft pair (i, j) we can decompose the incrossing integral using the total probability theorem as follows:

$$I_{in}^{ij}(t_0, t_1) = \sum_{\kappa \in \mathcal{K}} \int_{\tau^{ij}}^{t_1} \phi^{ij}(t \mid \kappa_{\tau^{ij}}^{ij} = \kappa)dt \cdot P\{\kappa_{\tau^{ij}}^{ij} = \kappa\} \tag{13}$$

with $\phi^{ij}(t \mid \kappa_{\tau^{ij}}^{ij} = \kappa)$ the conditional incrossing risk, defined by

$$\phi^{ij}(t \mid \kappa_{\tau^{ij}}^{ij} = \kappa) \triangleq \lim_{\Delta \downarrow 0} \frac{P(y_{t-\Delta}^{ij} \notin D^{ij}, y_t^{ij} \in D^{ij} \mid \kappa_{\tau^{ij}}^{ij} = \kappa)}{\Delta}.$$

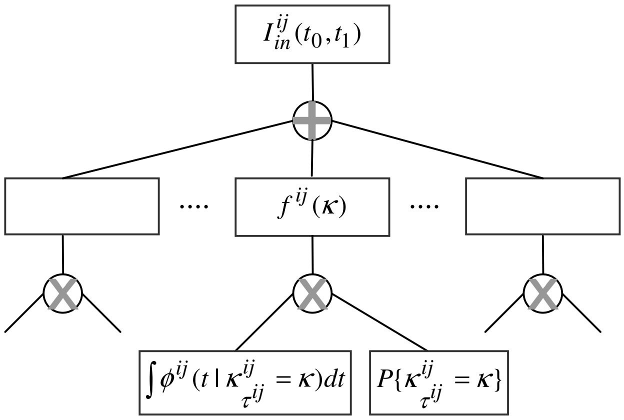

Figure 4 Collision risk tree

In Figure 4, equation (13) is presented in the form of a tree, where

$$f^{ij}(\kappa) = \int_{\tau^{ij}}^{t_1} \phi^{ij}(t \mid \kappa_{\tau^{ij}}^{ij} = \kappa)dt \cdot P\{\kappa_{\tau^{ij}}^{ij} = \kappa\} \cdot$$

This tree has a clear resemblance with the well-known fault tree. However, because of the underlying stochastic and physical relations, our new tree differs significantly and is called a collision risk tree. The collision risk tree decomposition in (13) allows evaluating the incrossing integral in two steps: first the probabilities $P\{\kappa_{\tau^{ij}} = \kappa\}$ and next the conditional incrossing integrals $\int_{\tau^{ij}}^{t_1} \phi^{ij}(t \mid \kappa_{\tau^{ij}}^{ij} = \kappa)dt$ for each $\kappa \in \mathcal{K}$. If the evaluation of $\int_{\tau^{ij}}^{t_1} \phi^{ij}(t \mid \kappa_{\tau^{ij}}^{ij} = \kappa)dt$ is as demanding as the direct evaluation of $\int_{\tau^{ij}}^{t_1} \phi^{ij}(t)dt$ is, then nothing is gained with this decomposition. However, by choosing the event sequence classification process $\{\kappa_t^{ij}\}$ and the stopping time τ^{ij} properly, it may be possible to simplify numerical evaluation of the incrossing risk considerably. The key to realise this is that the relevant state space to evaluate the integration of each $\phi^{ij}(t \mid \kappa_{\tau^{ij}}^{ij} = \kappa)$ over (τ^{ij}, t_1) should be much smaller than the state space needed to evaluate the integration of $\phi^{ij}(t)$ directly over (t_0, t_1). An additional advantage is that it becomes clear how much the contribution to the risk is per κ-value.

5. Results for an en-route ATC example

As an illustrative example, we show some results of applying the risk equations and risk decomposition approach of Sections 3 and 4 to a specific conventional en-route ATC situation, with two opposite streams of air traffic at the same flight level (see Figure 5).

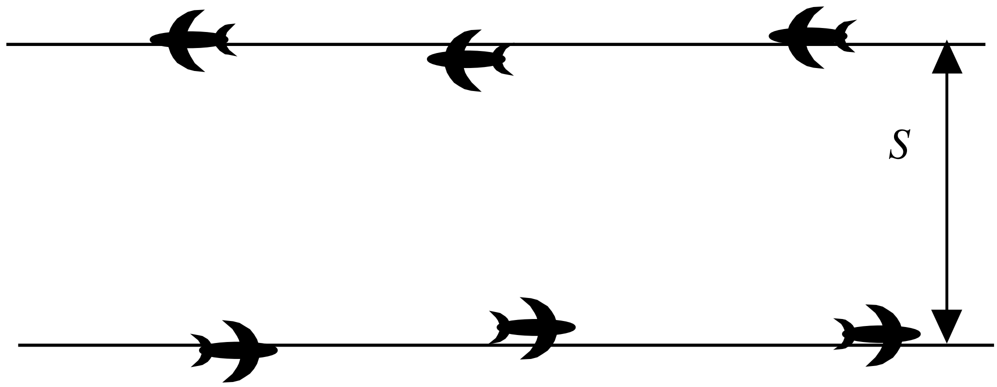

Figure 5 Opposite direction traffic in a dual lane route with lane spacing *S*.

See Everdij & Blom (2002) and Blom et al. (2003) for further explanation of this example. Here we restrict ourselves to giving the risk evaluation and composition results for varying spacing S values.

Let $\Re^i$ denote the expected number of incrossings per hour (=T) between aircraft i and an opposite flying aircraft. Then we have:

$$\Re^i = \sum_j I_{in}^{ij}(t_1 - T, t_1)$$

Let N_{flow} be the aircraft flow per hour per lane and in equation (1) let for all j: $f^j = f^i$, $g^j = g^i$ and $\{w_t^j\}$ be probabilistically equivalent to $\{w_t^i\}$, then

$$\Re^i = 2N_{flow} I_{in}^{ij}(t_1 - T, t_1) \tag{14}$$

with j one selected opposite flying aircraft that encounters aircraft i clearly within the time period.

Stopping time used

Let τ^{ij} be the first moment of overlap in along-lane direction between aircraft i and aircraft j, i.e.

$$\tau^{ij} \triangleq \min\{t_1, \inf\{t \geq t_1 - T; \left|y_{1,t}^{ij}\right| \leq d_1^{ij} + \Delta\}\}$$

with $y_{1,t}^{ij}$ the along distance component of y_t^{ij}, $d_1^{ij} = \frac{1}{2}s_1^i + \frac{1}{2}s_1^j$ and Δ a small positive value. With this stopping time, no collision between aircraft pair (i, j) can occur before τ^{ij}. Hence, substitution of (13) in (14) yields:

$$\Re^i = 2N_{flow} \sum_{\kappa \in \mathcal{K}} \int_{\tau^{ij}}^{t_1} \phi^{ij}(t \mid \kappa_{\tau^{ij}}^{ij} = \kappa) dt \cdot P\{\kappa_{\tau^{ij}}^{ij} = \kappa\} \tag{15}$$

Event sequence classification

For all t, we define the event sequence classification process κ_t^{ij} as a mapping of θ_t into

$\mathcal{K} \triangleq \mathcal{K}_{CN} \times \mathcal{K}_{CC} \times (\mathcal{K}_{AB})^2 \times (\mathcal{K}_{DM})^2$, where the set names *CN*, *CC, AB* and *DM* stand for:

- *CN* = Common Navigation modes {*CN Up*, *CN Down*}
- *CC* = Common Communication modes {*CC Up*, *CC Down*}
- *AB* = Aircraft Behaviour modes (*Nominal* or Deviating from ATC intent, with two Deviating modes: *Non-Nominal* drift away and *Turning* away)
- *DM* = Decision Making Loop modes, which covers surveillance, controller, radio-communication and crew (all being *Up* or at least one being *Down*).

Numerical results

For the model considered it appeared that, for the *CC*×*CN* values of κ, $P\{\kappa^{ij}_{\tau^{ij}} = \kappa\}$ could be obtained through Markov chain analysis of the behaviour of an independent Markov chain part of $\{\theta_t\}$. For the other κ-values *CC*×*CN* conditional Monte Carlo simulation have been run. Table 5 illustrates the $P\{\kappa^{ij}_{\tau^{ij}}\}$ outcomes for some clusters of κ-values:

I. Both aircraft in *AB Nominal* and *DM* being *Up* or *Down.*
II. At least one aircraft in *AB Turning* and *DM* being *Up* or *Down*.
III. All other combinations.

	CN Up		*CN Down*	
$(AB\times DM)^2$	*CC Up*	*CC Down*	*CC Up*	*CC Down*
I	$9.99\cdot10^{-1}$	$2.50\cdot10^{-4}$	0.0	0.0
II	$8.90\cdot10^{-5}$	$8.58\cdot10^{-8}$	$4.29\cdot10^{-10}$	$1.07\cdot10^{-13}$
III	$4.49\cdot10^{-4}$	$1.12\cdot10^{-7}$	$2.50\cdot10^{-6}$	$6.25\cdot10^{-10}$

Table 5 Common event sequence probabilities for clusters of κ-values in $\mathcal{K}$. For the model considered there is no *S* dependency.

Next, numerical results for $\int_{\tau^{ij}}^{t_1} \phi^{ij}(t \mid \kappa^{ij}_{\tau^{ij}} = \kappa)dt$ are obtained as function of spacing *S* for all κ values. The numerical evaluation is done through five steps:

1. Importance sampling based Monte Carlo simulation of sets of particles[*)] per κ-value to get an empirical density approximation for $p_{y^{ij}_{\tau^{ij}}, v^{ij}_{\tau^{ij}} \mid \kappa^{ij}_{\tau^{ij}}}(\cdot \mid \kappa)$ for each κ-value.
2. Gaussian sum density fitting of the resulting sets of particles per κ-value.
3. Numerical evaluation of (8) using the Gaussian sum characterisation for (8) in Blom & Bakker (2002);
4. Numerical integration over (τ^{ij}, t_1). The effective integration time is of the order $\Delta / E\{v^{ij}_{1,t}\} < 0.5$ s. On this short time interval eq. (1) is assumed to be approximated by the following ODE[**)]:
   $dy^{ij}_t = v^{ij}_t dt$
   $dv^{ij}_t = 0$
5. Repeat steps 3 and 4 for all relevant *S*-values.

Table 6 illustrates the $\int_{\tau^{ij}}^{t_1} \phi^{ij}(t \mid \kappa^{ij}_{\tau^{ij}})dt$ outcomes for clusters of κ-values in $\mathcal{K}$ and for *S* = 20 km.

[*)] A particle is a simulation sample with an importance weight attached to it.

[**)] This ODE implies that the D^{ij}-box has at most one incrossing. Hence, $I^{ij}_{in} = P^{ij}_{col}$.

	CN Up		*CN Down*	
$(AB\times DM)^2$	*CC Up*	*CC Down*	*CC Up*	*CC Down*
I	$2.0\cdot10^{-16}$	$1.1\cdot10^{-14}$	n.a.	n.a.
II	$5.2\cdot10^{-9}$	$6.2\cdot10^{-9}$	$1.4\cdot10^{-9}$	$2.9\cdot10^{-8}$
III	$2.6\cdot10^{-5}$	$1.5\cdot10^{-4}$	$1.6\cdot10^{-4}$	$1.6\cdot10^{-4}$

Table 6 Conditional incrossing integral for clusters of κ-values in $\mathcal{K}$ and spacing S = 20 km.

Composition using eq. (15)

Solving (15) by substituting N_{flow} = 3.6 aircraft per hour and the numerical results obtained for $P\{\kappa^{ij}_{\tau^{ij}} = \kappa\}$ and for $\int_{\tau^{ij}}^{t_1} \phi^{ij}(t \mid \kappa^{ij}_{\tau^{ij}} = \kappa)dt$, yields $\mathfrak{R}^i$. Figure 6 illustrates the outcomes as a function of S and for four selected clusters of κ-values in $\mathcal{K}$.

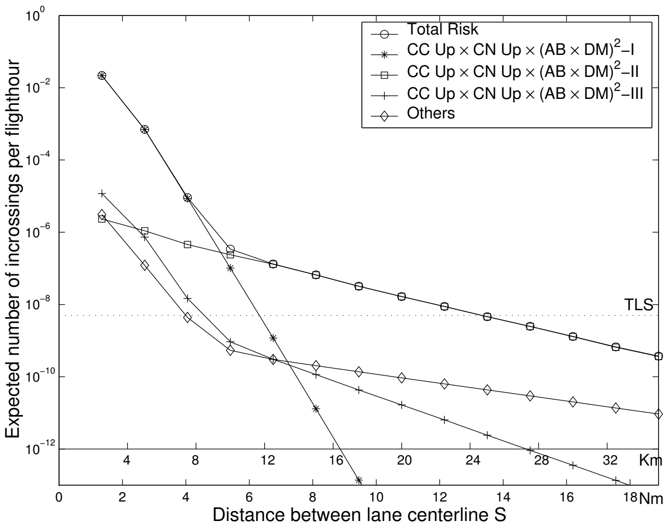

Figure 6 $\mathfrak{R}^i$ and the contributions to it from four clusters of κ values. The horizontal line represents ICAO's applicable Target Level of Safety (TLS) (ICAO, 1998).

In Figure 6, the curve for $\mathfrak{R}^i$ reaches the TLS line at about 24 km. This means that for the mathematical model, a safe spacing value would be 24 km. One should be aware that Figure 6 and Table 5 and Table 6 just illustrate the type of outputs one can get with the mathematical model. For the assessment against reality see Everdij & Blom (2002).

From these figures, it can be seen that the stopping time based risk decomposition works for this example. Figure 6 shows that Aircraft Behaviour combination Nominal × Sharp-turn is the main risk contributor with $\mathfrak{R}^i$ reaches the TLS. From an additional evaluation it followed that the Decision Making loop combination Up × Up most often applies in that situation. Therefore, in the model, the spacing (S) critical role is played by the risk contribution from the event to switch suddenly from Nominal to a Sharp-turn Aircraft Behaviour while Navigation, Communication and Decision Making loop conditions are nominal.

Everdij and Blom (2002) give the results of a bias and uncertainty assessment of the instantiated model, taking into account all assumptions made during the modelling and evaluation plus the possible differences in the used parameter values.

Numerical accuracy and simulation load

To get the results for all *S*-values, a total of 10^7 aircraft flighthours has been Monte Carlo simulated. This comes down to an average of 10^6 aircraft flighthours per κ-value. The numerical accuracy is 10^{-10}/flighthour. To get a similar accuracy through counting collisions during a standard Monte Carlo simulation, 10^{11} flighthours need to be simulated per *S*-value and for an almost twice as large state space. This is a factor 2.8×10^5 higher. Moreover, it doesn't provide insight in the role played by the κ-value conditions.

6. Concluding Remarks

Increasing air traffic capacity without sacrificing the required level of safety often is the key driver behind the development of advanced operational concepts for Air Traffic Management. During this development process there is need to receive feedback about the capacity/safety criticalities of the operational concept design. In support of this need, the paper has studied the development of a stochastic modelling approach towards the assessment of mid-air collision risk between aircraft for ATM operational concepts. The choice for collision risk has been backed up in section 2, where it is explained that the international civil aviation community is accustomed to maintain safety statistics and requirements in terms of these types of accidents. In sections 3 and 4, in-crossing risk and its decomposition has been studied within the setting of a stochastic differential equation with switching coefficients. The novelty of the approach over approaches known from the literature is twofold:

1) It allows to include Brownian motion in the evolution equations, and
2) It proposes a Markov (stopping) time based decomposition of the risk.

In Section 5 this novel approach has been illustrated to work for a particular en-route example.

There are many interesting directions for which it is valuable to extend the in-crossing risk model equations:

- Characterisation of large classes of SDE's the solutions of which are semimartingale strong Markov processes on a hybrid state space. One of the well known class of Davis (1984): Piecewise Deterministic Markov Process (PDMP), and extension of the mid-air incrossing risk equations of this paper to cover these hybrid state Markov processes.
- Development of risk equations for other types of accident risk. A relevant example of this type is the stochastic analysis based equations for wake vortex induced accident risk (Appendix in Kos et al., 2001), with an illustrative practical example in (Van Baren et al., 2002).
- Development of systematic ways to specify a mathematical model for an operational concept that has to be assessed on accident risk. One such development is the Dynamically Coloured Petri Net including a characterisation how a DCPN relates to a PDMP (Everdij & Blom, 2000).
- Further development of accident risk decomposition and particle filtering methods, and develop ways to combine the Markov (stopping) time based risk decomposition and/or the particle filtering methods with the analytical approaches towards solving forward Kolmogorov equations (e.g. Labeau et al., 2000).
- Incorporation of all these improvements within the TOPAZ (Traffic Organisation and Perturbation AnalyZer) accident risk assessment methodology and tool set (Blom et al., 2001a). For a realistic application of this tool set see (Blom et al., 2001c).

In collaboration with several European universities and research institutes, several of these further developments are currently under study within the HYBRIDGE project of the European Commission-IST.

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Appendix A Hybrid state Markov processes with hybrid jumps

The symbols used in this appendix are defined and used separately from the symbols used in the main text of this report.

A.1 Introduction

Many modelling and control studies for complex dynamical multi-agent systems have in common that they make use of continuous-time strong Markov processes the state of which is hybrid, i.e. one state component evolves in an Euclidean space, the other state component evolves in a discrete set, and each component may influence the evolution of the other component. Recently, Hu et al. (2000) noticed that there is a need to formally characterize hybrid state processes of which an Euclidean valued jump may depend of the simultaneous switching. For short, we refer to such simultaneous jumps with switching dependency as hybrid jumps.

There are two types of hybrid jumps possible: those that happen at instants of hitting some boundary, and those that happen at a sudden instant (i.e. Poisson type). A well-known class of semimartingale Markov processes are the Piecewise Deterministic Markov processes (Davis, 1984, 1993; Vermes, 1985). They incorporate both types of hybrid jumps, however they do not include diffusion. Moreover, their specific formulation does not allow a straightforward inclusion of diffusion.
For the combination of switching diffusion and hybrid jumps that happen when hitting some boundary, the situation has been improved by Borkar et al. (1991) and by Bensoussan & Menaldi (2000). The aim of this paper is to improve the situation for hybrid jumps that happen suddenly, i.e. are of Poisson type.

The approach taken is to study a stochastic differential equation (SDE) on a hybrid state space that is driven by Brownian motion and Poisson random measure. The particular SDE studied is of Itô-Skorohod type,

$$d\xi_t = \alpha(\xi_t)dt + \beta(\xi_t)dw_t + \int_U \psi(\xi_{t-},u)\, p_{\mathrm{P}}(dt,du)$$

where $\{w_t\}$ is a Brownian motion, and p_{P} is a Poisson random measure on $(0,\infty)\times U$, with U a Borel measurable space. If p_{P} generates a multivariate point $(\{t\},\{u\})$, then the path of $\{\xi_t\}$ has a discontinuity:

$$\xi_t = \xi_{t-} + \psi(\xi_{t-},u)\,,$$

with: $\xi_{t-} = \lim_{\Delta\downarrow 0} \xi_{t-\Delta}$.

The classical result for the existence of a pathwise unique solution of the Itô-Skorohod equation requires that ψ satisfies some local Lipschitz condition (Gihman and Skorohod, 1972; Ikeda & Watanabe, 1989). This Lipschitz condition essentially restricts the SDE solution to diffusions with Markov switching coefficients (Brockett & Blankenship, 1977) and with controlled coefficients (Ghosh et al, 1993, 1997), but does not allow hybrid jumps. Some non-classical pathwise uniqueness results for a discontinuous ψ have been developed by Lepeltier and Marchal (1976), Jacod and Protter (1982). We use these results as starting point to study the Itô-Skorohod equation.

This appendix is organised as follows. Appendix A.2 introduces the specific SDE results of Lepeltier & Marchal (1976). In appendix A.3, we incorporate the jump intensity modelling approach of Jacod & Protter (1982) in this SDE. In appendix A.4, we use this SDE to characterize a hybrid state strong Markov process which has hybrid jumps. Next, in appendix A.5, we show how these results extend the class of jump linear Gaussian systems. Finally, in appendix A.6, we characterise a General Stochastic Hybrid Process as the unique solution of such an SDE.

A.2 The SDE of Lepeltier and Marchal

Throughout this appendix we work within a complete stochastic basis $(\Omega,\Im,\boldsymbol{F},\boldsymbol{P},\boldsymbol{T})$, with measurable space $(\Omega,\Im)$, right continuous filtration $\boldsymbol{F}$, probability measure $\boldsymbol{P}$ and time index $\boldsymbol{T} = \Re_+ = [0,\infty)$, that is endowed with an m-dimensional standard Wiener process, $\{w_t\}$, and an independent homogeneous Poisson random measure (e.g. Jacod & Shiryaev, 1987, pp.70-71), $p_{\mathrm{P}}(dt,du)$ on $\Re_+ \times U$, with intensity measure $\nu(dt,du)=dt\cdot m(du)$.

First, we consider the following stochastic differential equation (SDE) in $\Re_+ \times \Re^n$,

$$d\xi_t = \alpha(\xi_t)dt + \beta(\xi_t)dw_t + \int_{U_1} \psi(\xi_{t-},u)\,p_P(dt,du) + \int_{U_2} \psi(\xi_{t-},u)\,p_P(dt,du), \tag{1}$$

with U_1 and U_2 such that $U_1 \cap U_2 = \emptyset$ and $U_1 \cup U_2 = U$, ξ_0 is an $\mathcal{F}_0$-measurable $\Re^n$-valued random variable, while α, β and ψ are measurable mappings of appropriate dimensions (with domains $\Re^n$, $\Re^n \times \Re^m$ and $\Re^n$, respectively).

Although, in the sequel, we are not really using the third right hand term, we start from (1) to notice the main difference between the roles played by the third and fourth right hand terms. The setup commonly used is to assume conditions such that $\left\{\int_0^t \int_{U_1} \psi(\xi_{s-},u)[p_P(ds,du) - \nu(ds,du)]\right\}$ is a local martingale, while the process $\left\{\int_0^t \int_{U_2} \psi(\xi_{s-},u)\,p_P(ds,du)\right\}$ has finite variation over each finite interval. The classical reference for an SDE of type (1) is Gihman and Skorohod (1972), who studied existence of a unique solution of (1) for the situation $U_2 = \emptyset$. Extensions of their results have been developed by Lepeltier and Marchal (1976) with $U_2 \neq \emptyset$, i.e. $U_1 = \{u;\ |u|\le 1\}$ and $U_2 = \{u;\ 1<|u|<\infty\}$. Under a non-Lipschitz $\psi(\cdot,u)$ for $u \in U_2$, they showed that (1) still has a unique solution which is a strong Markov process.

The interesting aspect of this is that the coefficients of the fourth right hand term of (1) may be discontinuous in ξ. This allows us to construct a class of hybrid state Markov processes that is larger than the class of solutions of systems with Markovian switching coefficients. For this construction we restrict our attention to the situation that $\psi(\xi,u) = 0$, for all $u \in U_1$, by which the third term of (1) is zero, i.e.:

$$d\xi_t = \alpha(\xi_t)dt + \beta(\xi_t)dw_t + \int_{U_2} \psi(\xi_{t-},u)\,p_{\mathrm{P}}(dt,du). \tag{2}$$

In line with Lepeltier and Machal (1976) the following assumptions are adopted:

A'.1. There is a constant K such that, for all $\xi \in \Re^n$,

$$|\alpha(\xi)|^2 + \|\beta(\xi)\|^2 + \int_{U_1} |\psi(\xi,u)|^2 m(du) \le K(1+|\xi|^2),$$

with $|a|^2 = \sum_i (a_i)^2$, and $\|b\|^2 = \sum_{i,j} (b_{ij})^2$.

A'.2. For all $k \in \aleph = \{0,1,2,\ldots\}$ there exists a constant L_k such that, for all ξ and y in the ball $B_k = \{x \in \Re^n; |x| \le k+1\}$,

$$|\alpha(\xi)-\alpha(y)|^2 + \|\beta(\xi)-\beta(y)\|^2 + \int_{U_1} |\psi(\xi,u)-\psi(y,u)|^2 m(du) \le L_k |\xi - y|^2.$$

A'.3. For every $k \in \aleph$ there exists a constant M_k, such that

$$\sup_{|\xi|\le k} \int_{U_2} |\psi(\xi,u)| m(du) \le M_k .$$

Proposition 2.1

Let A'.1, A'.2, and A'.3 be satisfied and let $U_2 = \{u;\ 1<|u|<\infty\}$. Then equation (2) has for every initial condition $\xi_0(\omega) = \xi \in \Re^n$ a pathwise unique solution, $\{\xi_t\}$, which is càdlàg and adapted. Moreover, there exists a measurable random function $f(t,\xi,\omega)$ such that $\xi_t(\cdot) = f(t,\xi,\cdot)$ almost surely for every t.

Proof: Lepeltier and Marchal (1976, Theorem III_4, pp. 84-85, and Corollary III_5, p. 86).

Proposition 2.2

Let the conditions of Proposition 2.1 be satisfied. Let $|\psi(\xi,u)| = 0$ or >1 for all $\xi \in \Re^n$ and $u \in U_2$, and let $\beta(\xi)$ be twice continuously differentiable in ξ. Then $\{\xi_t\}$ is a strong Markov process.

Proof: Lepeltier and Marchal (1976, Theorem III_6, p. 86).

A.3 Jump intensity model of Jacod and Protter

Jacod and Protter (1982; Protter, 1983) developed an elegant approach to explicitly model the jump intensity of $\{\xi_t\}$ in (2). Following this, we adopt the following compositions of $\psi(\xi,u)$ and $m(du)$:

$$m(du) = du_1 \cdot \mu(d\underline{u}),\ u_1 \in [0,\infty),\ \underline{u} \in \Re^d,$$
$$\psi(\xi,u) = \mathbf{1}_{(0,\Lambda(\xi)]}(u_1 - 1)\varphi(\xi,\underline{u}),$$

where $\underline{u} = \mathrm{Col}\{u_2,\ldots,u_{d+1}\}$, μ is a probability measure on $\Re^d$, Λ is a measurable mapping of $\Re^n$ into $[0,\infty)$, φ is a measurable mapping of $\Re^n \times \Re^d$ into $\Re^n$, and

$$\mathbf{1}_A(a) = 1, \quad a \in A,$$
$$= 0, \quad \text{else}.$$

With this, (2) becomes:

$$d\xi_t = \alpha(\xi_t)dt + \beta(\xi_t)dw_t + \int_{U_2} \varphi(\xi_{t-},\underline{u})\mathbf{1}_{(0,\Lambda(\xi_{t-})]}(u_1 - 1)\, p_{\mathrm{P}}(dt,du), \tag{3}$$

and the intensity measure of $p_{\mathrm{P}}(dt,du)$ equals $dt \cdot du_1 \cdot \mu(d\underline{u})$.

Next, we introduce the following assumptions:

A.1. There is a constant K such that, for all $\xi \in \Re^n$,
$|\alpha(\xi)|^2 + \|\beta(\xi)\|^2 \le K(1+|\xi|^2)$.

A.2. For all $k \in \aleph$ there exists a constant L_k such that, for all ξ and y in the ball $B_k = \{x \in \Re^n;\ |x| \le k+1\}$,
$|\alpha(\xi)-\alpha(y)|^2 + \|\beta(\xi)-\beta(y)\|^2 \le L_k|\xi - y|^2$,

A.3. $\Lambda(\cdot)$ is a bounded continuous mapping on $\Re^n$ with upper bound a constant C.

A'.4. For every $k \in \aleph$ there exists a constant M_k, such that
$\sup_{|\xi|\le k} \int_{\Re^d} |\varphi(\xi,\underline{u})| \mu(d\underline{u}) \le M_k$.

A'.5. $|\varphi(\xi,\underline{u})| = 0$ or >1 for all $\xi \in \Re^n$ and $\underline{u} \in \Re^d$.

Proposition 3.1
Let α and β satisfy A.1 and A.2, Λ satisfies A.3, φ satisfies A'.4 and A'.5 and the intensity measure of $p_P(dt,du)$ equals $dt \cdot du_1 \times \mu(d\underline{u})$. Then for every initial condition $\xi_0(\omega) = \xi \in \Re^n$, equation (3) has a pathwise unique solution, $\{\xi_t\}$, which is càdlàg and adapted. Moreover, if $\beta(\xi)$ is twice continuously differentiable in ξ, then $\{\xi_t\}$ is a strong Markov process.

Proof:
Due to A.3 the third right hand term of (3) can be written as:

$$\int_{(1,C+1]\times\Re^d} \varphi(\xi_{t-},\underline{u}) \mathbf{1}_{(0,\Lambda(\xi_{t-})]}(u_1 - 1) p_P(dt,du).$$

For every $\xi \in \Re^n$ and every $u_1 \in (-\infty,\infty)$, we define the mapping $\chi(\xi,u_1)$ as: $\chi(\xi,u_1) \triangleq 1_{[u_1,\infty)}(\Lambda(\xi))$. Hence, χ is a measurable mapping in ξ and u_1.

Because $1_{[u_1,\infty)}(s) = 1_{(-\infty,s]}(u_1)$ and $\Lambda(\xi)$ is continuous in ξ (A.3), we get:

$$1_{[u_1,\infty)}(\Lambda(\xi)) = 1_{(-\infty,\Lambda(\xi)]}(u_1) \text{ for all } \xi,\ u_1.$$

Substituting this in the definition of χ yields:

$$\chi(\xi,u_1) = 1_{(-\infty,\Lambda(\xi)]}(u_1).$$

Hence, the third right hand term of (3) can be replaced by

$$\int_{(1,C+1]\times\Re^d} \varphi(\xi_{t-},\underline{u}) \chi(\xi_{t-},u_1 - 1) p_P(dt,du).$$

This implies that (3) is an equation of type (2). Finally, A'.4 implies that condition A'.3 of Proposition 2.1 is satisfied. Due to A'.5 the strong Markov property follows from proposition 2.2.

QED

Next we consider a more general situation in which there are N jump intensities that influence the evolution of the process $\{\xi_t\}$. Similar as before, we introduce the following decomposition of $m(du)$ and $\psi(\xi,u)$:

$$m(du) = du_1 \times \mu(d\underline{u}),\ u_1 \in [0,\infty),\ \underline{u} \in \Re^d \quad (4)$$

$$\psi(\xi,u) = \sum_{i=1}^{N} \mathbf{1}_{(\Sigma_{i-1}(\xi),\Sigma_i(\xi)]}(u_1 - 1)\phi(\eta_i,\xi,\underline{u}), \tag{5}$$

$$\Sigma_i(\xi) = \Lambda(\xi)\sum_{j=1}^{i}\rho(\eta_j,\xi),\ i > 0, \qquad = 0, \quad i = 0, \tag{6}$$

where $\underline{u}$ refers to all components of u except the first one, $\eta_i \in \Re$ for all i, with $|\eta_i - \eta_j| \geq 1$ if $i \neq j$, Σ_1 through Σ_N are measurable mappings of $\Re^n$ into $[0,\infty)$, ϕ is a measurable mapping of $\Re\times\Re^n\times\Re^d$ into $\Re^n$, μ is a probability measure, and ρ is a measurable mapping of $\Re\times\Re^n$ into $[0,\infty)$, such that

$$\sum_{i=1}^{N}\rho(\eta_i,\xi) = 1, \text{for all } \xi \in \Re^n. \tag{7}$$

With this, (2) becomes:

$$d\xi_t = \alpha(\xi_t)dt + \beta(\xi_t)dw_t + \int_{U_2}\sum_{i=1}^{N}\phi(\eta_i,\xi_{t-},\underline{u})\mathbf{1}_{(\Sigma_{i-1}(\xi_{t-}),\Sigma_i(\xi_{t-})]}(u_1 - 1)p_{\mathrm{P}}(dt,du), \tag{8}$$

with the intensity measure of $p_{\mathrm{P}}(dt,du)$ equal to $dt\cdot du_1\cdot\mu(d\underline{u})$.

Now, we introduce the following assumptions:

<u>A.4.</u> $\rho(\eta,\cdot)$ is a continuous mapping for all $\eta \in \Re$,

<u>A.5.</u> For all $k \in \aleph$ there exists a constant M_k such that, $\sup_{|\xi|\leq k}\int_{\Re^d}|\phi(\eta,\xi,\underline{u})|\mu(d\underline{u}) \leq M_k$, for all $\eta \in \Re$,

<u>A.6.</u> $|\phi(\eta,\xi,\underline{u})| = 0$ or >1 for all $\eta \in \{\eta_i ; i=1,\ldots,N\}$, $\xi \in \Re^n$ and $\underline{u} \in \Re^d$

and subsequently we extend the characterisation of Proposition 3.1 to the above situation of multiple jump intensities.

<u>Proposition 3.2</u>
Let α and β satisfy <u>A.1</u> and <u>A.2</u>, Λ satisfies <u>A.3</u>, Σ_i satisfies (6), ρ satisfies (7) and <u>A.4</u>, ϕ satisfies <u>A.5</u> and <u>A.6</u>, and the intensity measure of $p_{\mathrm{P}}(dt,du)$ equals $dt\cdot du_1\cdot\mu(d\underline{u})$. Then for every initial condition $\xi_0(\omega) = \xi \in \Re^n$, equation (8) has a pathwise unique solution, $\{\xi_t\}$, which is càdlàg and adapted. Moreover, if $\beta(\xi)$ is twice continuously differentiable in ξ, then $\{\xi_t\}$ is a strong Markov process.

<u>Proof:</u> Similar to the proof of Proposition 3.1.

Next, we give a semi-martingale characterization and the generator of the solution $\{\xi_t\}$ of equation (8).

<u>Proposition 3.3</u>

Let the assumptions of Proposition 3.2 hold true. Then for all $\xi_0 \in \Re^n$, $\{\xi_t\}$ is a semi-martingale and a strong Markov process, and its generator, $\mathcal{A}$, satisfies:

$$\mathcal{A} f = \mathcal{L} f + \mathcal{J} f \text{ , for all } f \in D(\mathcal{A}) \supset C^2(\Re^n) \text{ ,} \tag{9}$$

where

$$\mathcal{L} f(\xi) = \sum_{i=1}^{n} \alpha_i(\xi) f_{\xi_i}(\xi) + \tfrac{1}{2} \sum_{i,j=1}^{n} [\beta(\xi)\beta(\xi)^T]_{ij} f_{\xi_i \xi_j}(\xi), \tag{10}$$

$$\mathcal{J} f(\xi) = \Lambda(\xi) \int_{\Re^n} [f(\zeta) - f(\xi)] Q(d\zeta;\xi), \tag{11}$$

and for all Borel $A \subset \Re^n$,

$$Q(A;\xi) = \sum_{i=1}^{N} \rho(\eta_i,\xi) \int_{\Re^d} \mathbf{1}_A(\xi + \phi(\eta_i,\xi,\underline{u}))\mu(d\underline{u}) \text{ .} \tag{12}$$

<u>Proof:</u>

Due to <u>A.3</u>, <u>A.4</u>, and <u>A.5</u>, the predictable part $\{a_t\}$ of $\{\xi_t\}$ satisfies

$$a_t = \int_0^t \alpha(\xi_s)ds + \int_0^t \int_{U_2} \sum_{i=1}^{N} \phi(\eta_i,\xi_{s-},\underline{u}) \mathbf{1}_{(\Sigma_{i-1}(\xi_{s-}),\Sigma_i(\xi_{s-})]}(u_1)du_1 \cdot \mu(d\underline{u})ds$$

$$= \int_0^t \alpha(\xi_s)ds + \int_0^t \Lambda(\xi_{s-}) \sum_{i=1}^{N} \rho(\eta_i,\xi_{s-}) \int_{\Re^d} \phi(\eta_i,\xi_{s-},\underline{u})\mu(d\underline{u})ds,$$

up to indistinguishability. This shows that $\{a_t\}$ is of finite variation on any finite time interval. Since $\{\xi_t - a_t - \xi_0\}$ is a local martingale and a_t is predictable, this means that $\{\xi_t\}$ is a special semimartingale (Jacod & Shiryaev, 1987, p.43) and that $\{a_t\}$ defines the (unique) canonical martingale decomposition of $\{\xi_t\}$. The generator $\mathcal{A}$ follows from Itô's differentiation rule for discontinuous semimartingales (e.g. Jacod & Shiryaev, 1987, p. 57).

QED

A.4 Hybrid state space

Now we are prepared to consider the hybrid state space situation such that for every ω, $\xi_t(\omega) \in \mathcal{M} \times \Re^{n-1}$, with $\mathcal{M} = \{\eta_i; i = 1,\dots,N\}$, $|\eta_i - \eta_j| \geq 1$ for $i \neq j$. First, we assume that the first component of $\{\xi_t\}$ is a pure jump process, i.e. for all $\xi \in \Re^n$,

$$\alpha_1(\xi) = 0 \tag{13.a}$$

$$\beta_1(\xi) = 0 \tag{13.b}$$

$$\phi_1(\eta,\xi,\underline{u}) = \eta - \xi_1 \text{ , for all } \eta \in \Re \text{ and } \underline{u} \in \Re^d \text{ .} \tag{13.c}$$

Substitution of equations (13.a,b,c) into (8) yields:

$$d\xi_{1,t} = \int_{(0,\infty)} \sum_{i=1}^{N} (\eta_i - \xi_{1,t-}) \mathbf{1}_{(\Sigma_{i-1}(\xi_{t-}),\Sigma_i(\xi_{t-})]}(u_1) p_{\mathrm{P}}(dt, du_1 \times \Re^d), \tag{14.a}$$

$$d\underline{\xi}_t = \underline{\alpha}(\xi_t)dt + \underline{\beta}(\xi_t)dw_t + \\ + \int_{(0,\infty)}\int_{\Re^d}\sum_{i=1}^{N}\underline{\phi}(\eta_i,\xi_{t-},\underline{u})\mathbf{1}_{(\Sigma_{i-1}(\xi_{t-}),\Sigma_i(\xi_{t-})]}(u_1)\,p_P(dt,du_1\times d\underline{u}), \tag{14.b}$$

where $\xi_{1,t}$ denotes the first component of ξ_t, and $\underline{\xi}_t$ denotes the other components.

Theorem 4.1

Let the conditions of Proposition 3.2 hold true, and let α_1, β_1 and ϕ_1 satisfy (13.a,b,c). Then for every initial condition $\xi_0(\omega)\in\Re^n$, equation (14.a,b) has a pathwise unique solution, $\{\xi_t\}$, which is càdlàg and adapted. Moreover, if $\xi_0(\omega)\in\mathcal{M}\times\Re^{n-1}$ for all ω, with $\mathcal{M}=\{n_i; i=1,\ldots,N\}$, then $\{\xi_t\}$ is a semi-martingale and a strong Markov process assuming values in the hybrid state space $\mathcal{M}\times\Re^{n-1}$, and its generator, $\mathcal{A}$, satisfies:

$$\mathcal{A}f = \mathcal{L}f + \mathcal{J}f \text{, for all } f\in D(\mathcal{A})\supset C^2(\Re^n), \tag{15}$$

where

$$\mathcal{L}f(\xi) = \sum_{i=2}^{n}\alpha_i(\xi)f_{\xi_i}(\xi) + \tfrac{1}{2}\sum_{i,j=2}^{n}[\beta(\xi)\beta(\xi)^T]_{ij}f_{\xi_i\xi_j}(\xi), \tag{16}$$

$$\mathcal{J}f(\xi) = \Lambda(\xi)\sum_{\eta\in\mathcal{M}}\int_{\Re^{n-1}}[f(\mathrm{Col}\{\eta,x\}) - f(\xi)]Q(\{\eta\}\times dx;\xi), \tag{17}$$

and for all Borel $\underline{A}\subset\Re^{n-1}$ and $\eta\in\mathcal{M}$:

$$Q(\{\eta\}\times\underline{A};\xi) = \rho(\eta,\xi)\int_{\Re^d}\mathbf{1}_{\underline{A}}(\underline{\xi}+\underline{\phi}(\eta,\xi,\underline{u}))\mu(d\underline{u}). \tag{18}$$

Proof: Since α_1, β_1 and ϕ_1 in (13.a,b,c) satisfy the conditions of Proposition 3.2, the existence of a unique solution and the adapted, càdlàg and strong Markov properties all follow from Propositions 3.2 and 3.3. From equation (14.a) and the initial condition $\xi_{1,0}(\omega)\in\mathcal{M}$ it can be shown that $\xi_{1,t}(\omega)\in\mathcal{M}$ for all ω and $t\geq 0$. Hence the state space of the Markov process $\{\xi_t\}$ is of the hybrid form $\mathcal{M}\times\Re^{n-1}$, and this yields the specific characterisation of the generator.

QED

Corollary 4.2

Under the conditions of Theorem 4.1, the solution of (14.a,b) is indistinguishable from the solution of the following set of equations:

$$d\xi_{1,t} = \sum_{i=1}^{N}(\eta_i - \xi_{1,t-})\,p_P(dt,(\Sigma_{i-1}(\xi_{t-}),\Sigma_i(\xi_{t-})]\times\Re^d), \tag{19.a}$$

$$d\underline{\xi}_t = \underline{\alpha}(\xi_t)dt + \underline{\beta}(\xi_t)dw_t + \int_{\Re^d}\underline{\phi}(\xi_{1,t},\xi_{t-},\underline{u})\,p_P(dt,(0,\Lambda(\xi_{t-})]\times d\underline{u}). \tag{19.b}$$

Proof:

Rewriting of (19.a) yields (14.a) up to indistinguishability. Since the first two right hand terms of (19.b) and (14.b) are equal, it remains to show that the third right hand term in (19.b) yields the third right hand term in (14.b) up to indistinguishability:

$$\int_{\Re^d} \underline{\phi}(\xi_{1,t},\xi_{t-},\underline{u})\, p_{\mathrm{P}}(dt,(0,\Lambda(\xi_{t-})]\times d\underline{u}) =$$
$$= \int_{\Re^d} \underline{\phi}(\xi_{1,t},\xi_{t-},\underline{u})\mathbf{1}_{(0,\Lambda(\xi_{t-})]}(u_1)\, p_{\mathrm{P}}(dt,du_1\times d\underline{u}) =$$
$$= \int_{\Re^d} \underline{\phi}(\xi_{1,t},\xi_{t-},\underline{u})\sum_{i=1}^{N}\mathbf{1}_{(\Sigma_{i-1}(\xi_{t-}),\Sigma_i(\xi_{t-})]}(u_1)\, p_{\mathrm{P}}(dt,du_1\times d\underline{u}) =$$
$$= \int_{\Re^d} \sum_{i=1}^{N}\left[\underline{\phi}(\xi_{1,t},\xi_{t-},\underline{u})\mathbf{1}_{(\Sigma_{i-1}(\xi_{t-}),\Sigma_i(\xi_{t-})]}(u_1)\right] p_{\mathrm{P}}(dt,du_1\times d\underline{u}) =$$
$$= \int_{\Re^d} \sum_{i=1}^{N}\left[\underline{\phi}(\xi_{1,t-}+d\xi_{1,t},\xi_{t-},\underline{u})\mathbf{1}_{(\Sigma_{i-1}(\xi_{t-}),\Sigma_i(\xi_{t-})]}(u_1)\right] p_{\mathrm{P}}(dt,du_1\times d\underline{u}) =$$
$$= \int_{\Re^d} \sum_{i=1}^{N}\left[\underline{\phi}(\xi_{1,t-}+(\eta_i-\xi_{1,t-}),\xi_{t-},\underline{u})\mathbf{1}_{(\Sigma_{i-1}(\xi_{t-}),\Sigma_i(\xi_{t-})]}(u_1)\right] p_{\mathrm{P}}(dt,du_1\times d\underline{u}) =$$
$$= \int_{\Re^d} \sum_{i=1}^{N}\left[\underline{\phi}(\eta_i,\xi_{t-},\underline{u})\mathbf{1}_{(\Sigma_{i-1}(\xi_{t-}),\Sigma_i(\xi_{t-})]}(u_1)\right] p_{\mathrm{P}}(dt,du_1\times d\underline{u})$$

QED

Remark:
We notice the interesting aspect that $\xi_{1,t}$ appears in the coefficient of (19.b)'s third right hand term. This means that this coefficient anticipates a switching from $\xi_{1,t-}$ to $\xi_{1,t}$, and thus a jump of $\{\underline{\xi}_t\}$ anticipates a simultaneous transition of $\{\xi_{1,t}\}$; i.e. $\{\xi_t\}$ has hybrid jumps.

A.5 *Jump linear Gaussian system with hybrid jumps*

For $\underline{\phi} = 0$, system (19.b) has been very well studied, in particular for the case that $\{\xi_{1,t}\}$ is a Markov process and that $\underline{\xi}_t$ is conditionally Gaussian given the σ-algebra generated by $\{\xi_{1,s}; s \leq t\}$. It would be interesting to see what happens with (19.b) under similar conditions, but with $\underline{\phi} \neq 0$. For the conditional Gaussian property of $\underline{\xi}_t$, the differentials in (19.b) should be linear in $(\underline{\xi}_t, dw_t, \underline{u})$, the mapping $\underline{\phi}(\theta,(\theta,\cdot),\underline{u})$ should vanish for all θ, and $\underline{\mu}$ should be Gaussian. For Markovian $\{\xi_{1,t}\}$ both $\Lambda(\xi)$ and $\rho(\theta,\xi)$ should be $\underline{\xi}$-invariant for all (θ,ξ_1).

Hence, we introduce the following assumptions;

L0 $\underline{\alpha}(\theta,x) = A_c(\theta) + A_x(\theta)x$
$\underline{\beta}(\theta,x) = B(\theta)$
$\underline{\phi}(\theta,(\eta,x),\underline{u}) = C_c(\theta,\eta) + C_x(\theta,\eta)x + C_u(\theta,\eta)\underline{u}$

L1 $C_c(\theta,\theta) = C_x(\theta,\theta) = C_u(\theta,\theta) = 0$ for all $\theta \in \mathcal{M}$,

L2 $\underline{\mu}$ is Gaussian with finite mean and finite covariance.

L3 $\Lambda(\xi)$ and $\rho(\theta,\xi)$ are bounded and $\underline{\xi}$-invariant for all $(\theta,\xi_1) \in \mathcal{M} \times \mathcal{M}$.

With this, and introducing $x_t = \underline{\xi}_t$ and $\theta_t = \xi_{1,t}$, (19.b) becomes:

$$dx_t = A_c(\theta_t)dt + A_x(\theta_t)x_t dt + B(\theta_t)dw_t + \\ + \int_{\Re^d} [C_c(\theta_t,\theta_{t-}) + C_x(\theta_t,\theta_{t-})x_{t-} + C_u(\theta_t,\theta_{t-})\underline{u}]\, p_P(dt,(0,\Lambda(\theta_{t-},0)]\times d\underline{u}). \tag{20}$$

The first three terms at the right hand side of equation (20) are the well-known terms for linear Gaussian systems with Markovian switching coefficients. The third term, however, is new; it allows the process $\{x_t\}$ to jump simultaneously with and depending of the switching of $\{\theta_t\}$. It can easily be verified that L0, L1, L2, and L3 imply that the conditions of Theorem 4.1 and Corollary 4.2 are satisfied:
L0 implies A.1 and A.2
L3 implies A.3 and A.4
L0 and L2 imply A.5
L1 implies A.6

A.6 General Stochastic Hybrid Process

Now we assume that for each $\theta \in \mathcal{M}$ there is an open connected subset E^θ of $\Re^{n-1}$, the boundary of which is denoted as ∂E^θ. We consider a sequence $\{\{\xi_t^i\}; i = 0,1,\ldots\}$ of processes:
$\{\xi_t^0\}$ is the solution of (19a,b) for $t \geq \tau_0 = 0$, with $\xi_0^0(\omega) \in \{\{\theta\}\times E^\theta; \theta \in \mathcal{M}\}$

$$\tau_1 \underline{\underline{\Delta}} \inf\{t > \tau_0; \xi_t^0 \in \{\{\theta\}\times\partial E^\theta; \theta \in \mathcal{M}\}\} \tag{21}$$

$\{\xi_t^1\}$ is indistinguishable from $\{\xi_t^0\}$ for $t < \tau_1$, and is the solution of (19a,b) for $t \geq \tau_1$, with initial condition $\xi_{\tau_1}^1(\omega)$ satisfying

$$P\{\xi_{1,\tau_1}^1 = \theta, \underline{\xi}_{\tau_1}^1 \in \underline{A}^\theta \mid \xi_{\tau_1-}^0 = \xi\} = Q(\{\theta\}\times\underline{A}^\theta, \xi) \tag{22}$$

(with $\theta \in \mathcal{M}$ and $\underline{A}^\theta$ a Borel subset of E^θ),
and so on. Hence for $i = 1,2,3,\ldots$,

$$\tau_i \underline{\underline{\Delta}} \inf\{t > \tau_{i-1}; \xi_t^i \in \{\{\theta\}\times\partial E^\theta; \theta \in \mathcal{M}\}\}, \tag{23}$$

$\{\xi_t^i\}$ is indistinguishable from $\{\xi_t^{i-1}\}$ for $t < \tau_i$, and is the solution of (19a,b) for $t \geq \tau_i$, with initial condition $\xi_{\tau_i}^i(\omega)$ satisfying

$$P\{\xi_{1,\tau_i}^i = \theta, \underline{\xi}_{\tau_i}^i \in \underline{A}^\theta \mid \xi_{\tau_i-}^{i-1} = \xi\} = Q(\{\theta\}\times\underline{A}^\theta, \xi). \tag{24}$$

Corollary 6.1

Let the conditions of Theorem 4.1 hold true. Let for each $\theta \in \mathcal{M}$, E^θ be an open subset of $\Re^{n-1}$ with boundary ∂E^θ, and let $(x + \underline{\phi}(\eta,\theta,x,\underline{u})) \in E^\eta$ for each $x \in E^\theta$, $\underline{u} \in \Re^d$. Let the collection of processes $\{\{\xi_t^i\}, i = 0,1,2,...\}$ be defined by (19a,b) and (21) through (24). Then for every initial condition $\xi_0^0(\omega) \in \Re^n$, equations (19a,b) and (21) through (24) have pathwise unique solutions $\{\xi_t^i\}$, for i=0,1,2,…, which is càdlàg and adapted.

Now define the process $\{\xi_t^*\}$ as follows:

$$\xi_t^* = \begin{cases} \xi_t^0 & \text{for all } t \in [0,\tau_1) \\ \xi_t^i & \text{for all } t \in [\tau_i,\tau_{i+1}),\ i = 1,2,... \end{cases} \tag{25}$$

Next, we adopt the following assumptions:

B.1 $\{\xi_t^*(\omega)\}$ hits the boundary $\partial E \underline{\underline{\Delta}} \bigcup_{\theta \in \mathcal{M}} \{\theta\} \times \partial E^{\theta}$ a finite number of times on any finite time interval,

B.2 $\{\tau_i\}$ is a sequence of predictable stopping times,

B.3 $\underline{\xi} + \underline{\phi}(\eta,(\theta,\underline{\xi}),\underline{u}) \in E^{\eta}$ for each $\underline{\xi} \in E^{\theta}$, $\underline{u} \in \Re^d$, and $\eta,\theta \in \mathcal{M}$.

Theorem 6.2

Let the conditions of Theorem 4.1 hold true. Let for each $\theta \in \mathcal{M}$, E^{θ} be an open subset of $\Re^{n-1}$ with boundary ∂E^{θ}, and let $(x + \underline{\phi}(\eta,\theta,x,\underline{u})) \in E^{\eta}$ for each $x \in E^{\theta}$, $\underline{u} \in \Re^d$. Let $\{\xi_t^*\}$ be defined by equations (19a,b) and (21) through (25), let assumptions **B.1**, **B.2** and **B.3** hold true, and let $\xi_0^*(\omega) \in E \underline{\underline{\Delta}} \bigcup_{\theta \in \mathcal{M}} \{\theta\} \times E^{\theta}$ for all ω. Then $\{\xi_t^*\}$ is a Markov process assuming values in the hybrid state space $\mathcal{M} \times \Re^{n-1}$, and its generator $\mathcal{A}$ satisfies:

$$\mathcal{A} f = \mathcal{L} f + \mathcal{J} f$$

for all $f \in \left\{ f' \in C^2(\Re^n); f'(\xi) = \int_E f'(\xi') Q(d\xi',\xi) \text{ for all } \xi \in \partial E \right\}$.

Proof:

From **B.1** and Corollary 6.1 it follows that $\{\xi_t^*\}$ is the pathwise unique solution of equations (19a,b) and (21) through (25). Due to **B.2** $\{\xi_t^*\}$ admits a unique canonical martingale decomposition. Hence, together with **B.3**, for the particular forms of f considered, the generator $\mathcal{A}$ follows from Itô's differentiation rule for discontinuous semi-martingales.

QED

Remark:

It should be noticed that Theorem 6.1 is less powerful than Theorem 4.1 is on two points:
- The strong Markov property has not been proven.
- The domain of the generator is rather limited, similar to PDP (Davis, 1984).

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Appendix B Acronyms

ATC	Air Traffic Control
ATM	Air Traffic Management
CCA	Common Cause Analysis
CRM	Collision Risk Model
DCPN	Dynamically Coloured Petri Net
ETA	Event Tree Analysis
FMEA	Failure Mode and Effect Analysis
FTA	Fault Tree Analysis
HMI	Human Machine Interface
ICAO	International Civil Aviation Organisation
JAA	Joint Aviation Authorities
NASPAC	National Airspace Systems Performance Analysis Capability
NLR	National Aerospace Laboratory NLR
PHA	Preliminary Hazard Analysis
PDMP	Piecewise Deterministic Markov Process
RAMS	Reorganized ATC Mathematical Simulator
ODE	Ordinary Differential Equation
SDE	Stochastic Differential Equation
TAAM	Total Airspace and Airport Modeller
TLS	Target Level of Safety
TOPAZ	Traffic Organization and Perturbation AnalyZer