COLLISION RISK MODELING OF AIR TRAFFIC

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Abstract
Air Traffic Management (ATM) involves interactions between multiple human operators, procedures and technical systems, all of which are highly distributed. This yields that providing safety is more than making sure that each of the ATM elements functions properly safe; it is the complex interaction between them that determines safety. The assessment of isolated indicators falls short in covering the complex interactions between procedures, human operators and technical systems in safety-critical non-nominal situations. To improve this situation, this paper develops an approach towards the modelling and assessment of risk of mid-air collision between aircraft.

1. Introduction

By its very nature Air Traffic Management (ATM) is a highly distributed safety critical operation. Each aircraft has its own crew, and each crew is communicating with and receives safety critical instructions from multiple human operators in different centres on the ground. The implication is that safety of air traffic is the result of interactions between multiple human operators, procedures (including spacing and separation criteria), and technical systems (hardware and software) all of which are highly distributed. Providing safety is more than making sure that each of these elements function properly and safely. Since the interactions between the various elements of ATM significantly determine safety, it is imperative to understand the safety impact of these interactions, particularly in relation to non-nominal situations.

Traditional ATM design approaches tend first to design advanced ATM that provides sufficient capacity, and next to extent the design with safety features. The advantage of this approach is that ATM developments can be organised around the clusters of individual elements, i.e., the communication cluster, the navigation cluster, the surveillance cluster, the automation tools cluster, the human machine interfaces (HMIs), the advanced procedures, etc. The disadvantage of this traditional approach is that it fails to address the impact of interactions between ATM elements on safety.

A goal directed approach would be to design an ATM operational concept that is inherently safe at the capacity-

level required. From this perspective, safety assessment might be one of the primary filters in the development of advanced ATM designs. An early filtering of ATM design concepts on safety grounds can potentially avoid a costly development program, or an even more costly implementation program that turns out to be less effective than expected. Although understanding this idea is principally not very difficult, it can be brought into practice only when an ATM safety assessment approach is available that provides appropriate feedback to the ATM designers at an early stage of the concept development (Fig. 1). This feedback should not only provide information on whether the design is safe enough, but it should also identify the safety-capacity bottlenecks.
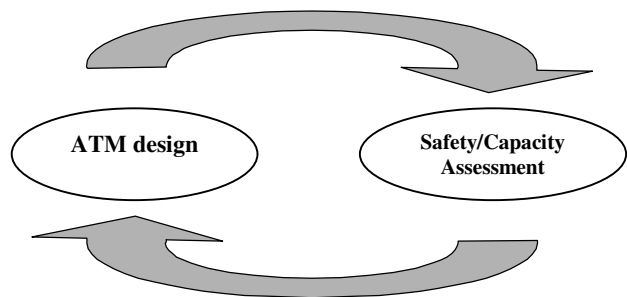

Figure 1: Safety feedback based ATM design.

For oceanic air traffic, the civil aviation community has developed a mathematical model to estimate mid-air collision risk levels as a function of spacing (ICAO, 1988). This model is known as the Reich collision model (Reich, 1964). Following Hsu (1981), in mathematical terms the Reich model assumes that the physical shape of each aircraft is a box, having a fixed x,y,z orientation, and the collision risk between two boxes is approximated by integrating the in-crossing rate over the time period in which these boxes may be close to each other. Unfortunately, this Reich model does not adequately cover situations where ground controllers monitor the air traffic through radar surveillance and provide tactical instructions to the aircraft crews.

The aim of the current paper is to improve the modelling and assessment of collision risk between aircraft by studying the problem within the framework of hybrid-state Markov processes. This framework has been well developed for applications to other safety critical industries, e.g. nuclear, chemical. As explained in a recent overview (Labeau et al., 2000), the particular processes studied are ordinary differential equations (ODE) with switching coefficients, such that the resulting hybrid state process is Markov. For risk

evaluation of this class of hybrid-state Markov processes several combinations of analytical and numerical techniques have been developed. The main extension of the current paper is that in contrast to the ODE's with switching coefficients, we consider stochastic differential equations (SDE) with switching coefficients. This allows Brownian motion terms, e.g. to represent the effect of random wind disturbances on aircraft trajectories.

In addition to the mathematical challenges of modelling collision risk in air traffic, there is the challenge to specify an appropriate mathematical model of an air traffic operation that covers all relevant elements and the interactions between these elements. For air traffic, this issue is covered by complementary studies, e.g. Corker (2000), Blom et al. (2001, 2003a), Everdij & Blom (2002), Stroeve et al. (2003), and falls outside the scope of the present study.

The paper is organised as follows. Section 2 develops mid-air collision risk equations. Section 3 develops a stopping-time based risk decomposition. Section 4 illustrates some results of the approach of sections 2 and 3 for a realistic application. Section 5 draws conclusions.

2. Mid-air collision risk equations

Throughout this and the next sections, all stochastic processes are defined on a complete stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, P, \mathsf{T})$ with $(\Omega, \mathcal{F}, P)$ a complete probability space, and $\mathbb{F}$ is an increasing sequence of sub-σ-algebra's on the positive time line $\mathsf{T}=\mathbb{R}_+$, i.e. $\mathbb{F} \overset{\Delta}{=} \{\mathcal{I}, (\mathcal{F}_t, t \in \mathsf{T}), \mathcal{F}\}$, $\mathcal{I}$ containing all P-null sets of $\mathcal{F}$ and $\mathcal{I} \subset \mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ for every $s < t$.

Consider an M-aircraft evolution model that is represented by stochastic differential equations[1] with switching coefficients, one for each of the M aircraft, i.e. for $i = 1,\dots,M$,
$$dx_t^i = f^i(x_t, \theta_t, t) + g^i(x_t, \theta_t, t)dw_t^i, \tag{1}$$
with $x_t \overset{\Delta}{=} \mathrm{Col}\{x_t^1,\dots,x_t^M\}$, $\theta_t \overset{\Delta}{=} \mathrm{Col}\{\theta_t^1,\dots,\theta_t^M\}$, $w_t \overset{\Delta}{=} \mathrm{Col}\{w_t^1,\dots,w_t^M\}$, $\{w_t^i\}$ an n-dimensional standard Brownian motion, x_t^i assumes values in $\mathbb{R}^n$ and θ_t^i a finite (N) state process such that $\{x_t, \theta_t\}$ is a semi-martingale and a strong Markov process. The mappings f and g may represent planning and control strategies. Some elements of x_t^i form the 3D position of aircraft i,
$$y_t^i = Hx_t^i, \tag{1'}$$
with H a 3×n-matrix. To avoid Brownian motion behaviour in positions, we adopt the assumption

A.1 $Hg^i(x_t, \theta_t, t) = 0$ for $i = 1,\dots,M$

Under assumption **A.1**, we get for $i = 1,\dots,M$,
$$dy_t^i = v_t^i dt \text{ with } v_t^i \overset{\Delta}{=} Hf^i(x_t, \theta_t, t)$$

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[1] Labeau et al. (2000) assume $g = 0$, i.e. no diffusion.

Next, with y_t^i and y_t^j representing the positions of the centres of aircraft pair (i,j), the relative 3D position is represented by the process $y_t^{ij} = y_t^i - y_t^j$, and the relative velocity is represented by the process $v_t^{ij} = v_t^i - v_t^j$. Hence
$$dy_t^{ij} = v_t^{ij} dt \tag{2}$$

A collision means that $\{y_t^{ij}\}$ enters a closed collision area D^{ij} around the origin; i.e. an area where aircraft i and j are not separated anymore. Under the assumption that the length of the aircraft equals the width of the aircraft, and that the volume of an aircraft is represented by a box the orientation of which does not change in time, then the size of D^{ij} is the sum of the size of two individual aircraft, i.e.
$$D^{ij} = D_1^{ij} \times D_2^{ij} \times D_3^{ij},$$
with $D_k^{ij} = [-m_k^{ij}, m_k^{ij}]$, $m_k^{ij} = \frac{1}{2}(s_k^i + s_k^j)$, s_1^i the length, s_2^i the width, s_3^i the height of aircraft i and $s_1^i = s_2^i$. If the relative position $\{y_t^{ij}\}$ enters D^{ij} at time τ, i.e. if $y_{\tau-\Delta}^{ij} \notin D^{ij}$ and $y_\tau^{ij} \in D^{ij}$ for $\Delta \downarrow 0$, then we say an incrossing event occurred. For equation (1) we assume that D^{ij} is transient (i.e. non-absorbing).

Next, we define for each (i,j) an indicator process $\{\chi_t^{ij}\}$ as follows:
$$\chi_t^{ij} = \begin{cases} 1 & \text{if } y_t^{ij} \in D^{ij}, \\ 0 & \text{else.} \end{cases}$$

C.1 For any (i,j) the indicator process $\{\chi_t^{ij}\}$ has finite variation over any finite interval.

Lemma 1
Under assumption **C.1** the indicator process $\{\chi_t^{ij}\}$ admits on any finite interval a unique decomposition:
$$\chi_t^{ij} = \chi_{t_0}^{ij} + \chi_t^{ij+} - \chi_t^{ij-}, \tag{3}$$
with $\{\chi_{t_0}^{ij+} = \chi_{t_0}^{ij-} = 0\}$, while $\{\chi_t^{ij+}\}$ and $\{\chi_t^{ij-}\}$ are increasing processes such that,
$$\int_{t_0}^{t_1} |d\chi_s^{ij}| = \chi_{t_1}^{ij+} + \chi_{t_1}^{ij-}.$$

Proof: With $\{y_t^{ij}\}$ progressively measurable for all t, and D^{ij} a Borel set, the indicator process $\{\chi_t^{ij}\}$ is also progressively measurable for all t. Due to assumption **C.1** any realisation $\{\chi_t^{ij}(\omega)\}$ is a real-valued measurable function with finite variation for all t, which implies decomposition (3) (Wong and Hajek, 1985, p.218). Q.E.D.

Remark 1: Notice that $\{\chi_t^{ij+}\}$ and $\{\chi_t^{ij-}\}$ count the in-crossings and out-crossings respectively of $\{y_t^{ij}\}$ in D^{ij}.

Next, we define $I_{in}^{ij}(t_0,t_1)$ as the expected number of incrossings between the two aircraft considered during $[t_0, t_1]$ $(t_0 < t_1 < \infty)$, i.e.,

$$I_{in}^{ij}(t_0,t_1) \triangleq E\{\chi_{t_1}^{ij+} - \chi_{t_0}^{ij+}\}, \qquad (4)$$

and define the collision probability $P_{col}^{ij}(t_0,t_1)$ by

$$P_{col}^{ij}(t_0,t_1) \triangleq P\{\chi_{t_1}^{ij+} \neq \chi_{t_0}^{ij+}\} \qquad (5)$$

Remark 2: Equation (5) implies that the first incrossing on a given interval is the collision on that interval.

Furthermore, define τ_0 as the moment of the first incrossing after t_0, i.e. $\tau_0 \triangleq \inf(t > t_0, \chi_t^{ij+} \neq \chi_{t-}^{ij+})$.

<u>Theorem 1</u>
Under assumption **C.1**, the collision risk $P_{col}^{ij}(t_0,t_1)$ defined in (4) satisfies:

$$P_{col}^{ij}(t_0,t_1) = \frac{I_{in}^{ij}(t_0,t_1)}{1 + \int_{t_0}^{t_1} I_{in}^{ij}(t,t_1 \mid \tau_0 = t) \cdot p_{\tau_0 \mid \tau_0 \leq t_1}(t)dt}.$$

Proof: See Blom et al. (2003b).

C.2 For all (i,j), $\Delta > 0$, $E\{(\chi_{t+\Delta}^{ij-} - \chi_t^{ij-})(\chi_{t+\Delta}^{ij+} - \chi_t^{ij+})\} = o(\Delta)$

<u>Theorem 2</u>
Under assumptions **C.1** and **C.2**, equation (4) yields:

$$I_{in}^{ij}(t_0,t_1) = \int_{t_0}^{t_1} E\{d\chi_t^{ij+}\} = \int_{t_0}^{t_1} \phi^{ij}(t)dt \qquad (6)$$

with $\phi^{ij}(t)$ the incrossing rate, which is defined, if the limit exists, as

$$\phi^{ij}(t) \triangleq \lim_{\Delta \downarrow 0} \frac{P\{y_{t-\Delta}^{ij} \notin D^{ij}, y_t^{ij} \in D^{ij}\}}{\Delta} \qquad (7)$$

Proof: See Blom et al. (2003b).

Next, some assumptions are introduced under which $\phi^{ij}(t)$ is characterised. These assumptions are:

A.2 $P\{y_t^{ij} \in D^{ij}, (y_t^{ij} - \Delta v_t^{ij}) \notin D^{ij}, y_{t-\Delta}^{ij} \in D^{ij}\} -$
   $P\{y_t^{ij} \in D^{ij}, (y_t^{ij} - \Delta v_t^{ij}) \in D^{ij}, y_{t-\Delta}^{ij} \notin D^{ij}\} = o(\Delta)$

A.3 For any $k \in \{1,2,3\}$, there is a constant L_k such that for all t and for all $y_k \in [-m_k^{ij}, m_k^{ij}]$: $E\{(v_{k,t}^{ij})^2\} \leq L_k$ and
   $E\{(v_{k,t}^{ij})^2 \mid y_{k,t}^{ij} = y_k\} \leq L_k$.

A.4 A rather technical assumption on the joint density function of the pair (y_t^{ij}, v_t^{ij}) (see Bakker & Blom, 1993).

<u>Theorem 3</u>
Under assumptions **A.1**, **A.2**, **A.3** and **A.4**, the incrossing rate $\phi^{ij}(t)$ defined in (7) satisfies:

$$\phi^{ij}(t) = \sum_{k=1}^{3} \int_{\underline{D}_k^{ij}} \left\{ \int_0^{\infty} v \, p_{\underline{y}_{k,t}^{ij}, y_{k,t}^{ij}, v_{k,t}^{ij}}(\underline{y}, -m_k^{ij}, v)dv + \int_{-\infty}^{0} -v \, p_{\underline{y}_{k,t}^{ij}, y_{k,t}^{ij}, v_{k,t}^{ij}}(\underline{y}, m_k^{ij}, v)dv \right\} d\underline{y} \qquad (8)$$

where

$$\underline{D}_1^{ij} \triangleq D_2^{ij} \times D_3^{ij}, \quad \underline{D}_2^{ij} \triangleq D_1^{ij} \times D_3^{ij}, \quad \underline{D}_3^{ij} \triangleq D_1^{ij} \times D_2^{ij},$$
$$\underline{y}_{1,t}^{ij} \triangleq (y_{2,t}^{ij}, y_{3,t}^{ij}), \quad \underline{y}_{2,t}^{ij} \triangleq (y_{1,t}^{ij}, y_{3,t}^{ij}), \quad \underline{y}_{3,t}^{ij} \triangleq (y_{1,t}^{ij}, y_{2,t}^{ij})$$

Proof: See Bakker & Blom (1993, Theorem 1)

Remark 3: Equations similar to (8) have been derived by Leadbetter (1966, 1973) and by Marcus (1977) for a one-dimensional process and by Belyaev (1968) for a multi-dimensional process.

Remark 4: In Blom & Bakker (2002), the incrossing rate is further characterised for Gaussian and Gaussian mixture shapes of $p_{y_t^{ij}, v_t^{ij}}(\cdot)$.

3. Stopping time based decomposition

Theorem 3 shows that $\phi^{ij}(t)$ can be evaluated as a function of the probability density of the joint relative state (y_t^{ij}, v_t^{ij}). In general, a characterisation of this probability density is complex, especially since there are combinatorially many types of non-nominal events. In order to improve this situation, we introduce a stopping time based approach for decomposing the incrossing risk for a pair of aircraft. Following Section 3, the evolution of the M-aircraft situation is modelled as a Markov process $\{\xi_t\} = \{x_t, \theta_t\}$ in a hybrid state space $X = (\mathbb{R}^n \times \mathbb{M})^M$. From the theory of Markov processes, e.g. Davis (1993), it follows that for a time homogeneous Markov process the evolution of the density distribution $p_{\xi_t}(\cdot)$ of the joint process can be characterised by a Chapman-Kolmogorov equation

$$P\{\xi_t \in A\} = \int_X P\{\xi_t \in A \mid \xi_0 = \xi\}P\{\xi_0 \in d\xi\}, \ t \geq 0, \qquad (10)$$

for any Borel set $A \subset X$.

The first step is to recognise that if the strong Markov property holds true for $\{\xi_t\}$, then equation (10) holds true for any stopping time τ as well:

$$P\{\xi_{\tau+t} \in A\} = \int_X P\{\xi_t \in A \mid \xi_0 = \xi\}P\{\xi_\tau \in d\xi\}, \ t \geq 0 \qquad (11)$$

which for example means that, more colloquially, Monte Carlo simulations of a strong Markov process may be restarted from an empirical distribution that has been obtained for any stopping time. Now for a stopping time $\tau^{ij} \in [t_0, t_1]$ that is smaller than the first incrossing moment τ_0^{ij} between aircraft pair (i,j) on $[t_0, t_1]$, i.e. $t_0 < \tau^{ij} < \tau_0^{ij}$, eq. (6) becomes

$$I_{in}^{ij}(t_0,t_1) = \int_{t_0}^{\tau^{ij}} \phi^{ij}(t)dt + \int_{\tau^{ij}}^{t_1} \phi^{ij}(t)dt = \int_{\tau^{ij}}^{t_1} \phi^{ij}(t)dt \qquad (12)$$

Next, we introduce a conditioning on classes of non-nominal events. To do so, we define an event sequence classification process $\{\kappa_t^{ij}\}$ assuming values in a discrete set $\mathcal{K}$, and such that κ_t^{ij} is a function of θ_t, i.e. $\kappa_t^{ij} = K^{ij}(\theta_t)$, with K^{ij} an application specific measurable mapping of θ_t into $\mathcal{K}$. Hence, $\{\xi_t, \kappa_t^{ij}\}$ too is a strong Markov process. Then for any stopping time τ^{ij} for the aircraft pair (i,j) we can decompose the incrossing integral using the total probability theorem as follows:

$$I_{in}^{ij}(t_0,t_1) = \sum_{\kappa \in \mathcal{K}} \int_{\tau^{ij}}^{t_1} \phi^{ij}(t \mid \kappa_{\tau^{ij}}^{ij} = \kappa)dt \cdot P\{\kappa_{\tau^{ij}}^{ij} = \kappa\} \qquad (13)$$

with $\phi^{ij}(t \mid \kappa_{\tau^{ij}}^{ij} = \kappa)$ the conditional incrossing risk, defined by

$$\phi^{ij}(t \mid \kappa_{\tau^{ij}}^{ij} = \kappa) \triangleq \lim_{\Delta \downarrow 0} \frac{P\{y_{t-\Delta}^{ij} \notin D^{ij}, y_t^{ij} \in D^{ij} \mid \kappa_{\tau^{ij}}^{ij} = \kappa\}}{\Delta}.$$
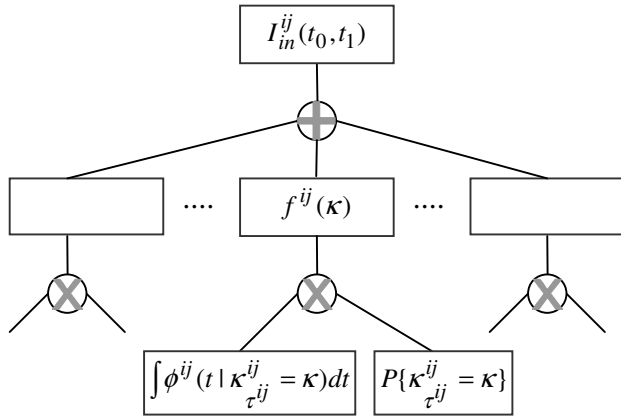

Figure 4: Collision risk tree.

In Figure 4, Equation (13) is presented in the form of a tree, where

$$f^{ij}(\kappa) = \int_{\tau^{ij}}^{t_1} \phi^{ij}(t \mid \kappa_{\tau^{ij}}^{ij} = \kappa) \, dt \cdot P\{\kappa_{\tau^{ij}}^{ij} = \kappa\}$$

This tree has a clear resemblance with the well-known fault tree. However, because of the underlying stochastic and physical relations, our new tree differs significantly and is called a collision risk tree. The collision risk tree decomposition in (13) allows evaluating the incrossing integral in two steps: first the probabilities $P\{\kappa_{\tau^{ij}} = \kappa\}$ and next the conditional incrossing integrals $\int_{\tau^{ij}}^{T} \phi^{ij}(t \mid \kappa_{\tau^{ij}}^{ij} = \kappa)dt$ for each $\kappa \in \mathcal{K}$. If the evaluation of $\int_{\tau^{ij}}^{T} \phi^{ij}(t \mid \kappa_{\tau^{ij}}^{ij} = \kappa)dt$ is as demanding as the direct evaluation of $\int_{\tau^{ij}}^{t_1} \phi^{ij}(t)dt$ is, then nothing is gained with this decomposition. However, by choosing the event sequence classification process $\{\kappa_t^{ij}\}$ and

the stopping time τ^{ij} properly, it may be possible to simplify numerical evaluation of the incrossing integral considerably. The key to realise this is that the relevant state space to evaluate the integration of each $\phi^{ij}(t \mid \kappa_{\tau^{ij}}^{ij} = \kappa)$ over (τ^{ij}, t_1) should be much smaller than the state space needed to evaluate the integration of $\phi^{ij}(t)$ directly over (t_0, t_1). An additional advantage is that it becomes clear how much the contribution to the risk is per κ-value.

4. Results for an en-route ATC example

As an illustrative example, we show some results of applying the risk equations and risk decomposition approach of sections 3 and 4 to a specific conventional en-route ATC situation, with two opposite streams of air traffic at the same flight level (see Figure 5).
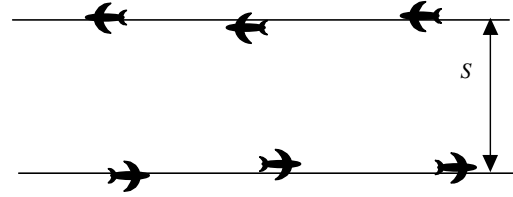

Figure 5: Opposite direction traffic in a dual lane route with lane spacing S

See Everdij & Blom (2002) and Blom et al. (2003a) for further explanation of this example. Here we restrict ourselves to giving the risk evaluation and composition results for varying spacing S values.

Let $\mathfrak{R}^i$ denote the expected number of incrossings per hour ($=T$) between aircraft i and an opposite flying aircraft. Then we have:

$$\mathfrak{R}^i = \sum_j I_{in}^{ij}(t_1 - T, t_1)$$

Let N_{flow} be the aircraft flow per hour per lane and in eq. (1) let for all i, j: $f^i = f^j$, $g^i = g^j$ and $\{w_t^i\}$ and $\{w_t^j\}$ are probabilistically equivalent, then

$$\mathfrak{R}^i = 2N_{flow}I_{in}^{ij}(t_1 - T, t_1) \qquad (14)$$

with j one selected aircraft that encounters aircraft i clearly within the time period.

Stopping time used

Let τ^{ij} be the first moment of overlap in along-lane direction between aircraft i and aircraft j, i.e.

$$\tau^{ij} \triangleq \min\{t_1, \inf\{t \geq t_1 - T; \left| y_{1,t}^{ij} \right| \leq d_1^{ij} + \Delta\}\}$$

with $y_{1,t}^{ij}$ the along distance component of y_t^{ij}, $d_1^{ij} = \frac{1}{2}s_1^i + \frac{1}{2}s_1^j$ and Δ a small positive value. With this stopping time, no collision between aircraft pair (i, j) can occur before τ^{ij}. Hence, substitution of (13) in (14) yields:

$$\mathfrak{R}^i = 2N_{flow} \sum_{\kappa \in \mathcal{K}} \int_{\tau^{ij}}^{t_1} \phi^{ij}(t \mid \kappa_{\tau^{ij}}^{ij} = \kappa)dt \cdot P\{\kappa_{\tau^{ij}}^{ij} = \kappa\} \qquad (15)$$

Event sequence classification

For all t, we define the event sequence classification process κ_t^{ij} as a mapping of θ_t into

$$\mathcal{K} \triangleq \mathcal{K}_{CN} \times \mathcal{K}_{CC} \times (\mathcal{K}_{AB})^2 \times (\mathcal{K}_{DM})^2,$$

where the set names *CN, CC, AB* and *DM* stand for:

- *CN* = Common Navigation modes {*CN Up, CN Down*}
- *CC* = Common Communication modes {*CC Up, CC Down*}
- *AB* = Aircraft Behaviour modes (*Nominal* or Deviating from ATC intent, with two Deviating modes: *Non-Nominal* drift away and *Turning* away)
- *DM* = Decision Making Loop modes, which covers surveillance, controller, radio-communication and crew (all being *Up* or at least one being *Down*).

Numerical results

For the model considered it appeared that, for the *CC×CN* values of κ, $P\{\kappa_{\tau^{ij}}^{ij} = \kappa\}$ could be obtained through Markov chain analysis of the behaviour of an independent Markov chain part of $\{\theta_t\}$. For the other κ-values *CC×CN* conditional Monte Carlo simulation have been run. Table 5 illustrates the $P\{\kappa_{\tau^{ij}}^{ij}\}$ outcomes for some clusters of κ-values:

I. Both aircraft in *AB Nominal* and *DM* being *Up* or *Down*.
II. At least one aircraft in *AB Turning* and *DM* being *Up* or *Down*.
III. All other combinations.

$(AB \times DM)^2$	CN Up		CN Down	
	CC Up	CC Down	CC Up	CC Down
I	$9.99 \cdot 10^{-1}$	$2.50 \cdot 10^{-4}$	0.0	0.0
II	$8.90 \cdot 10^{-5}$	$8.58 \cdot 10^{-8}$	$4.29 \cdot 10^{-10}$	$1.07 \cdot 10^{-13}$
III	$4.49 \cdot 10^{-4}$	$1.12 \cdot 10^{-7}$	$2.50 \cdot 10^{-6}$	$6.25 \cdot 10^{-10}$

Table 5 Common event sequence probabilities for clusters of κ-values in $\mathcal{K}$. For the model considered there is no S dependency.

Next, numerical results for $\int_{\tau^{ij}}^{t1} \phi^{ij}(t \mid \kappa_{\tau^{ij}}^{ij} = \kappa)dt$ are obtained as function of spacing S for all κ values. The numerical evaluation is done through five steps:

1. Importance sampling based Monte Carlo simulation of sets of particles[*] per κ-value to get an empirical density approximation for $p_{y_{\tau^{ij}}^{ij}, v_{\tau^{ij}}^{ij} \mid \kappa_{\tau^{ij}}^{ij}}(\cdot \mid \kappa)$ for each κ-value.
2. Gaussian sum density fitting of the resulting sets of particles per κ-value.
3. Numerical evaluation of (8) using the Gaussian sum characterisation for (8) in Blom & Bakker (2002);
4. Numerical integration over (τ^{ij}, t_1). The effective integration time is of the order $\Delta / E\{v_{1,t}^{ij}\} < 0.5$ s. On this

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[*] A particle is a simulation sample with an importance weight attached to it.

short time interval eq. (1) is assumed to be approximated by the following ODE[**]:

$$dy_t^{ij} = v_t^{ij} dt$$
$$dv_t^{ij} = 0$$

5. Repeat steps 3 and 4 for all relevant *S*-values.

Table 6 illustrates the $\int_{\tau^{ij}}^{t1} \phi^{ij}(t \mid \kappa_{\tau^{ij}}^{ij})dt$ outcomes for clusters of κ-values in $\mathcal{K}$ and for $S = 20$ km.

$(AB \times DM)^2$	CN Up		CN Down	
	CC Up	CC Down	CC Up	CC Down
I	$2.0 \cdot 10^{-16}$	$1.1 \cdot 10^{-14}$	n.a.	n.a.
II	$5.2 \cdot 10^{-9}$	$6.2 \cdot 10^{-9}$	$1.4 \cdot 10^{-9}$	$2.9 \cdot 10^{-8}$
III	$2.6 \cdot 10^{-5}$	$1.5 \cdot 10^{-4}$	$1.6 \cdot 10^{-4}$	$1.6 \cdot 10^{-4}$

Table 6 Conditional incrossing integral for clusters of κ-values in $\mathcal{K}$ and spacing S = 20 km.

Composition using eq. (15)

Solving (15) by substituting $N_{flow} = 3.6$ aircraft per hour and the numerical results obtained for $P\{\kappa_{\tau^{ij}}^{ij} = \kappa\}$ and for $\int_{\tau^{ij}}^{t1} \phi^{ij}(t \mid \kappa_{\tau^{ij}}^{ij} = \kappa)dt$, yields $\mathfrak{R}^i$. Figure 6 illustrates the outcomes as a function of S and for four selected clusters of κ-values in $\mathcal{K}$.
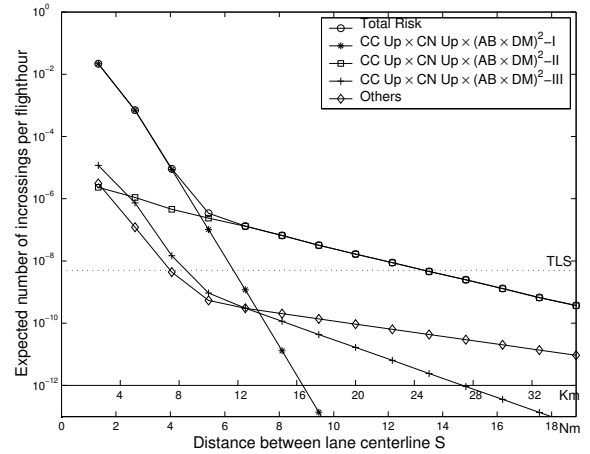

Figure 6. $\mathfrak{R}^i$ and the contributions to it from four clusters of κ values. The horizontal line represents ICAO's applicable Target Level of Safety (TLS) (ICAO, 1998).

In Figure 6, the curve for $\mathfrak{R}^i$ reaches the TLS line at about 24 km. This means that for the mathematical model, a safe spacing value would be 24 km. One should be aware that Figure 6 and Table 5 and Table 6 just illustrate the type of outputs one can get with the mathematical model. For the assessment against reality, see Everdij & Blom (2002).

[**] This ODE implies that the D^{ij}-box has at most one incrossing. Hence, $I_{in}^{ij} = P_{col}^{ij}$.

Numerical accuracy and simulation load
To get the results for all *S*-values, a total of 10^7 aircraft flighthours has been Monte Carlo simulated. This comes down to an average of 10^6 aircraft flighthours per κ-value. The numerical accuracy is 10^{-10}/flighthour. To get a similar accuracy through counting collisions during a standard Monte Carlo simulation, 10^{11} flighthours need to be simulated per *S*-value and for an almost twice as large state space. This is a factor 2.8×10^5 higher. Moreover, it doesn't provide insight in the role played by the κ-value conditions.

5. Concluding Remarks

Increasing air traffic capacity without sacrificing the required level of safety often is the key driver behind the development of advanced operational concepts for ATM. During this development process there is need to receive feedback about the capacity/safety criticalities of the operational concept design. In support of this need, the paper has studied the development of a stochastic modelling approach towards the assessment of mid-air collision risk between aircraft for ATM operational concepts. In sections 2 and 3, collision risk and its decomposition has been studied within the setting of a stochastic differential equation with switching coefficients. The novelty of the approach over approaches known from the literature is twofold:
1) It includes Brownian motion in the evolution equations;
2) It introduced a stopping time based risk decomposition.
In Section 4 this novel approach has been illustrated to work well for a particular en-route example.

There are several interesting directions that ask for a further development of the stochastic analysis approach to accident risk modelling in air traffic:
- Characterisation of large classes of SDE's the solutions of which are semimartingale strong Markov processes on a hybrid state space.
- Development of representation formalisms to specify a mathematical model for an operational concept that has to be assessed on accident risk.
- Further development of accident risk decomposition and novel Monte Carlo simulation methods, and ways to combine these with the analytical approaches towards solving Chapman Kolmogorov equations.
- Development of mathematical equations for other types of accident risk in air traffic, e.g. the stochastic analysis based framework for wake vortex induced accident risk.

In collaboration with several European universities and research institutes, these directions are currently under study within the HYBRIDGE project of the European Commission.

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