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STOCHASTIC HYBRID PROCESSES WITH HYBRID JUMPS

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Abstract: Switching Diffusion processes can be represented as pathwise unique solutions of SDE's in a hybrid state space that are driven by Brownian motion and Poisson random measure. This paper extends these SDE's to switching jump-diffusions, the jumps of which i) happen simultaneously with mode switching, and ii) depend on the mode after the switching. Jumps satisfying both i) and ii) are referred to as hybrid jumps. Because of ii) there is an anticipation effect in the SDE, which makes the hybrid jump extension challenging.

Keywords: Markov models, Hybrid models, Modelling, Poisson processes, Discrete event systems, Continuous time systems

1. INTRODUCTION

For the performance optimization of dynamical multi-agent systems such as air traffic management, communication networks, electric power networks and nuclear and chemical plants, the stochastic analysis framework for switching diffusion processes has proven to be very useful in addressing these processes in their natural continuous-time setting. Illustrative are the planning and scheduling study of flexible manufacturing systems by Gershwin (1989), the hierarchical control study of air traffic management by Tomlin et al. (1998), and the accident risk estimation study of nuclear/chemical plants by Labeau et al. (2000).

In the context of multi-sensor multi-target tracking (Sworder and Boyd, 1999) and accident risk modelling for air traffic management (Blom et al., 2001) there appeared a need for switching jump-diffusion processes, the jumps of which i) happen simultaneously with mode switching and ii) depend on the mode after the switching. Jumps satisfying both i) and ii) are referred to as hybrid jumps. A hybrid jump example in air traffic is the "top of descend", i.e. the moment at which a pilot switches from level flying to descend towards the airport. At such moment both the configuration mode of the aircraft switches and the rate of descend jumps to a value that belongs to the descend mode. Another air traffic modelling example is a sudden failure of an engine; this is typically modelled as a sudden switch to a failure mode and a simultaneous jump of the thrust of the aircraft to a value that belongs to the failure mode. As illustrated by these two examples, there are two types of hybrid jumps: a forced jump that happens at an instant of hitting some boundary, and a Poisson type of jump that happens at a sudden instant.

A well-known and very powerful class of stochastic processes with both types of hybrid jumps is the Piecewise Deterministic Markov Process (PDP) of Davis (1984). A PDP has the strong Markov property, its control is well developed (e.g. Vermes, 1985; Davis, 1993), the relation to stochastic hybrid automata is well defined (e.g. Branicky, 1995; Pola et al., 2003), and the relation to Petri nets has been established (Everdij & Blom, 2003). Unfortunately, the PDP formulation does not allow a straightforward

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inclusion of Brownian motion without loosing PDP's nice properties.

For the combination of switching diffusion and hybrid jumps that happen when hitting some boundary, the situation has been improved significantly by Borkar et al. (1991), Bensoussan & Menaldi (2000) and Hu et al. (2000). The aim of this paper is to improve the situation for hybrid jumps that happen suddenly, i.e. their timing is of Poisson type. The basis for this improvement has been developed in (Blom, 1990).

The paper is organised as follows. Section 2 presents the main result of the paper, while Sections 3, 4 and 5 provide a step by step development of this main result. Section 3 introduces a specific SDE of Lepeltier & Marchal (1976). Section 4 incorporates in this SDE the jump intensity modelling approach of Jacod & Protter (1982). Section 5 uses the SDE to characterize a switching jump diffusion with hybrid jumps as a strong hybrid state Markov process.

2. THE MAIN RESULT

Throughout this paper we work within a complete stochastic basis $(\Omega,\mathfrak{I},\boldsymbol{F},\boldsymbol{P},\boldsymbol{T})$, with measurable space $(\Omega,\mathfrak{I})$, right continuous filtration $\boldsymbol{F}$, probability measure $\boldsymbol{P}$ and time index $\boldsymbol{T}=\Re_+=[0,\infty)$, that is endowed with an m-dimensional standard Wiener process, $\{w_t\}$, and an independent homogeneous Poisson random measure $p_{\mathrm{P}}(dt,du)$ (e.g. Jacod & Shiryaev, 1987, pp.70-71).

Let $\{x_t\}$ be an $\Re^n$ valued stochastic process satisfying the Itô-Skorohod type of equation

$$dx_t = a(\theta_t,x_t)dt + b(\theta_t,x_t)dw_t + \int_{\Re^d} c(\theta_t,\theta_{t-},x_{t-},\underline{u})\cdot p_{\mathrm{P}}(dt,(0,\Lambda(\theta_{t-},x_{t-})]\times d\underline{u}) \qquad \text{(I)}$$

with $x_{t-}\triangleq\lim_{\Delta\downarrow 0} x_{t-\Delta}$, a, b, c are measurable mappings of appropriate dimensions, $p_{\mathrm{P}}(dt,du_1\times d\underline{u})$ has intensity measure $\nu(dt,du_1\times d\underline{u})=dt.du_1.\mu(d\underline{u})$, and $\{\theta_t\}$ is an $M=\{\eta_1,\ldots,\eta_N\}$ valued process satisfying:

$$p\{\theta_{t+\Delta}=\theta \mid \theta_t=\eta, x_s,\theta_s, s\le t\} = \lambda_{\eta\theta}(x_t)\Delta + o(\Delta), \quad \text{for } \eta,\theta\in M,\ \eta\neq\theta \qquad \text{(II')}$$

with $\lambda_{\eta\theta}\ge 0$, $\eta\neq\theta$.

If p_{P} generates a multivariate point $(\{t\},\{u_1\},\{\underline{u}\})$ then the path of $\{x_t\}$ may have discontinuity (=jump):

$$x_t - x_{t-} = \begin{cases} c(\theta_t,\theta_{t-},x_{t-},\underline{u}) & \text{if } u_1\in(0,\Lambda(\theta_{t-},x_{t-})] \\ 0 & \text{else} \end{cases}$$

This means there are u_1 values that cause a hybrid jump in the sense that $\theta_t\neq\theta_{t-}$, $x_t\neq x_{t-}$ and $x_t - x_{t-}$ depends on θ_t. The question now is if system (I,II') behaves well as a stochastic process, i.e. if it has a well defined solution, and if the process $\{x_t,\theta_t\}$ is a strong hybrid state Markov process. Because a hybrid jump has an anticipation effect, this is a rather challenging question. To answer this question, in the sequel of this paper the following theorem is proven for a specific version of (II'):

$$d\theta_t = \sum_{i=1}^{N}(\eta_i - \theta_{t-})\cdot p_{\mathrm{P}}(dt,(\Sigma_{i-1}(\theta_{t-},x_{t-}),\Sigma_i(\theta_{t-},x_{t-})]\times\Re^d) \qquad \text{(IIa)}$$

with

$$\Sigma_i(\theta,x) = \begin{cases} \Lambda(\theta,x)\sum_{j=1}^{i}\rho(\eta_j,\theta,x) & i>0 \\ 0 & i=0 \end{cases} \qquad \text{(IIb)}$$

<u>Theorem 1</u>

Let a and b satisfy <u>B.1</u> and <u>B.2</u>, let c satisfy <u>B.3</u>, let $\lambda_{\theta\eta}$ satisfy <u>B.4</u>, and the intensity measure of $p_{\mathrm{P}}(dt,du_1\times d\underline{u})$ equals $dt.du_1.\mu(d\underline{u})$. Then for every initial condition $\theta_0(w)=\theta\in M$ and $x_0(w)=x\in\Re^n$, equation (I,IIa,b) has a pathwise unique solution, $\{x_t,\theta_t\}$ which is a strong Markov process.

<u>B.1</u> There is a constant K such that, for all $x\in\Re^n$ and $\theta\in M$,
$$|a(\theta,x)|^2 + \|b(\theta,x)\|^2 \le K(1+|x|^2).$$
with $|a|^2=\sum_i (a_i)^2$, and $\|b\|^2=\sum_{i,j}(b_{ij})^2$.

<u>B.2</u> For all $k\in\aleph$ there exists a constant L_k such that, for all x and y in the ball $\{x\in\Re^n;\ |x|\le k+1\}$,
$$|a(\theta,x)-a(\theta,y)|^2 + \|b(\theta,x)-b(\theta,y)\|^2 \le L_k|x-y|^2.$$

<u>B.3</u> For every $k\in\aleph$ there exists a constant M_k, such that
$$\sup_{|x|\le k}\int_{\Re^d}|c(\eta,\theta,x,\underline{u})|\mu(d\underline{u})\le M_k \quad \text{for all } \eta,\theta\in M.$$

<u>B.4</u> $\lambda_{\theta\eta}(x)=\Lambda(\theta,x)\rho(\eta,\theta,x)$, η, $\theta\in M$, $x\in\Re^n$, with $\Lambda(\theta,\cdot)$ and $\rho(\eta,\theta,\cdot)$ continuous mappings of $\Re^n$ into $[0,\infty)$ such that $\Lambda(\theta,x)\le C$ and $\sum_{\eta\in M}\rho(\eta,\theta,x)=1$, for all θ and x.

3. THE SDE OF LEPELTIER & MARCHAL

As a first step in deriving Theorem 1, we consider the following stochastic differential equation (SDE) in $\Re_+\times\Re^n$,

$$d\xi_t = \alpha(\xi_t)dt + \beta(\xi_t)dw_t + \int_{U_1}\psi(\xi_{t-},u)p_P(dt,du) + \int_{U_2}\psi(\xi_{t-},u)p_P(dt,du), \qquad (1)$$

with U_1 and U_2 such that $U_1 \cap U_2 = \emptyset$ and $U_1 \cup U_2 = U$, ξ_0 is an $\mathcal{F}_0$-measurable $\Re^n$-valued random variable, while α, β and ψ are measurable mappings of appropriate dimensions (with domains $\Re^n$, $\Re^n \times \Re^m$ and $\Re^n$, respectively), $\{w_t\}$ an m-dimensional standard Wiener process, and $p_P(dt,du)$ an independent homogeneous Poisson random measure (e.g. Jacod & Shiryaev, 1987, pp.70-71), on Borel $\Re_+ \times U$, with intensity measure $\nu(dt,du)=dt \cdot m(du)$.

Although, in the sequel, we are not really using the third right hand term, we start from (1) to notice the main difference between the roles played by the third and fourth right hand terms. The set-up commonly used is to assume conditions such that

$$\left\{ \int_0^t \int_{U_1} \psi(\xi_{s-},u)[p_P(ds,du) - \nu(ds,du)] \right\}$$

is a local martingale, while the process

$$\left\{ \int_0^t \int_{U_2} \psi(\xi_{s-},u) p_P(ds,du) \right\}$$

has finite variation over each finite interval.

The classical result for the existence of a pathwise unique solution of the Itô-Skorohod equation requires that ψ satisfies some local Lipschitz condition (Gihman and Skorohod, 1972; Ikeda & Watanabe, 1989). This Lipschitz condition essentially restricts the SDE solution to diffusions with Markov switching coefficients (Brockett & Blankenship, 1977; Skorohod, 1989) and with controlled coefficients (Ghosh et al., 1993, 1997). Results for a discontinuous ψ have been developed by Lepeltier and Marchal (1976). They studied (1) under the assumptions of $U_1 = \{u;\ |u| \le 1\}$, $U_2 = \{u;\ 1 < |u| < \infty\}$, and a non-Lipschitz $\psi(\cdot,u)$ for $u \in U_2$, and showed that it has a pathwise unique solution.

Proposition 2.1

Let A'.1, A'.2, and A'.3 be satisfied and let $U_1 = (-\infty,0] \times \Re^d$ and $U_2 = (0,\infty) \times \Re^d$. Then equation (1) has for every initial condition $\xi_0(\omega) = \xi \in \Re^n$ a pathwise unique solution, $\{\xi_t\}$, which is càdlàg and adapted. Moreover, there exists a measurable random function $f(t,\xi,\omega)$ such that $\xi_t(\cdot) = f(t,\xi,\cdot)$ almost surely for every t.

A'.1. There is a constant K such that, for all $\xi \in \Re^n$,

$$|\alpha(\xi)|^2 + \|\beta(\xi)\|^2 + \int_{U_1} |\psi(\xi,u)|^2 m(du) \le K(1+|\xi|^2).$$

A'.2. For all $k \in \aleph = \{0,1,2,\ldots\}$ there exists a constant L_k such that, for all ξ and y in the ball

$$\boldsymbol{B}_k = \{x \in \Re^n; |x| \le k+1\},$$

$$|\alpha(\xi) - \alpha(y)|^2 + \|\beta(\xi) - \beta(y)\|^2 + \int_{U_1} |\psi(\xi,u) - \psi(y,u)|^2 m(du) \le L_k |\xi - y|^2.$$

A'.3. For every $k \in \aleph$ there exists a constant M_k, such that

$$\sup_{|\xi| \le k} \int_{U_2} |\psi(\xi,u)| m(du) \le M_k .$$

Proof: See Lepeltier and Marchal (1976, Th. III_4).

4. JUMP INTENSITY OF JACOD & PROTTER

From here on, we restrict our attention to the situation that $\psi(\xi,u) = 0$ for all $u \notin (1,\infty) \times \Re^d \subset U_2$, by which the third term of (1) is zero, i.e.:

$$d\xi_t = \alpha(\xi_t)dt + \beta(\xi_t)dw_t + \int_{U_2} \psi(\xi_{t-},u) p_P(dt,du) \tag{2}$$

Jacod and Protter (1982; Protter, 1983) developed an elegant approach to explicitly model the jump intensity of $\{\xi_t\}$ in (2). Therefore they adopted the following kind of compositions of $\psi(\xi,u)$ and $m(du)$:

$$m(du) = du_1 \cdot \mu(d\underline{u}),\ u_1 \in \Re,\ \underline{u} \in \Re^d,$$

$$\psi(\xi,u) = \mathbf{1}_{(0,\Lambda(\xi)]}(u_1 - 1)\varphi(\xi,\underline{u}),$$

where $\underline{u} = \text{Col}\{u_2,\ldots,u_{d+1}\}$, μ is a probability measure on $\Re^d$, Λ is a measurable mapping of $\Re^n$ into $[0,\infty)$, φ is a measurable mapping of $\Re^n \times \Re^d$ into $\Re^n$, and

$$\mathbf{1}_A(a) = 1, \quad a \in A,$$
$$= 0, \quad \text{else.}$$

With this, (2) becomes:

$$d\xi_t = \alpha(\xi_t)dt + \beta(\xi_t)dw_t + \int_{U_2} \varphi(\xi_{t-},\underline{u}) \mathbf{1}_{(0,\Lambda(\xi_{t-})]}(u_1 - 1) p_P(dt,du), \tag{3}$$

and the intensity measure of $p_P(dt,du)$ equals $dt \cdot du_1 \cdot \mu(d\underline{u})$.

Next, we introduce the following assumptions:

A.1. There is a constant K such that, for all $\xi \in \Re^n$,

$$|\alpha(\xi)|^2 + \|\beta(\xi)\|^2 \le K(1+|\xi|^2).$$

A.2. For all $k \in \aleph$ there exists a constant L_k such that, for all ξ and y in the ball $\boldsymbol{B}_k = \{x \in \Re^n;\ |x| \le k+1\}$,

$$|\alpha(\xi) - \alpha(y)|^2 + \|\beta(\xi) - \beta(y)\|^2 \le L_k |\xi - y|^2,$$

A.3. $\Lambda(\cdot)$ is a bounded continuous mapping on $\Re^n$ with upper bound a constant C.

A'.4. For every $k \in \aleph$ there exists a constant M_k, such that

$$\sup_{|\xi| \le k} \int_{\Re^d} |\varphi(\xi,\underline{u})| \mu(d\underline{u}) \le M_k .$$

Proposition 3.1
Let α and β satisfy A.1 and A.2, Λ satisfies A.3, φ satisfies A'.4, and the intensity measure of $p_P(dt,du)$ equals $dt\cdot du_1 \times \mu(d\underline{u})$. Then for every initial condition $\xi_0(\omega) = \xi \in \Re^n$, equation (3) has a pathwise unique solution, $\{\xi_t\}$, which is càdlàg and adapted. Moreover, there exists a measurable random function $f(t,\xi,\omega)$ such that $\xi_t(\cdot) = f(t,\xi,\cdot)$ almost surely for every t.

Proof: Due to A.3 the third right hand term of (3) is equal to:

$$\int_{(0,C]\times\Re^d} \varphi(\xi_{t-},\underline{u})\mathbf{1}_{(0,\Lambda(\xi_{t-})]}(u_1)\, p_P(dt,du).$$

By defining the mapping $\chi(\xi,u_1)$, for every $\xi \in \Re^n$ and every $u_1 \in [0,\infty)$, by

$$\chi(\xi,u_1) \underline{\underline{\Delta}} 1_{(0,\Lambda(\xi)]}(u_1) = 1_{[u_1,\infty)}(\Lambda(\xi)),$$

the third right hand term of (3) can be replaced by

$$\int_{(0,C]\times\Re^d} \varphi(\xi_{t-},\underline{u})\chi(\xi_{t-},u_1)\, p_P(dt,du).$$

This implies that (3) is an equation of type (2). Due to A.3, $\chi(\xi,u_1)$ is measurable in (ξ, u_1). Together with A'.4 this implies that condition A'.3 of Proposition 2.1 is satisfied. QED

Next we consider a more general situation in which there are N jump intensities that influence the evolution of the process $\{\xi_t\}$. Similar as before, we introduce the following compositions of $m(du)$ and $\psi(\xi,u)$:

$$m(du) = du_1 \times \mu(d\underline{u}),\ u_1 \in [1,\infty),\ \underline{u} \in \Re^d \tag{4}$$

$$\psi(\xi,u) = \sum_{i=1}^{N} \mathbf{1}_{(\Sigma_{i-1}(\xi),\Sigma_i(\xi)]}(u_1 - 1)\phi(\eta_i,\xi,\underline{u}), \tag{5}$$

$$\Sigma_i(\xi) = \Lambda(\xi)\sum_{j=1}^{i} \rho(\eta_j,\xi),\ i > 0, \qquad = 0,\quad i = 0, \tag{6}$$

where $\underline{u}$ refers to all components of u except the first one, $\eta_i \in \Re$ for all i, $\eta_i \neq \eta_j$ if $i \neq j$, Σ_1 through Σ_N are measurable mappings of $\Re^n$ into $[0,\infty)$, ϕ is a measurable mapping of $\Re\times\Re^n\times\Re^d$ into $\Re^n$, μ is a probability measure, ρ is a measurable mapping of $\Re\times\Re^n$ into $[0,\infty)$, such that

$$\sum_{i=1}^{N} \rho(\eta_i,\xi) = 1, \text{for all}\ \xi \in \Re^n. \tag{7}$$

With this, (2) becomes:

$$d\xi_t = \alpha(\xi_t)dt + \beta(\xi_t)dw_t + \int_{U_2} \sum_{i=1}^{N} \phi(\eta_i,\xi_{t-},\underline{u})\cdot \cdot\mathbf{1}_{(\Sigma_{i-1}(\xi_{t-}),\Sigma_i(\xi_{t-})]}(u_1 - 1)\, p_P(dt,du), \tag{8}$$

where $p_P(dt,du)$ has intensity measure $dt\cdot du_1\cdot\mu(d\underline{u})$.

Now, we introduce the following assumptions:

A.4. $\rho(\eta,\cdot)$ is a continuous mapping for all $\eta\in\Re$,

A.5. For all $k \in \aleph$ there exists a constant M_k such that,

$$\sup_{|\xi|\le k} \int_{\Re^d} |\phi(\eta,\xi,\underline{u})|\mu(d\underline{u}) \le M_k, \text{ for all } \eta\in\Re,$$

and subsequently we extend Proposition 3.1 to the situation of multiple jump intensities.

Proposition 3.2
Let α and β satisfy A.1 and A.2, Λ satisfies A.3, Σ_i satisfies (6), ρ satisfies (7) and A.4, ϕ satisfies A.5, and the intensity measure of $p_P(dt,du)$ equals $dt\cdot du_1\cdot\mu(d\underline{u})$. Then for every initial condition $\xi_0(\omega) = \xi \in \Re^n$, equation (8) has a pathwise unique solution, $\{\xi_t\}$, which is càdlàg and adapted. Moreover, there exists a measurable random function $f(t,\xi,\omega)$ such that $\xi_t(\cdot) = f(t,\xi,\cdot)$ almost surely for every t.

Proof: Similar to the proof of Proposition 3.1.

Next, we give a strong Markov characterization of the solution $\{\xi_t\}$ of equation (8).

Proposition 3.3
Let the assumptions of Proposition 3.2 hold true. Then for all $\xi_0 \in \Re^n$, $\{\xi_t\}$ is a strong Markov process, and its generator, $\mathcal{A}$, satisfies:

$$\mathcal{A}f = \mathcal{L}f + \mathcal{J}f, \tag{9}$$

for all $f \in D(\mathcal{A}) \supset C^2(\Re^n)$, where

$$\mathcal{L}f(\xi) = \sum_{i=1}^{n} \alpha_i(\xi)f_{\xi_i}(\xi) + +\frac{1}{2}\sum_{i,j=1}^{n}[\beta(\xi)\beta(\xi)^T]_{ij} f_{\xi_i\xi_j}(\xi), \tag{10}$$

$$\mathcal{J}f(\xi) = \Lambda(\xi)\int_{\Re^n}[f(\zeta) - f(\xi)]Q(d\zeta;\xi), \tag{11}$$

with for all Borel $A \subset \Re^n$,

$$Q(A;\xi) = = \sum_{i=1}^{N} \rho(\eta_i,\xi)\int_{\Re^d} \mathbf{1}_A(\xi + \phi(\eta_i,\xi,\underline{u}))\mu(d\underline{u}) \tag{12}$$

Proof: The existence of a measurable f in Proposition 3.2 implies the Markov property for $\{\xi_t\}$. Due to A.3, A.4, and A.5, the predictable part $\{a_t\}$ of $\{\xi_t\}$ satisfies

$$a_t = \int_0^t \alpha(\xi_s)ds + \int_0^t\int_{U_2}\sum_{i=1}^{N}\phi(\eta_i,\xi_{s-},\underline{u})\cdot \cdot\mathbf{1}_{(\Sigma_{i-1}(\xi_{s-}),\Sigma_i(\xi_{s-})]}(u_1 - 1)du_1\cdot\mu(d\underline{u})ds$$

$$= \int_0^t \alpha(\xi_s)ds + \int_0^t \Lambda(\xi_{s-})\sum_{i=1}^{N}\rho(\eta_i,\xi_{s-})\cdot \cdot\int_{\Re^d}\phi(\eta_i,\xi_{s-},\underline{u})\mu(d\underline{u})ds,$$

up to indistinguishability. This shows that $\{a_t\}$ is of finite variation on any finite time interval. Since $\{\xi_t - a_t - \xi_0\}$ is a local martingale and a_t is predictable, this means that $\{\xi_t\}$ is a special semimartingale (Jacod & Shiryaev, 1987, p.43) and that $\{a_t\}$ defines the (unique) canonical martingale decomposition of $\{\xi_t\}$. The uniqueness of the canonical martingale decomposition implies a unique associated martingale problem, and since (8) has a pathwise unique solution, the associated martingale problem has a unique solution too. Hence, $\{\xi_t\}$ is a strong Markov process (e.g. Lepeltier & Marchal, 1976, Th. II 12). The generator $\mathcal{A}$ follows from Itô's differentiation rule for discontinuous semimartingales (e.g. Jacod & Shiryaev, 1987).

QED

5. HYBRID STATE SPACE

Now we are prepared to consider the hybrid state space situation such that for every ω, $\xi_t(\omega) \in \mathcal{M} \times \Re^{n-1}$, with $\mathcal{M} = \{\eta_i; i = 1,\ldots,N\}$, $n_i \neq n_j$ for $i \neq j$. First, we assume that the first component of $\{\xi_t\}$ is a pure jump process, i.e. for all $\xi \in \Re^n$:

$$\alpha_1(\xi) = 0 \qquad (13.a)$$

$$\beta_1(\xi) = 0 \qquad (13.b)$$

$$\phi_1(\eta, \xi, \underline{u}) = \eta - \xi_1 \text{, for all } \eta \in \Re \text{ and } \underline{u} \in \Re^d . \qquad (13.c)$$

Substitution of equations (13.a,b,c) into (8) yields:

$$d\xi_{1,t} = \int_{(0,\infty)} \sum_{i=1}^{N} (\eta_i - \xi_{1,t-}) \cdot \mathbf{1}_{(\Sigma_{i-1}(\xi_{t-}), \Sigma_i(\xi_{t-})]}(u_1) \cdot p_{\mathrm{P}}(dt, du_1 \times \Re^d) \qquad (14.a)$$

$$d\underline{\xi}_t = \underline{\alpha}(\xi_t)dt + \underline{\beta}(\xi_t)dw_t + \int_{(0,\infty)} \int_{\Re^d} \sum_{i=1}^{N} \underline{\phi}(\eta_i, \xi_{t-}, \underline{u}) \cdot \mathbf{1}_{(\Sigma_{i-1}(\xi_{t-}), \Sigma_i(\xi_{t-})]}(u_1) \cdot p_{\mathrm{P}}(dt, du_1 \times d\underline{u}) \qquad (14.b)$$

where $\xi_{1,t}$ denotes the first component of ξ_t, and $\underline{\xi}_t$ denotes the other components.

Theorem 4.1

Let the conditions of Proposition 3.2 hold true, and let α_1, β_1 and ϕ_1 satisfy (13.a,b,c). Then for every initial condition $\xi_0(\omega) \in \Re^n$, equation (14.a,b) has a pathwise unique solution, $\{\xi_t\}$, which is càdlàg and adapted. Moreover, if $\xi_0(\omega) \in \mathcal{M} \times \Re^{n-1}$ for all ω, with $\mathcal{M} = \{n_i; i = 1,\ldots,N\}$, then $\{\xi_t\}$ is a strong Markov process assuming values in the hybrid state space $\mathcal{M} \times \Re^{n-1}$, and its generator, $\mathcal{A}$, satisfies:

$$\mathcal{A} f = \mathcal{L} f + \mathcal{J} f , \qquad (15)$$

for all $f \in D(\mathcal{A}) \supset C^2(\Re^n)$, where

$$\mathcal{L} f(\xi) = \sum_{i=2}^{n} \alpha_i(\xi) f_{\xi_i}(\xi) + \tfrac{1}{2} \sum_{i,j=2}^{n} [\beta(\xi)\beta(\xi)^T]_{ij} f_{\xi_i \xi_j}(\xi), \qquad (16)$$

$$\mathcal{J} f(\xi) = \Lambda(\xi) \sum_{\eta \in \mathcal{M}} \int_{\Re^{n-1}} [f(\mathrm{Col}\{\eta, x\}) - f(\xi)] Q(\{\eta\} \times dx; \xi), \qquad (17)$$

and for all Borel $\underline{A} \subset \Re^{n-1}$ and $\eta \in \mathcal{M}$:

$$Q(\{\eta\} \times \underline{A}; \xi) = = \rho(\eta, \xi) \int_{\Re^d} \mathbf{1}_{\underline{A}}(\underline{\xi} + \underline{\phi}(\eta, \xi, \underline{u})) \mu(d\underline{u}) \qquad (18)$$

Proof: Since α_1, β_1 and ϕ_1 in (13.a,b,c) satisfy the conditions of Proposition 3.2, the existence of a unique solution and the adapted, càdlàg and strong Markov properties all follow from Propositions 3.2 and 3.3. From equation (14.a) and the initial condition $\xi_{1,0}(\omega) \in \mathcal{M}$ it can be shown that $\xi_{1,t}(\omega) \in \mathcal{M}$ for all ω and $t \geq 0$. Hence the state space of the Markov process $\{\xi_t\}$ is of the hybrid form $\mathcal{M} \times \Re^{n-1}$, and this yields the specific characterisation of the generator.

QED

Corollary 4.2

Under the conditions of Theorem 4.1, the solution of (14.a,b) is indistinguishable from the solution of the following set of equations:

$$d\xi_{1,t} = \sum_{i=1}^{N} (\eta_i - \xi_{1,t-}) \cdot p_{\mathrm{P}}(dt, (\Sigma_{i-1}(\xi_{t-}), \Sigma_i(\xi_{t-})] \times \Re^d), \qquad (19.a)$$

$$d\underline{\xi}_t = \underline{\alpha}(\xi_t)dt + \underline{\beta}(\xi_t)dw_t + \int_{\Re^d} \underline{\phi}(\xi_{1,t}, \xi_{t-}, \underline{u}) p_{\mathrm{P}}(dt, (0, \Lambda(\xi_{t-})] \times d\underline{u}). \qquad (19.b)$$

Proof: Rewriting of (19.a) yields (14.a). Since the first two right hand terms of (19.b) and (14.b) are equal, it remains to show that the third right hand term in (19.b) yields the third right hand term in (14.b) up to indistinguishability:

$$\int_{\Re^d} \underline{\phi}(\xi_{1,t}, \xi_{t-}, \underline{u}) p_{\mathrm{P}}(dt, (0, \Lambda(\xi_{t-})] \times d\underline{u}) =$$

$$= \int_{(0,\infty)} \int_{\Re^d} \underline{\phi}(\xi_{1,t}, \xi_{t-}, \underline{u}) \mathbf{1}_{(0, \Lambda(\xi_{t-})]}(u_1) p_{\mathrm{P}}(dt, du_1 \times d\underline{u}) =$$

$$= \int_{(0,\infty)} \int_{\Re^d} \underline{\phi}(\xi_{1,t}, \xi_{t-}, \underline{u}) \cdot \sum_{i=1}^{N} \mathbf{1}_{(\Sigma_{i-1}(\xi_{t-}), \Sigma_i(\xi_{t-})]}(u_1) p_{\mathrm{P}}(dt, du_1 \times d\underline{u}) =$$

$$= \int_{(0,\infty)} \int_{\Re^d} \sum_{i=1}^{N} \left[\underline{\phi}(\xi_{1,t}, \xi_{t-}, \underline{u}) \mathbf{1}_{(\Sigma_{i-1}(\xi_{t-}), \Sigma_i(\xi_{t-})]}(u_1) \right] \cdot p_{\mathrm{P}}(dt, du_1 \times d\underline{u}) =$$

$$= \int_{(0,\infty)} \int_{\Re^d} \sum_{i=1}^{N} \Big[\phi(\xi_{1,t-} + d\xi_{1,t}, \xi_{t-}, \underline{u}) \cdot$$

$$\cdot \mathbf{1}_{(\Sigma_{i-1}(\xi_{t-}), \Sigma_i(\xi_{t-})]}(u_1) \Big] \cdot p_{\mathrm{P}}(dt, du_1 \times d\underline{u}) =$$

$$= \int_{(0,\infty)} \int_{\Re^d} \sum_{i=1}^{N} \Big[\phi(\xi_{1,t-} + (\eta_i - \xi_{1,t-}), \xi_{t-}, \underline{u}) \cdot$$

$$\cdot \mathbf{1}_{(\Sigma_{i-1}(\xi_{t-}), \Sigma_i(\xi_{t-})]}(u_1) \Big] \cdot p_{\mathrm{P}}(dt, du_1 \times d\underline{u}) =$$

$$= \int_{(0,\infty)} \int_{\Re^d} \sum_{i=1}^{N} \Big[\phi(\eta_i, \xi_{t-}, \underline{u}) \mathbf{1}_{(\Sigma_{i-1}(\xi_{t-}), \Sigma_i(\xi_{t-})]}(u_1) \Big] \cdot$$

$$\cdot p_{\mathrm{P}}(dt, du_1 \times d\underline{u})$$

QED

Remark: Theorem 4.1 and Corollary 4.2 imply Theorem 1 in Section 2.

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