Theoretical Foundations of Stochastic Hybrid Systems*

Manuela L. Bujorianu† and John Lygeros‡

†Department of Engineering
University of Cambridge
Cambridge, CB2 1PZ, UK
email: lmb56@eng.cam.ac.uk
‡Department of Electrical and Computer Engineering
University of Patras
Patras, GR26500, Greece
email: lygeros@ee.upatras.gr

Abstract: In this paper we set up a mathematical structure, called Markov String, to obtaining a very general class of models for stochastic hybrid systems. Markov Strings are, in fact, a class of Markov processes, obtained by a mixing mechanism of stochastic processes, introduced by Meyer. We prove that Markov strings are strong Markov processes with the cadlag property. We then show how a very general class of stochastic hybrid processes can be embedded in the framework of Markov Strings. This class, which is referred to as the General Stochastic Hybrid Systems (GSHS), includes as special cases all the classes of stochastic hybrid processes, proposed in the literature.

Keywords: stochastic hybrid systems, Markov string, Markov processes, strong Markov property, cadlag.

1 Introduction

We formulate a very general class of Markov processes, which will be called *Markov Strings*, loosely based on the so-called "melange" operation of Markov processes [10]. We start with a countable family of Markov processes with some nice properties: the strong Markov property, the càdlàg property. For each process belonging to this family, we underlie the associated probabilistic elements: probability space, natural filtration, translation operator, probabilities on the trajectories. We suppose that a stopping time associated to each process, with memoryless property, and a renewal kernel are priory given. The stopping times play the role of the jump times from one process to another and the renewal kernel gives the distribution of the post-jump location. The probabilistic construction of the Markov String is natural:
1. start with one process which belongs to the given family;

*This work was supported by the European Commission under project HYBRIDGE, IST-2001-32460.

2. kill the starting process at the time T_1 of the first jump;
3. jump according to the renewal kernel;
4. restart an other process (belonging to the given family) from the new location;
5. proceed until the jump time of the new process and kill again, etc.

The pieced together process obtained by the above procedure is called Markov String. Its jump structure is completely described by a renewal kernel, which is priory given, and the family of stopping times described above. To eliminate pathological solutions that take an infinite number of discrete transitions in a finite amount of time (known as Zeno solutions) we impose that the resulted Markov String has finitely many jumps in finite time. We prove that the above Markov Strings, as stochastic processes, enjoy useful properties like the (strong) Markov property and the càdlàg property.

Markov Strings differ from the class of processes considered in [10] in that:
1. The jump times are essentially given stopping times, *not necessarily the life times of the component processes*; 2. After a jump the string restarts following an other process which might be different from the pre-jump process.

The mixing ("melange") operation in [10] is only sketched and the author claims that it can be obtained using the renewal ("renaissance") operation. We consider that the passing from renewal to mixing is not straightforward. It is necessary to emphases the construction of all probabilistic elements associated with the resulted string. Lifting the renewal construction to the mixing construction, remarkable changes should be introduced in the Markov string definitions of the state space, probability space, probabilities on the trajectories.

We then show how a very general class of stochastic hybrid processes can be embedded in the framework of Markov Strings. This class, which is referred to as the *General Stochastic Hybrid Systems* (GSHS), allows:
1. Diffusion processes in the continuous evolution; 2. Discrete transitions which are *spontaneous* (according to a transition rate) or *forced* (driven by a boundary hitting time); 3. Probabilistic reset of the discrete and continuous state as a result of discrete transitions.

The class of GSHS includes as special cases all the classes of stochastic hybrid processes, proposed in the literature, as Piecewise Deterministic Markov Processes (PDMP) [6, 4, 11], Stochastic Hybrid Systems (SHS) [8] and Switching Diffusion Processes (SDP) [7]. We show that the class of GSHS inherits the strong Markov and càdlàg properties from Markov Strings.

2 Markov Strings

In this section we define the Markov string notion, which, roughly speaking, is a stochastic process obtained by mixing some given Markov processes.

We prove that if we start with a countable family of 'nice' Markov processes then the Markov string resulted will inherit the properties of its components.

2.1 The Ingredients

Suppose that $\mathbb{M}^i = (\Omega^i, \mathcal{F}^i, \mathcal{F}^i_t, x^i_t, \theta^i_t, P^i, P^i_{x^i})$, $i \in Q$ is a countable family of Markov processes. We denote the state space of each $\mathbb{M}^i$ by $(X^i, \mathcal{B}^i)$ and assume that $\mathcal{B}^i$ is the Borel σ-algebra of X^i if X^i is a topological Hausdorff space. Let Δ be the cemetery point for all X^i, $i \in Q$, which is an adjoined point to X^i, $X^i_\Delta = X^i \cup \{\Delta\}$. The existence of Δ is assumed in order to have a probabilistic interpretation of $P^i_{x^i}(x^i_t \in X^i) < 1$, i.e. at some 'termination time' $\zeta^i(\omega_i)$ the process $\mathbb{M}^i$ escapes to and is trapped at Δ. For each

$i \in Q$, the elements $\mathcal{F}^i$, $\mathcal{F}_t^{i,0}$, $\mathcal{F}_t^i$, $\theta_t^i, P^i, P_{x^i}^i$ have the usual meaning as in the Markov process theory [3]. Also, in this paper, we make use some standard notions in the Markov process theory as: underlying probability space, natural filtration, translation operator, Wiener probabilities, admissible filtration, stopping time, strong Markov property [3]. Let (P_t^i) denote the operator semigroup associated to $\mathbb{M}^i$ which maps $\mathcal{B}^i(X^i)$ into itself given by

$$P_t^i f(x^i) = E_{x^i}^i f(x_t^i),$$

where $E_{x^i}^i$ is the expectation w.r.t. $P_{x^i}^i$. Then a function f is *p-excessive* $(p > 0)$ w.r.t. $\mathbb{M}^i$ if $f \geq 0$ and $e^{-pt} P_t^i f \leq f$ for all $t \geq 0$ and $e^{-pt} P_t^i f \nearrow f$ as $t \searrow 0$.

Assumption 1 *For each $i \in Q$, we suppose that:*
1. $\mathbb{M}^i$ is a strong Markov process. 2. P^i is a complete probability.
3. The state space X^i is a topological space which is homeomorphic to a Borel subset of a complete separable metric space (Borel space).
4. $\mathbb{M}^i$ enjoys the càdlàg property, i.e. for each $\omega_i \in \Omega^i$, the sample path $t \mapsto x_t^i(\omega_i)$ is right continuous on $[0, \infty)$ and has left limits on $(0, \infty)$ (inside X_Δ^i)..
5. The p-excessive functions of $\mathbb{M}^i$ are P^i-a.s. right continuous on trajectories.

Part 3. implies that the underlying probability space Ω^i can be assumed to be $D_{[0,\infty)}(X^i)$, the space of functions mapping $[0, \infty)$ to X^i which are right continuous functions with left limits. In the terminology of [9], parts 1., 3. and 5. of the Assumption 1 imply that each $\mathbb{M}^i$ is a *right process.*

Using this family of Markov processes $\{\mathbb{M}^i\}_{i \in Q}$, we define a new Markov process whose realizations consist of concatenations of realizations for different $\mathbb{M}^i$. To achieve this goal, we need to define the transition mechanism from one process to the others. The jumping mechanism will be driven by:
1. a sequence of stopping times, for each process is chosen a stopping time (which gives the jump temporal parameter),
2. a renewal kernel which gives the post jump location.

2.2 The Construction

Using the elements defined in the section 2.1 we construct a stochastic process $\mathbb{M} = (\Omega, \mathcal{F}, \mathcal{F}_t, x_t, \theta_t, P, P_x)$. The process $\mathbb{M}$ is obtained by the concatenation of the component processes $\mathbb{M}^i$ and will be called *Markov String*. Roughly speaking, this Markov string is constructed in such a way that its sample paths are obtained sticking the sample component paths between some jump times.

To completely define the Markov String we need to specify the following elements: 1. $(X, \mathcal{B})$ - the state space; 2. $(\Omega, \mathcal{F}, P)$ - the underlying probability space; 3. $\mathcal{F}_t$ - the natural filtration; 4. θ_t - the translation operator; 5. P_x - Wiener probabilities.

State Space $(X, \mathcal{B})$

The state space will be X defined as follows. X is constructed as the direct sum of spaces X^i, with the same cemetery point Δ, i.e.

$$X = \bigcup_{i \in Q} \{(i, x) | x \in X^i\}. \tag{1}$$

It is possible to define a metric ρ on X such that $\rho(x_n, x) \to 0$ as $n \to \infty$ with $x_n = (i_n, x_n^{i_n})$, $x = (i, x^i)$ if and only if there exists m such that $i_n = i$ for all $n \geq m$ and $x_{m+k}^i \to x^i$ as $k \to \infty$. The metric ρ restricted to any component X^i is equivalent to the usual component metric [6]. Each $\{i\} \times X^i$, being a Borel space, will be homeomorphic to a measurable subset of the Hilbert cube, $\mathcal{H}$ (Urysohn's theorem, Prop. 7.2 [2]). Recall that $\mathcal{H}$ is the product of countable many copies of $[0,1]$. The relation (1) implies that X will be, as well, homeomorphic to a measurable subset of $\mathcal{H}$. Thus X is a Borel space [2].

The space X can be endowed with the Borel σ-algebra $\mathcal{B}(X)$ generated by its metric topology. Moreover, we have

$$\mathcal{B}(X) = \sigma\{\bigcup_{i\in Q}\{i\} \times \mathcal{B}^i\}. \tag{2}$$

Then $(X, \mathcal{B}(X))$ is a Borel space, whose Borel σ-algebra $\mathcal{B}(X)$ restricted to each component X^i gives the initial σ-algebra $\mathcal{B}^i$ [6]. The above argument allows us to make the following remark:

Remark 1 *We can suppose, without loss of generality, that $X^i \cap X^j = \emptyset$ if $i \neq j$. Thus the relations (1) and (2) become*

$$X = \bigcup_{i\in Q} X^i; \tag{3}$$

$$\mathcal{B}(X) = \sigma(\bigcup_{i\in Q} \mathcal{B}^i). \tag{4}$$

Therefore, we can suppose, as well, that $\Omega^i \cap \Omega^j = \emptyset$ if $i \neq j$.

Probability Space

The space Ω can be thought as the space generated by the concatenation operation defined on the union of the spaces Ω^i (which pairwise disjoint), i.e. $\Omega = (\bigcup_{i\in Q}\Omega^i)^*$. Thus, the $\sigma-$algebra $\mathcal{F}$ on Ω will be the smallest $\sigma-$algebra on Ω such that the projection $\pi^i : \Omega \to \Omega^i$ are $\mathcal{F}/\mathcal{F}^i$ measurable, $i \in Q$. The probability P on $\mathcal{F}$ will be defined as a 'product measure'. Let $\widehat{\mathcal{F}}$ be the $\sigma(\bigcup\limits_{i\in Q} \mathcal{F}^i)$ defined on $\bigcup\limits_{i\in Q} \Omega^i$.

Recipe

Formally, in order to define the desired Markov string, $\mathbb{M}$, we need to give:
1. $(S^i)_{i\in Q}$, where, for each $i \in Q$, S^i is a stopping time of $\mathbb{M}^i$ with the 'memoryless' property, i.e.

$$S^i(\theta^i_t\omega_i) = S^i(\omega_i) - t, \, \forall t < S^i(\omega_i) \tag{5}$$

2. The jumping mechanism between the processes $\mathbb{M}^i$ is governed by a *renewal kernel* which is a Markovian kernel

$$\Psi : \{\bigcup_{i\in Q}\Omega^i\} \times \mathcal{B}(X) \to [0,1]$$

satisfying the following conditions: (a) If $S^i(\omega_i) = +\infty$ then $\Psi(\omega_i, \cdot) = \varepsilon_\Delta$; (b) If $t < S^i(\omega_i)$ then $\Psi(\theta^i_t\omega_i, \cdot) = \Psi(\omega_i, \cdot)$.
Notation. The cemetery point of the state space Ω^i is denoted by $[\Delta]^i$ and the cemetery point of Ω is denoted by $[\Delta]$. We use to denote by ω (resp. $\widehat{\omega}$ or ω_i) an arbitrary element of Ω (resp. $\bigcup\limits_{i\in Q}\Omega^i$ or Ω^i).

In the following, we give the procedure to construct a sample path of the stochastic process $(x_t)_{t>0}$ with values in X, starting from a fixed initial point $x_0 = x_0^{i_0} \in X^{i_0}$. Let

ω_{i_0} be a sample path of the process $(x_t^{i_0})$ starting with x_0. In fact, we give a recipe *to construct a Markov string starting with an initial path* ω_{i_0}. Let $T_1(\omega_{i_0}) = S^{i_0}(\omega_{i_0})$. The the event ω and the associated sample path are inductively defined. In the first step

$$\omega = \omega_{i_0}$$

The sample path $x_t(\omega)$ up to the first jump time is now defined as follows:

if $T_1(\omega) = \infty$: $x_t(\omega) = x_t^{i_0}(\omega_{i_0})$, $t \geq 0$
if $T_1(\omega) < \infty$: $x_t(\omega) = x_t^{i_0}(\omega_{i_0})$, $0 \leq t < T_1(\omega)$
x_{T_1} is a r.v. according to $\varepsilon_{\omega_{i_0}}\Psi$.

The process restarts from $x_{T_1} = x_1^{i_1}$ according to the same recipe, using now the process $(x_t^{i_1})$. Let ω_{i_1} be a sample of the process $(x_t^{i_1})$ starting with $x_1^{i_1}$. Thus, if $T_1(\omega) < \infty$ we define the next jump time

$$T_2(\omega_{i_0}, \omega_{i_1}) = T_1(\omega_{i_0}) + s_{i_2}(\omega_{i_2}).$$

Then, in the second step

$$\omega = \omega_{i_0} * \omega_{i_1}$$

The sample path $x_t(\omega)$ between the two jump times is now defined as follows:

if $T_2(\omega) = \infty$: $x_t(\omega) = x_{t-T_1}^{i_1}(\omega_{i_1})$, $t \geq T_1(\omega)$
if $T_2(\omega) < \infty$: $x_t(\omega) = x_t^{i_1}(\omega_{i_1})$, $0 \leq T_1(\omega) \leq t < T_2(\omega)$
x_{T_2} is a r.v. according to $\varepsilon_{\omega_{i_1}}\Psi$.

Generally, if $T_k(\omega) = T_k(\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_{k-1}}) <$ with

$$\omega = \omega_{i_0} * \omega_{i_1} * ... * \omega_{i_{k-1}}$$

then the next jump time is

$$T_{k+1}(\omega) = T_{k+1}(\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_k}) = T_k(\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_{k-1}}) + S^{i_k}(\omega_{i_k}) \tag{6}$$

The sample path $x_t(\omega)$ between the two jump times T_k and T_{k+1} is defined as:

$$\begin{array}{lcl} \text{if } T_{k+1}(\omega) & = & \infty : \; x_t(\omega) = x_{t-T_k}^{i_k}(\omega_{i_k}), t \geq T_{k+1}(\omega) \\ \text{if} T_{k+1}(\omega) & < & \infty : \; \begin{array}{l} x_t(\omega) = x_{t-T_k}^{i_k}(\omega_{i_k}), 0 \leq T_k(\omega) \leq t < T_{k+1}(\omega) \\ x_{T_{k+1}} \text{ is a r.v. according to } \varepsilon_{\omega_{i_k}}\Psi. \end{array} \end{array} \tag{7}$$

We have constructed a sequence of jump times $0 < T_1 < T_2 < ... < T_n < ...$ Let $T_\infty = \lim_{n\to\infty} T_n$. Then $x_t(\omega) = \Delta$ if $t \geq T_\infty$. A sample path until T_{k_0} (where $k_0 = \min\{k : S^{i_k}(\omega) = \infty\}$) of the process (x_t), starting from a fixed initial point $x_0 = (i_0, x_0^{i_0})$, is obtained as the concatenation:

$$\omega = \omega_{i_0} * \omega_{i_1} * ... * \omega_{i_{k_0-1}}.$$

We denote $N_t(\omega) = \sum I_{(t \geq T_k)}$ the number of jump times in the interval $[0, t]$. To eliminate pathological solutions that take an infinite number of discrete transitions in a finite amount of time (known as Zeno solutions) we impose the following assumption:

Assumption 2 *For every starting point $x \in X$, $EN_t < \infty$, for all $t \in \mathbb{R}_+$.*

Under the assumption 2, the underlying probability space Ω can be identified with $D_{[0,\infty)}(X)$.

Wiener Probabilities

One might define the expectation $E^x f$, $x \in X$, where f is a $\mathcal{F}$-measurable function on Ω, which depends only on a finite number of variables, by recursion on the number of variables.
Step1. If f depends only on ω_i, i.e. $f(\omega) = f_1(\omega_i)$ with f_1 a $\mathcal{F}^i$-measurable function on Ω^i, then
• if $x = x^i \in X^i$ then $E_x f = E^i_{x^i} f$, where $E^i_{x^i}$ is the expectation corresponding to the probability $P^i_{x^i}$;
• if $x = x^j \in X^j$, $j \neq i$ then $E_x f = 0$.
Step2. If f depends only on $\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_n}$, i.e. $f(\omega) = f_n(\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_n})$ with f_n a $\prod_{k=1}^{n} \mathcal{F}^{i_k}$-measurable function on $\prod_{k=1}^{n} \Omega^{i_k}$ then

$$\begin{aligned} f_{n-1}(\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_{n-1}}) &= \int_{\Omega^{i_n}} f_n(\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_{n-1}}, \omega_{i_n}) dP^{i_n}_{\Psi(\omega_{i_{n-1}}, \cdot)}(\omega_{i_n}); \\ g(\omega) &= f_{n-1}(\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_{n-1}}); \\ E_x f &= E_x g. \end{aligned} \tag{8}$$

Translation Operators

Let us define now the translation operator (θ_t) associated with (x_t). If $t \geq T_\infty(\omega)$, then we take $\theta_t(\omega) = [\Delta] = ([\Delta]^i)_{i \in Q}$. Otherwise, there exists k such that $T_k(\omega) \leq t < T_{k+1}(\omega)$. In this case we take

$$\theta_t(\omega) = (\theta^{i_k}_{t-T_k(\omega)}(\omega_{i_k}), \omega_{i_{k+1}}, ...). \tag{9}$$

Lemma 1 *(θ_t) is the translation operator associated with (x_t), i.e.*

$$\theta_s \circ \theta_t = \theta_{s+t};\ x_s \circ \theta_t = x_{s+t}.$$

Proof. If $t \geq T_\infty(\omega)$, then $\theta_t(\omega) = [\Delta]$ and $x_{s+t}(\omega) = \Delta = x_s(\theta_t(\omega))$.
Suppose that there exist $k, i \geq 0$ such that $T_k(\omega) \leq t < T_{k+1}(\omega)$ and $T_i(\theta_t \omega) \leq s < T_{i+1}(\theta_t \omega)$. Then

$$x_t(\omega) = x^{i_k}_{t-T_k}(\omega_{i_k});\ (x_s \circ \theta_t)(\omega) = x^{i_l}_{s-T_l}(\theta^{i_l}_{s-T_l} \omega_{i_l}).$$

Since $\theta_t(\omega)$ is given by (9) and T_{k+1} is given by (6) we obtain

$$\begin{aligned} T_{k+1}(\theta_t \omega) &= S^{i_k}(\theta^{i_k}_{t-T_k(\omega)}(\omega_{i_k})) = S^{i_k}(\omega_{i_k}) - (t - T_k(\omega)) \\ &= T_{k+1}(\omega) - t. \end{aligned}$$

Then

$$T_{i+1}(\theta_t \omega) = T_{k+i+1}(\omega) - t$$

Therefore

$$T_i(\theta_t \omega) \leq s < T_{i+1}(\theta_t \omega) \Leftrightarrow T_{k+i}(\omega) \leq s + t < T_{k+i+1}(\omega).$$

Markov Chain

Let (p_n) be a discrete time Markov chain associated to (x_t) with the state space $(\bigcup_{i \in Q} \Omega_i, \widehat{\mathcal{F}})$ and the underlying probability space $(\Omega, \mathcal{F})$. The chain (p_n) is essentially 'the $n-th$' step of the process (x_t). If its starting point is ω_{i_0} (a trajectory in Ω^{i_0} starting in $x^{i_0}_0$) then $p_n(\omega) = \omega_{i_n}$.
The transition kernel associated with (p_n) can be defined as follows:

$$H(\widehat{\omega}, A) = P_{\varepsilon_{\widehat{\omega}} \Psi}(A),\ A \in \widehat{\mathcal{F}}.$$

The construction of P^x from subsection 2.2 is such that
- H is the transition function of (p_n);
- P^x is the initial probability law of (p_n); i.e. if $\widehat{\omega} \in \bigcup_{i\in Q} \Omega_i$ which starts in $x \in X$

$$P^{\widehat{\omega}}(p_0 \in A) = P^x(A),\ A \in \mathcal{F}.$$

Let η_k be the projection $(p_0, p_1, ..., p_k)$, i.e. $\eta_k(\omega) = (\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_k})$.

Natural Filtrations

Let $(\mathcal{F}_t)$ be the natural filtration with respect to (x_t). The natural filtration $(\mathcal{F}_t)$ on Ω is built such that we have the following definition of $\mathcal{F}_t$-measurability:

Definition 2 *A $\mathcal{F}$-measurable function f on Ω is $\mathcal{F}_t$-measurable if the following property holds:*
For each k, the function $f \cdot I_{\{T_k(\omega)\le t<T_{k+1}(\omega)\}}$ is equal to $h \circ \eta_k$, where the function $h(\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_k})$ is such that for a fixed $(\widehat{\omega}_{i_0}, \widehat{\omega}_{i_1}, ..., \widehat{\omega}_{i_{k-1}})$ with $T_k(\widehat{\omega}_{i_0}, \widehat{\omega}_{i_1}, ..., \widehat{\omega}_{i_{k-1}}) \le t$, $\omega_{i_k} \mapsto h(\widehat{\omega}_{i_0}, \widehat{\omega}_{i_1}, ..., \widehat{\omega}_{i_{k-1}}, \omega_{i_k})$ is measurable with respect to $\mathcal{F}^{i_k}_{t-T_k}$.

Because the families of filtrations $(\mathcal{F}^i_t)$ are nondecreasing and right continuous, one can verify that the family $(\mathcal{F}_t)$ has the same properties, as follows.

Proposition 3 *(i) The family $(\mathcal{F}_t)$ is nondecreasing and right continuous.*
(ii) The random variables T_k are stopping times w.r.t. $(\mathcal{F}_t)$.
(iii) Let T a stopping time with respect to $(\mathcal{F}_t)$. For each $k \in \mathbb{N}$, $T \wedge T_k$ is a function on Ω which depends only on $\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_{k-1}}$. On the other hand, if $\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_{k-1}}$ are fixed, the function $(T \wedge T_{k+1} - T_k)^+$ with ω_{i_k} as argument is a stopping time with respect $(\mathcal{F}^{i_k}_t)$.

Proof. The proof can be obtained with small changes from the similar result proofs given in [10] for the case of rebirth processes.

Jump Process

Fix $x \in X$ and consider a Markov string (x_t) starting at x as constructed above. The associated *jump process* (η_t) takes values in $X \times \mathbb{Z}_+$ is defined as

$$\eta_t = \begin{bmatrix} x \\ 0 \end{bmatrix},\ t < T_1, ...,\ \eta_t = \begin{bmatrix} \eta^1_t \\ \eta^2_t \end{bmatrix} = \begin{bmatrix} x_{T_k} \\ k \end{bmatrix},\ T_k \le t < T_{k+1}.$$

We do not have a one-to-one correspondence between the sample paths of (x_t) and (η_t), as in the case of piecewise deterministic Markov processes [6]. Given the sample path $\{x_s, s \le t\}$, the finite set of jump times is $\{T_j,\ j = 1, ..., k\} = \{s \in (0, t] : x_s \ne x_{s-}\}$, and the sample path $\{\eta_s,\ s \le t\}$ is defined using the above recipe. But conversely, given $\{\eta_s, s \le t\}$, since $x_0 = \eta^1_0$ then it is possible to find more then one trajectories which start from x_0. In fact, a sample trajectory of (η_t) is associated with a family of sample paths of (x_t).
Therefore, the above jump process will not serve to study the Markov string, as in [6]. Its role is taken by the Markov chain constructed in section 2.2.

2.3 Basic Properties

Simple Markov Property

Mainly, in this section we prove that the Markov string (x_t) constructed in section 2.2 is a right Markov process. The proof engine is based on the Markov property of the discrete time Markov chain (p_n).

Remark 2 *For each k on the set $\{T_k(\omega) \leq t < T_{k+1}(\omega)\}$ we have: $x_t = x^{i_k}_{t-T_k} \circ p_k$.*

Proposition 4 *Any Markov string $\mathbb{M} = (\Omega, \mathcal{F}, \mathcal{F}_t, x_t, \theta_t, P, P_x)$, obtained using the procedure presented in section 2.2, is a Markov process.*

Proof. The simple Markov property of (x_t) is equivalent to the following implication [10]:
If f is a positive $\mathcal{F}_t$-measurable function and g is a $\mathcal{F}$-measurable function then

$$E^x[f \cdot g \circ \theta_t] = E^x[f \cdot E^{x_t}[g]]. \tag{10}$$

The identity (10) can be unfolded into two separated equalities

$$E^x[f \cdot g \circ \theta_t \cdot I_{\{t \geq T_\infty\}}] = E^x[f \cdot E^{x_t}[g] \cdot I_{\{t \geq T_\infty\}}] \tag{11}$$

$$E^x[f \cdot g \circ \theta_t \cdot I_{\{T_k(\omega) \leq t < T_{k+1}(\omega)\}}] = E^x[f \cdot E^{x_t}[g] \cdot I_{\{T_k(\omega) \leq t < T_{k+1}(\omega)\}}] \tag{12}$$

The identity (11) is clear because on $\{t \geq T_\infty\}$

$$E^{x_t}[g] = g([\Delta]);\ \theta_t(\omega) = [\Delta];\ x_t(\omega) = \Delta.$$

Let us prove now the identity (12). Let $\omega \in \Omega$. By the definition of $\mathcal{F}_t$ we have

$$f(\omega) \cdot I_{\{T_k(\omega) \leq t < T_{k+1}(\omega)\}}(\omega) = h(\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_k}) \tag{13}$$

where h is a measurable function as in the definition 2 and is equal to zero outside of the set $\{T_k(\omega) \leq t < T_{k+1}(\omega)\}$.

In order to prove (12) it is enough to treat the case when the function g depends only on a finite number of variables (because the expectation E^x is defined by the recursion (8)).

We start with the case when the function g depends only on a single variable, ω_{i_0}, i.e. $g(\omega) = a(\omega_{i_0})$, where a is $\mathcal{F}^{i_0}$-measurable on Ω^{i_0}. In this case, the left-hand side of (12) is equal to

$$E^x[f \cdot I_{\{T_k(\omega) \leq t < T_{k+1}(\omega)\}} \cdot a(\theta^{i_k}_{t-T_k(\omega)}(\omega_{i_k}))]. \tag{14}$$

Because the term between [...] depends only on $(\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_k})$, (14) becomes

$$E^x\{\int_{\Omega^{i_k}} h(\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_k}) \cdot a(\theta^{i_k}_{t-T_k(\omega)}(\omega_{i_k}))dP^{i_k}_{\Psi(\omega_{i_{k-1}}, \cdot)}(\omega_{i_k})\}. \tag{15}$$

Again, the integrand between $\{...\}$ depends only on $(\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_{k-1}})$. Since the function $\omega_{i_k} \to h(\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_k})$ is $\mathcal{F}^{i_k}_{t-T_k}$-measurable, we can use the Markov property of the process $\mathbb{M}^{i_k}$ and (15) becomes

$$\int_{\Omega^{i_k}} h(\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_k}) E^{i_k}_{x^{i_k}_{t-T_k}(\omega_{i_k})}[a] dP^{i_k}_{\Psi(\omega_{i_{k-1}}, \cdot)}(\omega_{i_k}). \tag{16}$$

Since $x_t(\omega) = x^{i_k}_{t-T_k}(\omega_{i_k})$ on $\{T_k(\omega) \le t < T_{k+1}(\omega)\}$ the computation of the right-hand side of (12) gives

$$E^x\{h(\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_k}) \cdot E^{i_k}_{x^{i_k}_{t-T_k}(\omega_{i_k})}[a]\} \tag{17}$$

Using the recursive procedure, as before, (17) gives (16).

Suppose now that (12) is established for all functions g which depend only on $(\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_{k-1}})$. We have to prove that (12) is true for

$$g(\omega) = g(\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_k});\ k > 0.$$

Let

$$c(\omega) = c(\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_{k-1}}) = \int_{\Omega^{i_k}} b(\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_k}) dP^{i_k}_{\Psi(\omega_{i_{k-1}}, \cdot)}(\omega_{i_k}).$$

Using the recursive procedure, one can check that the functions

$$h(...) g \circ \theta_t \quad \text{and} \quad h(...) c \circ \theta_t$$

have the same expectations.
On the other hand, the functions

$$h(...) E_{x_t}[g] \quad \text{and} \quad h(...) E_{x_t} c$$

have the same expectations. Since c depends only on $k-1$ variables, this implies (12) for the general case.

Cadlag Property

Proposition 5 *If $\mathbb{M} = (\Omega, \mathcal{F}, \mathcal{F}_t, x_t, \theta_t, P, P_x)$ is a Markov string as in section 2.2, then for all $\omega \in \Omega$ the trajectories $t \mapsto x_t(\omega)$ are right continuous on $[0, \infty)$ with left limits on $(0, \infty)$.*

Proof. The result is a direct consequence of two facts:
1. the sample paths of (x_t) are obtained by concatenation of sample paths of component process;
2. the component processes enjoy the càdlàg property.
Then the Markov string inherits the càdlàg property.

Strong Markov Property

Theorem 6 *Any Markov string $\mathbb{M} = (\Omega, \mathcal{F}, \mathcal{F}_t, x_t, \theta_t, P, P_x)$, obtained using the procedure presented in section 2.2, is a strong Markov process.*

Each T_k is a stopping time for (x_t) (see proposition 3 (ii)). For each $k \ge 1$, T_k can be obtained by the following recursion

$$T_{k+1} = T_k + S^{i_k} \circ \theta_{T_k}$$

Let us prove now that the process (x_t) is a strong Markov process. The filtration $(\mathcal{F}_t)$ is nondecreasing and right continuous (see proposition 3 (i)). Then the process (x_t) satisfies the right hypothesis.
Let (P_t) be the semigroup of the whole Markov process (x_t), $P_t g(x) = E_x g(x_t)$, where g is bounded $\mathcal{B}$-measurable function. Let $(U_p)_{p>0}$ the resolvent associated to the semigroup, i.e.

$$U_p g = \int_0^\infty e^{-pt} P_t g dt.$$

It is known that the strong Markov property is equivalent with each from the following assertions [9]:
1. If g is a positive bounded continuous function on X_Δ then $f = U_p g$ $(p > 0)$ is nearly Borel and right continuous on the process trajectories.
2. Each p-excessive function $(p > 0)$ is nearly Borel and right continuous on the process trajectories.
Recall that a real function defined on the state space X_Δ is nearly Borel for the process (x_t) if there exist two Borel function h and h' on X_Δ such that $h' \leq f \leq h$ and

$$P\{\omega | \exists t, h' \circ x_t(\omega) < h \circ x_t(\omega)\} = 0. \tag{18}$$

Let g be a positive bounded continuous function on X. We have $g = \sum_{i \in Q} g^i$, where $g^i = g|_{X^i}$ are bounded continuous functions on X^i. Then $P_t g = \sum_{i \in Q} P_t^i g^i$ and

$$U_p g = \int_0^\infty e^{-pt} P_t g dt = \sum_{i \in Q} \int_0^\infty e^{-pt} P_t^i g^i dt = \sum_{i \in Q} U_p^i g^i.$$

It is known that $f = U_p g$ $(p > 0)$ (the restriction to X) is p-excessive function with respect to (P_t) and for each $i \in Q$ and the function $f^i = U_p^i g^i$ is p-excessive function with respect to (P_t^i). Therefore, f^i is nearly Borel and right continuous on the trajectories of the process (x_t^i). It is clear from the construction that the function f is right continuous on the trajectories of the process (x_t).
Let $h^i, h^{i\prime}$ two Borel functions on X_Δ^i such that $h' \leq f^i \leq h^i$ and

$$h^{i\prime} \circ x_t^i(\omega_i) = h^i \circ x_t^i(\omega_i) \;\; P^i - a.s., \forall t \geq 0. \tag{19}$$

Let us consider the function h, h' defined as below:

$$h = \sum_{i \in Q} h^i, \; h' = \sum_{i \in Q} h^{i\prime} \; . \tag{20}$$

It is clear that

$$P\{\omega | \exists t \geq T_\infty, h' \circ x_t(\omega) < h \circ x_t(\omega)\} = 0.$$

Let us compute the probability of the following event:

$$A_k = \{\exists t | T_k \leq t < T_{k+1}, h' \circ x_t(\omega) < h \circ x_t(\omega)\}.$$

We have $A_k \in \mathcal{F}$. Let $a_k = I_{A_k}$ which depends only on $\omega_{i_0}, \omega_{i_2}, ..., \omega_{i_k}$. The recursive method to compute the probability of A_k on $\{T_k \leq t < T_{k+1}\}$ gives

$$\int_{\Omega^{i_k}} a_k(\omega_{i_0}, \omega_{i_2}, ..., \omega_{i_k}) dP^{i_k}_{\Psi(\omega_{i_{k-1}}, \cdot)}(\omega_{i_k}). \tag{21}$$

Since $a_k(\omega_{i_0}, \omega_{i_2}, ..., \omega_{i_k})$ on Ω^{i_k} is exactly the indicator function of

$$B = \{\omega_{i_k} | \exists u < S^{i_k}(\omega_{i_k}), h^{i_k\prime} \circ x_u^{i_k}(\omega) < h^{i_k} \circ x_u^{i_k}(\omega)\}$$

using (19) we obtain that the integral (21) is zero. Therefore the functions h, h' defined by (20) verify the condition (18). Then f will be a nearly Borel function relative to the process (x_t).

3 General Stochastic Hybrid Systems

3.1 Description

General Stochastic Hybrid Systems (GSHS) are a class of non-linear stochastic continuous-time hybrid dynamical systems. GSHS are characterized by an hybrid state defined by two components: the continuous state and the discrete state. The continuous state evolves in according to a stochastic differential equation (SDE) whose vector field and drift factor depend on the hybrid state, both continuous and discrete. Switching between two discrete states is governed by a probability law or occurs when the continuous state hits the boundary of its state space. Whenever a switching occurs, the hybrid state is reset instantly to a new state in according to a probability law which depends itself on the past hybrid state.

GSHS involve a hybrid state space, with both continuous and discrete states. The continuous and the discrete parts of the state variable have their own natural dynamics, but the main point is to capture the interaction between them.

The time t is measured continuously. The state of the system is represented by a continuous variable x and a discrete variable i. The continuous variable evolves in some "cells" X^i (open sets in the Euclidean space) and the discrete variable belongs to a countable set Q. The intrinsic difference between the discrete and continuous variables, consists of the way that they evolve through time. The continuous state is governed by an SDE that depends on the hybrid state. The discrete dynamics produces transitions in both (continuous and discrete) state variables x, i. Transitions occur when the continuous state hits the boundary of the state space (forced transitions) or according with a probability law. Whenever a transition occurs the hybrid state is reset instantly to a new value. The value of the discrete state after the transition is determined by the hybrid state before the transition. On the other hand, the new value of the continuous state obeys a probability law which depends on the last hybrid state. Thus, a sample trajectory has the form $(q_t, x_t, t \geq 0)$, where $(x_t, t \geq 0)$ is piecewise continuous and $q_t \in Q$ is piecewise constant. Let $(0 \leq T_1 \leq T_2 \leq ... \leq T_i \leq T_{i+1} \leq ...)$ be the sequence of jump times at which the continuous and the discrete part of the system interact. This time sequence is generated when the state of the system hits the boundary or according with a transition rate.

In fact, GSHS are a class of stochastic processes which generalizes piecewise deterministic Markov processes (PDMP) introduced by Davis in [6]. The difference between GSHS and PDMP is that for GSHS between two consecutive jumps the process is a diffusion whilst for PDMP the inter-jumps motion is deterministic, according to a vector field.

3.2 The Abstract Model

State space

Let Q be a countable set of discrete states, and let $d : Q \to \mathbb{N}$ and $\mathcal{X} : Q \to \mathbb{R}^{d(\cdot)}$ be two maps assigning to each discrete state $i \in Q$ an open subset X^i of $\mathbb{R}^{d(i)}$. We call the set

$$X(Q, d, \mathcal{X}) = \bigcup_{i \in Q} \{i\} \times X^i$$

the *hybrid state space* of the GSHS and $x = (i, x^i) \in X(Q, d, \mathcal{X})$ the *hybrid state*. The completion of the hybrid state space will be

$$\overline{X} = X \cup \partial X$$

where

$$\partial X = \bigcup_{i \in Q} \{i\} \times \partial X^i,$$

It is clear that, for each $i \in Q$, the state space X^i is a Borel space. It is possible to define a metric ρ on X in such a way the restriction of ρ to any component X^i is equivalent to the usual Euclidean metric (see section 2). Then $(X, \mathcal{B}(X))$ is a Borel space (see section 2 for the construction of $\mathcal{B}(X)$). Moreover, X is a homeomorphic with a Borel subset of a compact metric space (Lusin space) because it is a locally compact Hausdorff space with countable base (see [6] and the references therein).

Construction

Assumption 3 *Suppose that* $b : Q \times X^{(\cdot)} \to \mathbb{R}^{d(\cdot)}$, $\sigma : Q \times X^{(\cdot)} \to \mathbb{R}^{d(\cdot)\times m}$, $m \in \mathbb{N}$, *are bounded and Lipschitz continuous in* x.

This assumption ensures, for any $i \in Q$, the existence and uniqueness (Theorem 6.2.2. in [1]) of the solution for the following SDE

$$dx(t) = b(i, x(t))dt + \sigma(i, x(t))dW_t, \tag{22}$$

where $(W_t, t \geq 0)$ is the m-dimensional standard Wiener process in a complete probability space.
In this way, when i runs in Q, the equation (22) defines a family of diffusion processes $\mathbb{M}^i = (\Omega^i, \mathcal{F}^i, \mathcal{F}^i_t, x^i_t, \theta^i_t, P^i)$, $i \in Q$ with the state spaces $\mathbb{R}^{d(i)}$, $i \in Q$.

The jump (switching) mechanism between the diffusions is governed by two functions: the jump rate λ and the transition measure R. The jump rate $\lambda : X \to \mathbb{R}_+$ is a measurable function and the transition measure R maps X into the set $\mathcal{P}(X)$ of probability measure on $(X, \mathcal{B}(X))$.

One can consider the transition measure $R : \overline{X} \times \mathcal{B}(X) \to [0, 1]$ as a reset probability kernel such that: (i) for all $A \in \mathcal{B}(X)$, $R(\cdot, A)$ is measurable; (ii) for all $x \in \overline{X}$ the function $R(x, \cdot)$ is a probability measure.

Assumption 4 *(i)* $\lambda : X \to \mathbb{R}_+$ *is a measurable function such that* $t \to \lambda(x^i_t(\omega_i))$ *is integrable on* $[0, \varepsilon(x^i))$, *for some* $\varepsilon(x^i) > 0$, *for each* $x^i \in X^i$ *and each* ω_i *starting at* x^i.
(ii) For each $i \in Q$ *the restriction of* λ *to* X^i *is bounded. Let* $c^i = \sup_{x^i \in X^i} \lambda(x^i)$.

Since $\overline{X}$ is a Borel space, then $\overline{X}$ is homeomorphic to a subset of the Hilbert cube, $\mathcal{H}$. Therefore, its space of probabilities is homeomorphic to the space of probabilities of the corresponding subset of $\mathcal{H}$ (Lemma 7.10 [2]). There exists a measurable function $F : \mathcal{H} \times \overline{X} \to X$ such that

$$R(x, A) = \mathfrak{p}F^{-1}(A),\ A \in \mathcal{B}(X) \tag{23}$$

where $\mathfrak{p}$ is the probability measure on $\mathcal{H}$ associated to $R(x, \cdot)$ and

$$F^{-1}(A) = \{\boldsymbol{\omega} \in \mathcal{H} | F(\boldsymbol{\omega}, x) \in A\}.$$

The measurability of F is guaranteed by the measurability properties of R.

We construct an GSHS as a *Markov string* H which admits $(\mathbb{M}^i)$ as subprocesses. The sample path of the stochastic process $(x_t)_{t>0}$ with values in X, starting from a fixed initial point $x_0 = (i_0, x_0^{i_0}) \in X$ is defined as in subsection 2.2 using a particular sequence of stopping times and a particular renewal kernel. We have to precise, from the beginning, that the above recipe gives a sample path of GSHS starting with a initial diffusion path whose starting point is x_0. An arbitrary point x_0 does not define in a unique way a diffusion path!

Let ω_i a trajectory which starts in (i, x^i). Let $t_*(\omega_i)$ be the first hitting time of ∂X^i of the process (x^i_t). Let us define the function

$$F(t, \omega_i) = I_{(t<t_*(\omega_i))} \exp(-\int_0^t \lambda(i, x^i_s(\omega_i)))ds. \tag{24}$$

Using this function we define a stopping time S^i associated to the diffusions (x^i_t). In other words, F can be thought of as the *survivor function* for the stopping time. Obviously, the stopping time S^i is the minimum of two other stopping times:

1. the first exit time from X^i, i.e. $t_*|_{\Omega^i}$;

2. the the stopping time with the exponential survivor function (24), i.e.

$$S^i(\omega_i) = \inf\{t > 0 | F(t, \omega_i) \leq e^{-c^i t}\}$$

   or,

$$P^i[S^i > t] = P^i\{\omega_i | F(t, \omega_i) \geq e^{-c^i t}\}$$

The event ω and the associated sample path are inductively defined. In the first step $\omega = \omega_{i_0}$. The first jump time of the process is $T_1(\omega) = T_1(\omega_{i_0}) = S^{i_0}(\omega_{i_0})$. The sample path $x_t(\omega)$ up to the first jump time is now defined as follows:

if $T_1(\omega) = \infty:$ $\quad x_t(\omega) = (i_0, x^{i_0}_t(\omega_{i_0})),\ t \geq 0$

if $T_1(\omega) < \infty:$ $\quad x_t(\omega) = (i_0, x^{i_0}_t(\omega_{i_0})),\ 0 \leq t < T_1(\omega)$

$\quad x_{T_1} = F(\boldsymbol{\omega}, (i_0, x^{i_0}_{T_1}(\omega_{i_0})))$.

The process restarts from $x_{T_1} = (i_1, x^{i_1}_1)$ according to the same recipe, using now the process $x^{i_1}_t$. Let ω_{i_1} be a diffusion path starting in $x^{i_1}_1$. Then $\omega = \omega_{i_0} * \omega_{i_1}$. Then, if $T_1(\omega) < \infty$ we define the next jump time

$$T_2(\omega) = T_2(\omega_{i_0}, \omega_{i_1}) = T_1(\omega_{i_0}) + S^{i_1}(\omega_{i_1})$$

The sample path $x_t(\omega)$ between the two jump times is now defined as follows:

if $T_2(\omega) = \infty:$ $\quad x_t(\omega) = (i_1, x^{i_1}_{t-T_1}(\omega)),\ t \geq T_1(\omega)$

if $T_2(\omega) < \infty:$ $\quad x_t(\omega) = (i_1, x^{i_1}_t(\omega)),\ 0 \leq T_1(\omega) \leq t < T_2(\omega)$

$\quad x_{T_2}(\omega) = F(\boldsymbol{\omega}, (i_1, x^{i_1}_{T_2}(\omega)))$.

and so on. Let $T_1 < T_2 < ... < T_n < ...$ be the sequence of stopping times obtained by the above method. Let $T_\infty = \lim_{n\to\infty} T_n$. All probabilistic elements associated to (x_t) are constructed as in section 2.2. We suppose that the assumption 2 is in force.

Formal Definitions

We can introduce the following definition.

Definition 7 *A General Stochastic Hybrid Model (GSHS) is a collection* $H = ((Q, d, \mathcal{X}), b, \sigma, Init, \lambda, R)$ *where*

- Q *is a countable set of discrete variables;*
- $d : Q \to \mathbb{N}$ *is a map giving the dimensions of the continuous state spaces;*
- $\mathcal{X} : Q \to \mathbb{R}^{d(\cdot)}$ *maps each* $q \in Q$ *into an open subset* X^q *of* $\mathbb{R}^{d(q)}$*;*
- $b : X(Q, d, \mathcal{X}) \to \mathbb{R}^{d(\cdot)}$ *is a vector field;*
- $\sigma : X(Q, d, \mathcal{X}) \to \mathbb{R}^{d(\cdot)\times m}$ *is a* $X^{(\cdot)}$*-valued matrix,* $m \in \mathbb{N}$*;*

- *Init* $: \mathcal{B}(X) \to [0,1]$ *is an initial probability measure on* $(X, \mathcal{B}(S))$*;*
- $\lambda : \overline{X}(Q, d, \mathcal{X}) \to \mathbb{R}^+$ *is a transition rate function;*
- $R : \overline{X} \times \mathcal{B}(\overline{X}) \to [0,1]$ *is a transition measure.*

Definition 8 (GSHS Execution) *A stochastic process* $x_t = (q(t), x(t))$ *is called a GSHS execution if there exists a sequence of stopping times* $T_0 = 0 \leq T_1 \leq T_2 \leq \ldots$ *such that for each* $k \in \mathbb{N}$*,*
- $x_0 = (q_0, x_0^{q_0})$ *is a* $Q \times X$*-valued random variable extracted according to the probability measure Init;*
- *For* $t \in [T_k, T_{k+1})$*,* $q_t = q_{T_k}$ *is constant and* $x(t)$ *is a solution of the SDE:*

$$dx(t) = b(q_{T_k}, x(t))dt + \sigma(q_{T_k}, x(t))dW_t \tag{25}$$

where W_t *is a the* m*-dimensional standard Wiener;*
- $T_{k+1} = T_k + S^{i_k}$ *where* S^{i_k} *is according with the survivor function (24).*
- *The probability distribution of* $x(T_{k+1})$ *is governed by the law* $R\left((q_{T_k}, x(T_{k+1}^-)), \cdot\right)$*.*

3.3 Properties

GSHS, being constructed as particular Markov strings, they inherit the properties of these, namely they are *strong Markov processes* with *càdlàg property.*

Proposition 9 *Any General Stochastic Hybrid Model* H*, under the standard assumptions of section 3.2, is a Borel right process.*

Proof. To prove that H is a right Markov process, we have to verify the hypothesis of theorem 4. We can suppose without loss of generality that $\Omega^i \cap \Omega^j = \emptyset$. Then, the kernel Ψ can be defined as follows

$$\Psi : \{\bigcup_{i \in Q} \Omega^i\} \times \mathcal{B}(X) \to [0,1] \quad \text{such that} \quad \Psi(\omega_i, A) = R(x^i_{S^i(\omega_i)}, A)$$

We need to check that: If $0 < t < S^i(\omega_i)$ then $\Psi(\theta^i_t \omega_i, \cdot) = \Psi(\omega_i, \cdot)$, i.e.the stopping times (S^i) have the 'memoryless' property $R(x^i_{S^i(\theta^i_t \omega_i)}, \cdot) = R(x^i_{S^i(\omega_i)}, \cdot)$. In fact, we have to prove that, if $0 < t < t + s < S^i(\omega_i)$ then

$$P^{x^i}(S^i > t + s | S^i > t) = P^{x^i_t}(S^i > s) \tag{26}$$

Using the survivor function defined by (24), since $t^*(\theta^i_t \omega_i) = t_*(\omega_i) - t$ which implies that $t + s < t_*(\omega_i) \Leftrightarrow s < t_*(\theta^i_t \omega_i)$, we get

$$\begin{aligned}
\frac{F(t+s, \omega_i)}{F(t, \omega_i)} &= \frac{I_{\{t+s<t_*(\omega_i)\}} \exp(-\int_0^{t+s} \lambda(x^i_\tau(\omega_i))d\tau)}{I_{\{t<t_*(\omega_i)\}} \exp(-\int_0^{t} \lambda(x^i_\tau(\omega_i))d\tau)} \\
&= I_{\{t+s<t_*(\omega_i)\}} \exp(-\int_t^{t+s} \lambda(x^i_\tau(\omega_i))d\tau) \\
&= I_{\{t+s<t_*(\omega_i)\}} \exp(-\int_0^{s} \lambda(x^i_{\tau+t}(\omega_i))d\tau) \\
&= I_{\{t+s<t_*(\omega_i)\}} \exp(-\int_0^{s} \lambda(x^i_\tau \circ \theta^i_t(\omega_i))d\tau) \\
&= F(s, \theta^i_t(\omega_i))
\end{aligned}$$

The left hand side of (26) becomes

$$\begin{aligned}
P^{x^i}(S^i &> t + s | S^i > t) = P^{x^i}[F(t, \omega_i) F(s, \theta^i_t(\omega_i) \geq e^{-c^i(t+s)} | F(t, \omega_i) \geq e^{-c^i t}] \\
&= P^{x^i_t}[F(s, \theta^i_t(\omega_i)) \geq e^{-c^i s}]
\end{aligned}$$

and (26) is proved.

Thus, H is a Markov string obtained by mixing some diffusion processes. Since the state space is a Lusin space, H is a Borel right process.

4 Conclusions

In this paper we set up the notion of Markov string, which is roughly speaking, a concatenation of Markov processes. This notion arises as a result of our research on stochastic hybrid system modelling [8, 4, 5, 11] and it aims to be a very general formalization of all existing models of stochastic hybrid systems.

References

[1] L. Arnold. *Stochastic Differential Equations: Theory and Applications*. Wiley - Interscience, New York-London-Sydney, 1974.

[2] D.P. Bertsekas, S.E. Shreve. *Stochastic Optimal Control: The Discrete-Time Case*. Athena Scientific, 1996.

[3] R.M. Blumenthal, R.K. Getoor, *Markov Processes and Potential Theory*, Academic Press, New York and London, 1968.

[4] M.L. Bujorianu, J. Lygeros. *Reachability Questions in Piecewise Deterministic Markov Processes*. In O. Maler, A. Pnueli Eds., Proceedings *Hybrid Systems: Computation and Control*, 6th International Workshop, HSCC 2003, LNCS 2623, 2003, pp.126-140.

[5] M.L Bujorianu. *Extended Stochastic Hybrid Systems and their Reachability Problem*. 7th International Workshop: *Hybrid Systems: Computation and Control*, HSCC 2004, to appear in Springer LNCS series.

[6] M.H.A. Davis. *Markov Models and Optimization*, Chapman & Hall, London, 1993.

[7] M.K. Ghosh, A. Arapostathis, S.I. Marcus. *Ergodic Control of Switching Diffusions*. SIAM J. Control Optim. Vol. 35, No. 6, pp. 1952-1988,1997.

[8] J. Hu, J. Lygeros, S. Sastry. *Towards a Theory of Stochastic Hybrid Systems*. In N. Lynch and B. Krogh Eds., *Hybrid Systems: Computation and Control*, LNCS 1790, pp.160-173, 2000.

[9] P.A. Meyer. *Probability and Potentials*. Blaisdell, Waltham Mass, 1966.

[10] P.A. Meyer. *Renaissance, Recollectments, Mélanges, Ralentissement de Processus de Markov*. Ann. Inst. Fourier, Vol.25, pp.465-497, 1975.

[11] G. Pola, M.L. Bujorianu, J. Lygeros, M. D. Di Benedetto. *Stochastic Hybrid Models: An Overview with applications to Air Traffic Management*. ADHS, Analysis and Design of Hybrid System, Saint-Malo, 2003.