Invariant Measure of Stochastic Hybrid Processes

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***Abstract*— Stability of stochastic hybrid systems has recently been discussed by many authors, for example, Basak et al. (1996), Bensoussan and Menaldi (2000), Ji and Chizeck (1990), Mariton (1990), Mao et al. (1999, 2000) and Yuan and Mao (2003), to name a few. The aim of this paper is to study the invariant measure of non-linear stochastic hybrid systems.**

I. Introduction

Stability of stochastic hybrid systems has recently received a lot of attention. For example, Ji and Chizeck (1990) and Mariton (1990) studied the stability of a jump equation

$$dX(t) = A(r(t))X(t)dt, \tag{1}$$

where $r(t)$ is a Markov chain taking values in $S = \{1, 2, ..., N\}$. Mao (1999) investigated the exponential stability for general nonlinear stochastic differential equations with Markovian switching

$$dX(t) = f(X(t), t, r(t))dt + g(X(t), t, r(t))dB(t). \tag{2}$$

Shaikhet (1996) took the time delay into account and considered the stability of a semi-linear stochastic differential delay equation with Markovian switching, while Mao et al. (2000) investigated the stability of a nonlinear stochastic differential delay equation with Markovian switching.

Most of these papers are concerned with asymptotic stability in probability or in mean square (i.e. the solution will tend to zero in probability or in mean square). However, this asymptotic stability is sometimes too strong and in this case it is useful to know whether or not the solution will converge in distribution (not necessary to converge to zero). Basak et al. (1996) discussed such property for a semi-linear stochastic differential equation with Markovian switching of the form

$$dX(t) = A(r(t))X(t)dt + \sigma(X(t), r(t))dB(t), \tag{3}$$

and moreover, Yuan and Mao (2003) investigated this property for a general nonlinear stochastic differential equation with Markovian switching. For more information on stability and the hybrid systems the reader is referred to [4], [5], [10], [14], [16], [17] and the references therein.

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Our aim is to establish criteria on the existence of invariant measure for a non-linear stochastic hybrid system

$$X(t) = \int_0^t f(X(s), r(s))ds + \int_0^t g(X(s), r(s))dB(s) + \int_{[0,t]\times\mathbb{R}^d} h(X(s-), \rho)N(dsd\rho). \tag{4}$$

On the hand, it is well known that once the existence of the invariant measure of an SDE is established, we may compute it by solving the associated PDE, known as the forward equation or the Kolmogorov-Fokker-Planck equation ([9]). If the system is a linear SDE, one can explicitly solve the Fokker-Planck equation. But for the nonlinear SDE, especially in the case of Eq.(4), the situation becomes more complex and it is nontrivial to solve the PDEs. As an alternative, we derive the relationship between the probability density of Eq.(4) and that of the corresponding SDEs.

In Section 2 we shall give the formal definition of stochastic hybrid systems. In Section 3, a generalized Itô formula for Eq. (4) will be established and relationship of probability density and invariant measure among different stochastic systems will be discussed. Section 4 provides some sufficient conditions for the existence of invariant measure in terms of Lyapunov functions.

II. Stochastic Hybrid Systems

Throughout this paper, unless otherwise specified, we let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a complete probability space with a filtration $\mathcal{F}_t$ satisfying the usual conditions (i.e. it is increasing and right continuous while $\mathcal{F}_0$ contains all P-null sets). Let $B(t) = (B_t^1, \ldots, B_t^m)^T$ be an m-dimensional Brownian motion defined on the probability space. Let $|\cdot|$ denote the Euclidean norm for vectors or the trace norm for matrices.

Let $r(t), t \geq 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \ldots, N\}$ with generator $Q = (q_{ij})_{N\times N}$ given by

$$P\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} q_{ij}\Delta + o(\Delta) & \text{if } i \neq j, \\ 1 + q_{ij}\Delta + o(\Delta) & \text{if } i = j, \end{cases}$$

where $\Delta > 0$. Here $q_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while

$$q_{ii} = -\sum_{j\neq i} q_{ij}.$$

We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$. It is well known that almost every sample path of $r(t)$ is right continuous step function. Let $\Pi(\cdot)$ be a probability measure on the Borel subsets of $\mathbb{R}^d$

that has compact support Γ. Assume there exist mutually independent sequences of random varibles $\{\nu_n, 0 \le n < \infty\}$ and $\{\rho_n, 0 \le n < \infty\}$, where the ν_n are exponentially distributed with mean $\frac{1}{\lambda}$ and the ρ_n have distribution $\Pi(\cdot)$. Assume also that these random variables are independent of $B(\cdot)$ and $r(\cdot)$. Let $\tau_0 = 0$, $\tau_{n+1} = \tau_n + \nu_n$. Then τ_n will be the jump times of the process $X(t)$. Let $h : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^n$ be a bounded measure function. Starting with a given initial condition $X(0) = x_0, r(0) = i_0$, between the jumps of $[\tau_k, \tau_{k+1})$, the state equations are of the form

$$\begin{cases} dX(t) = f(X(t), r(t))dt + g(X(t), r(t))dB(t) \\ X(\tau_k) = X(\tau_k-) + h(X(\tau_k-), \rho_k), \quad t \in [\tau_k, \tau_{k+1}) \end{cases} \tag{5}$$

where

$$f : \mathbb{R}^n \times S \to \mathbb{R}^n, \quad g : \mathbb{R}^n \times S \to \mathbb{R}^{n\times m}.$$

The process thus constructed will be defined for all $t \ge 0$ since $\tau_n \to \infty$ as $n \to \infty$ a.s. The mutual independence of the components used to construct the process implies that $(X(t), r(t))$ is a time homogeneous Markov process ([6], [7]). In this paper we shall discuss the invariant measure for the stochastic differential equations with Poisson jump. Assume $\{(\nu_i, \rho_i)\}$ are the point masses of the Poisson random measure. Random variable $\inf\{\nu_i - t : \nu_i \ge t\}$ is exponentially distributed with mean $1/\lambda$. Let $N(\cdot)$ is a Poisson random measure with intensity measure $\lambda dt \times \Pi(dy)$. Given that (ν, ρ) is a point mass of $N(\cdot)$, ρ is distributed according to $\Pi(\cdot)$. Eq. (5) shall be the form of ([6])

$$X(t) = \int_0^t f(X(s), r(s))ds + \int_0^t g(X(s), r(s))dB(s) + \int_{[0,t]\times\mathbb{R}^d} h(X(s-), \rho)N(dsd\rho). \tag{6}$$

For the existence and uniqueness of the solution we shall impose a hypothesis ([7]):

(H) *Both f and g satisfy the local Lipschitz condition and the linear growth condition. That is, for each $k = 1, 2, \ldots$, there is an $L_k > 0$ such that*

$$|f(x, i) - f(y, i)| + |g(x, i) - g(y, i)| \le L_k|x - y|$$

for all $i \in S$ and those $x, y \in \mathbb{R}^n$ with $|x| \vee |y| \le k$; there is an $H > 0$ such that

$$|f(x, i)| + |g(x, i)| \le H(1 + |x|)$$

for all $x \in \mathbb{R}^n$ and $i \in S$; and there is moreover an $M > 0$ for all $x \in \mathbb{R}^n, z \in \mathbb{R}^d$ such that

$$|h(x, z)| \le M.$$

To state our main result, we still need a few more notations. Let $C^2(\mathbb{R}^n \times S; \mathbb{R}_+)$ denote the family of all non-negative functions $V(x, i)$ on $\mathbb{R}^n \times S$ which are continuously twice differentiable in x. Let

$$\lambda(x) = \lambda \int_{\{y:h(x,y)\neq 0\}} \Pi(dy) \le \lambda,$$

$$\bar{\Pi}(x, A) = \int_{\{y:h(x,y)\in A, h(x,y)\neq 0\}} \Pi(dy)$$

and

$$V_x(x, i) = \left(\frac{\partial V(x, i)}{\partial x_1}, \ldots, \frac{\partial V(x, i)}{\partial x_n}\right),$$

$$V_{xx}(x, i) = \left(\frac{\partial^2 V(x, i)}{\partial x_i \partial x_j}\right)_{n\times n}.$$

If $V \in C^2(\mathbb{R}^n \times S; \mathbb{R}_+)$, define operators from $\mathbb{R}^n \times S$ to $\mathbb{R}$ by

$$L^{(1)}V(x, i) = V_x(x, i)f(x, i) + \frac{1}{2}\text{trace}[g^T(x, i)V_{xx}(x, i)g(x, i)], \tag{7}$$

$$L^{(2)}V(x, i) = \lambda(x) \int_\Gamma [V(x + y, i) - V(x, i)]\bar{\Pi}(x, dy), \tag{8}$$

$$LV(x, i) = L^{(1)}V(x, i) + L^{(2)}V(x, i), \tag{9}$$

$$\mathcal{L}V(x, i) = LV(x, i) + \sum_{j=1}^N q_{ij}V(x, j). \tag{10}$$

We conclude this section by defining an invariant measure for Eq. (4) ([3]). Let X^{x_0,i_0} denote the solution with initial $X(0) = x_0, r(0) = i_0$ and $Y^{x_0,i_0}(t)$ denote the $\mathbb{R}^n \times S$-valued process $(X^{x_0,i_0}(t), r^{i_0}(t))$. Then $Y(t)$ is a time homogeneous Markov process. Let $P(t, x, i, dy \times \{j\})$ denote the transition probability of the process $Y(t)$.

Definition 2.1: The probability measure μ on $\mathbb{R}^n \times S$ is said to be the invariant measure of $Y(t) = (X(t), r(t))$ if for any $t > 0$

$$\sum_{l=1}^N \int_{\mathbb{R}^n} P(t, (x, l), A \times \{j\})\mu(dx \times \{l\}) = \mu(A \times \{j\}).$$

III. Transition Probability, Probability Density and Invariant Measure

In this section, we shall discuss the relationship among transition probability, probability density and invariant measures. For the future use we shall prove the generalized Itô formula as a lemma. Fix any x_0 and i_0 and write $X^{x_0,i_0}(t) = X(t)$ for simplicity.

Lemma 3.1: Let $V \in C^2(\mathbb{R}^n \times S; \mathbb{R}_+)$ and $\hat{\tau}_1, \hat{\tau}_2$ be bounded stopping times such that $\hat{\tau}_1 \le \hat{\tau}_2$ a.s. If $V(X(t), r(t))$ and $\mathcal{L}V(X(t), r(t))$ etc. are bounded on $t \in [\hat{\tau}_1, \hat{\tau}_2]$, then

$$EV(X(\hat{\tau}_2), r(\hat{\tau}_2)) = EV(X(\hat{\tau}_1), r(\hat{\tau}_1)) + E\int_{\hat{\tau}_1}^{\hat{\tau}_2} \mathcal{L}V(X(s), r(s))ds. \tag{11}$$

Proof For any $s < t$, define

$$J_V(i,s,t) = \sum_{s\le u\le t}[V(X(u,i)) - V(X(u-,i))] - \int_s^t \lambda(X(s))\int_\Gamma \big[V(X(u)+y, r(u)) - V(X(u), r(u))\big]\bar{\Pi}(X(u),dy)du. \quad (12)$$

Let $s=\sigma_0<\sigma_1<\cdots<\sigma_v<t$ be all the times when $r(u)$ has a jump. Applying the Itô formula to $V(x(u),i)$ on the intervals $[s,\sigma_1),(\sigma_1,\sigma_2),\dots,(\sigma_v,t]$, we have that

$$V(X(\sigma_1),i) - V(X(s),i) = \int_s^{\sigma_1} LV(X(u),i)du + \int_s^{\sigma_1} V_x(X(u),i)g(X(u),i)dB(s) + J_V(i,s,\sigma_1),$$

$$V(X(\sigma_{l+1}),i) - V(X(\sigma_l),i) = \int_{\sigma_l}^{\sigma_{l+1}} LV(X(u),i)du + \int_{\sigma_l}^{\sigma_{l+1}} V_x(X(u),i)g(X(u),i)dB(s) + J_V(i,\sigma_l,\sigma_{l+1}), \quad l=2,\dots,v-1,$$

$$V(X(t),i) - V(X(\sigma_v),i) = \int_{\sigma_v}^{t} LV(X(u),i)du + \int_v^t V_x(X(u),i)g(X(u),i)dB(s) + J_V(i,t,\sigma_v).$$

Setting $s=0$ and substituting $i=i_0$ in the first equation, $i=r(\sigma_l)$ in the second and $i=r(\sigma_v)$ in the third, we get

$$\begin{aligned} V(X(t),r(t)) - V(x_0,i_0) &= \int_0^t LV(X(s),r(s))ds + M(t) + \sum_{l=1}^{v}[V(X(\sigma_l),r(\sigma_l)) - V(X(\sigma_l),r(\sigma_l-))] \\ &= \int_0^t LV(X(s),r(s))ds + M(t) + \sum_{j=1}^{N}\int_0^t q_{r(s)j}V(X(s),j)ds \\ &= \int_0^t \mathcal{L}V(X(s),r(s))ds + M(t), \end{aligned}$$

where $M(t) = \int_0^t V_x(X(u),r(u))g(X(u),r(u))dB(s) + \sum_{v=1}^{l} J_V(r(\sigma_v),0,t)$ is a martingale. The required assertion follows taking the expectation both sides. □

For $t\in\mathbb{R}_+, i\in S, x\in\mathbb{R}^n$ and $A\in\mathcal{B}(\mathbb{R}^n)$, let $P^i(t,x,A)$ denote the transition probability of the following stochastic differential equation

$$dX(t) = f(X(t),i)dt + g(X(t),i)dB(t) + \int_{[0,t]\times\mathbb{R}^d} h(X(s-),\rho)N(dsd\rho), \quad (13)$$

and $P_0^i(t,x,A)$ denote the transition probability of the following stochastic differential equation

$$dX(t) = f(X(t),i)dt + g(X(t),i)dB(t). \quad (14)$$

Let $\tilde{\sigma}_i$ denote the sojourn time of $r(t)$ in state $i\in S$. Then, for any $t>0$, $A\in\mathcal{B}(\mathbb{R}^n)$ and $i,j\in S$, we have the well-known fact

$$P(\tilde{\sigma}_i > t) = e^{q_{ii}t}, \quad (15)$$

$$\begin{aligned} P(t,(x,i),A\times\{j\}) &= E_{x,i}[I_{\{X(s)\in A, r(s)=j\}}|\tilde{\sigma}_i>t]P_{x,i}(\tilde{\sigma}_i>t) + E_{x,i}[I_{\{X(s)\in A, r(s)=j\}}I_{\{\tilde{\sigma}_i<t\}}] \\ &= e^{q_{ii}t}P^i(t,x,A)\delta_{ij} - \int_0^t q_{ii}e^{q_{ii}s}ds \times \int_{\mathbb{R}^n} P^i(s,x,dy)\sum_{l\ne i} q_{il}P(t-s,(y,l),A\times\{j\}) \end{aligned} \quad (16)$$

and

$$\begin{aligned} P(t,(x,i),A\times\{j\}) &= E_{x,i}[I_{\{X(s)\in A, r(s)=j\}}|\tilde{\sigma}_i>t]P_{x,i}(\tilde{\sigma}_i>t) + E_{x,i}[I_{\{X(s)\in A, r(s)=j\}}I_{\{\tilde{\sigma}_i<t\}}] \\ &= e^{q_{ii}t}P^i(t,x,A)\delta_{ij} - \int_0^t q_{ii}e^{q_{ii}s}ds \times \int_{\mathbb{R}^n} \sum_{l\ne i} q_{lj}P(s,(x,i),dy\times\{l\})P^l(t-s,y,A). \end{aligned} \quad (17)$$

By (17), we have the following lemma.

Lemma 3.2: Let μ denote the invariant measure of $P(t,(x,i),A\times\{j\})$. Then for any $A\in\mathcal{B}(\mathbb{R}^n), j\in S$

$$\mu(A\times\{j\}) = \sum_{i=1}^N \int_{\mathbb{R}^n} e^{q_{ii}t}P^i(t,x,A)\delta_{ij}\mu(dx\times\{i\}) - \sum_{i=1}^N\int_0^t q_{ii}e^{q_{ii}s}ds\int_{\mathbb{R}^n}\sum_{l\ne i} q_{lj}P^l(t-s,x,A)\mu(dx\times\{i\}). \quad (18)$$

From (17) and (18), if we know the transition probability of the solution of Eq. (13), then we can obtain the transition probability and invariant measure of Y(t). Using (17) again, the relationship between probability densities is stated by the following lemma.

Lemma 3.3: Let p and p^i denote the transition probability density of $P(t,(x,i),\cdot\times\cdot)$ and $P^i(t,x,\cdot)$, respectively. Then for any $t\ge 0, x,y\in\mathbb{R}^n, i,j\in S$

$$p(t,(x,i),y\times\{j\}) = e^{q_{ii}t}p^i(t,x,y)\delta_{ij} - \int_0^t q_{ii}e^{q_{ii}s}ds \times \int_{\mathbb{R}^n}\sum_{l\ne i} q_{lj}p(s,(x,i),dz\times\{l\})p^l(t-s,z,y). \quad (19)$$

On the other hand, by [18, Theorem 14 on p35] and noting $\Pi(\mathbb{R}^d) = 1$, compute

$$
\begin{aligned}
P^i(t,x,A) &= e^{-t} P_0^i(t,x,A) \\
&+ \int_0^t du \int\int e^{-t} P_0^i(t-u, y+h(y,\theta), A) \\
&\times P_0^i(u,x,dy)\Pi(d\theta) + \sum_{n=2}^{\infty} \int_{0<u_1<\cdots<u_n<t} \\
&\times \int\int \cdots \int\int e^{-u_1} P_0^i(u_1,x,dy_1)\Pi(d\theta) \\
&\times e^{-(u_2-u_1)} P_0^i(u_2-u_1, y_1+h(y_1,\theta_1), dy_2) \\
&\times \cdots \times e^{-(u_n-u_{n-1})} \\
&\times P_0^i(u_n-u_{n-1}, y_{n-1}+h(y_{n-1},\theta_{n-1}), dy_n) \\
&\times e^{-(t-u_n)} P_0^i(t-u_n, y_n+h(y_n,\theta_n), A) \\
&\times \Pi(d\theta_1)\cdots\Pi(d\theta_n) du_1\cdots du_n. \qquad (20)
\end{aligned}
$$

Let p_0^i denote the probability density of $P_0^i(t,x,\cdot)$. By (16) and (20), we have

$$
\begin{aligned}
p(t,(x,i),y\times\{j\}) &= e^{q_{ii}t}\delta_{ij}e^{-t}p_0^i(t,x,y) \\
&+ \sum_{n=1}^{\infty}\delta_{ij}\int_{0<u_1<\cdots<u_n<t}\int\int\cdots\int\int e^{-u_1} \\
&\times p_0^i(u_1,x,dy_1)\Pi(d\theta)e^{-(u_2-u_1)} \\
&\times p_0^i(u_2-u_1, y_1+h(y_1,\theta_1), dy_2) \\
&\times \cdots \times e^{-(u_n-u_{n-1})} \\
&\times p_0^i(u_n-u_{n-1}, y_{n-1}+h(y_{n-1},\theta_{n-1}), dy_n) \\
&\times e^{-(t-u_n)} p_0^i(t-u_n, y_n+h(y_n,\theta_n), y) \\
&\times \Pi(d\theta_2)\cdots\Pi(d\theta_n) du_1\cdots du_n \\
&- \sum_{l\neq i} q_{il}q_{ii}\int_0^t e^{q_{ii}s}ds\int_{\mathbb{R}^n} e^{-s} \\
&\times p_0^i(t,x,dy)p(t-s,(y,l),y\times\{j\}) \\
&- \sum_{n=1}^{\infty}\sum_{l\neq i} q_{il}q_{ii}\int_0^t e^{q_{ii}s}ds \\
&\times \int_{\mathbb{R}^n} p(t-s,(y,l),y\times\{j\}) \\
&\times \int_{0<u_1<\cdots<u_n<s}\int\int\cdots\int\int e^{-u_1} \\
&\times p_0^i(u_1,x,dy_1)\Pi(d\theta)e^{-(u_2-u_1)} \\
&\times p_0^i(u_2-u_1, y_1+h(y_1,\theta_1), dy_2) \\
&\times \cdots \times e^{-(u_n-u_{n-1})} \\
&\times p_0^i(u_n-u_{n-1}, y_{n-1}+h(y_{n-1},\theta_{n-1}), dy_n) \\
&\times e^{-(s-u_n)} p_0^i(s-u_n, y_n+h(y_n,\theta_n), dy) \\
&\times \Pi(d\theta_1)\cdots\Pi(d\theta_n) du_1\cdots du_n. \qquad (21)
\end{aligned}
$$

Theorem 3.1: The transition probability density of $Y(t)$ and that of the corresponding SDEs are related by (21).

IV. Existence of Invariant Measure

In this section we will discuss the existence of invariant measure of $Y(t)$. Let us introduce more notations. Denote by $\mathcal{P}(\mathbb{R}^n \times S)$ the family of all probability measures on $\mathbb{R}^n \times S$. Denote by $\mathbb{L}$ the family of mappings $F : \mathbb{R}^n \times S \to \mathbb{R}$ satisfying

$$|F(x,i) - F(y,j)| \le |x-y| + |i-j| \quad \text{and} \quad |F(x,i)| \le 1.$$

For $P_1, P_2 \in \mathcal{P}(\mathbb{R}^n \times S)$ define metric $d_{\mathbb{L}}$ as follows:

$$
\begin{aligned}
d_{\mathbb{L}}(P_1,P_2) = \sup_{F\in\mathbb{L}} \Big| &\sum_{i=1}^N \int_{\mathbb{R}^n} F(x,i)P_1(dx,i) \\
&- \sum_{i=1}^N \int_{\mathbb{R}^n} F(x,i)P_2(dx,i)\Big|.
\end{aligned}
$$

It is known that the weak convergence of probability measures is a metric concept (Ikeda and Watanabe [7, Proposition 2.5]). In other words, a sequence $\{P_k\}_{k\ge1}$ of probability measures in $\mathcal{P}(\mathbb{R}^n \times S)$ converges weakly to a probability measure $P_0 \in \mathcal{P}(\mathbb{R}^n \times S)$ if and only if

$$\lim_{k\to\infty} d_{\mathbb{L}}(P_k, P_0) = 0.$$

Let us now define the weak convergence of $Y(t)$.

Definition 4.1: $Y(t)$ is said to be convergent weakly to $\pi(\cdot\times\cdot) \in \mathcal{P}(\mathbb{R}^n \times S)$ if the transition probability measure $P(t,x,i,dy\times\{j\})$ converges weakly to $\pi(dy\times\{j\})$ as $t\to\infty$ for every $(x,i) \in \mathbb{R}^n \times S$, that is

$$\lim_{t\to\infty}\Big(\sup_{F\in\mathbb{L}} |EF(Y(t) - E_\pi F|\Big) = 0,$$

where

$$E_\pi F = \sum_{i=1}^N \int_{\mathbb{R}^n} F(y,i)\pi(dy,i).$$

Obviously if $Y(t)$ converges weakly to a probability measure implies the existence of a unique invariant probability measure for $Y(t)$. To show this property we impose the following assumptions.

Assumption 4.1: For any $(x,i) \in \mathbb{R}^n \times S$, we have

$$\sup_{0\le t<\infty} E|X^{x,i}(t)|^2 < \infty. \qquad (22)$$

Assumption 4.2: For any $(x,y,i) \in \mathbb{R}^n \times S$, we have

$$\lim_{t\to\infty} E|X^{x,i}(t) - X^{y,i}(t)|^2 = 0. \qquad (23)$$

Using the same method of Yuan and Mao [19], we have the following results.

Theorem 4.1: Let (H) hold. Under Assumptions 4.1 & 4.2, $Y(t)$ converges weakly to a probability measure.

Theorem 4.1 implies that $Y(t)$ has a unique invariant measure under Assumptions 4.1 & 4.2. It is therefore necessary to establish sufficient criteria for these properties so that Theorem 4.1 is applicable. On the other hand, Assumption 4.1 is concerned with boundedness while Assumption 4.2 is associated with uniformly asymptotic stability. The importance of the study on both of them is therefore clear.

Lemma 4.1: Assume that there exists a function $V \in C^2(\mathbb{R}^n \times S; \mathbb{R}_+)$ and positive numbers c_1, c_2, β such that

$$c_1|x|^2 \le V(x,i) \qquad (24)$$

and

$$LV(x,i) \le -c_2 V(x,i) + \beta \tag{25}$$

for all $(x,i) \in \mathbb{R}^n \times S$. Then Assumption 4.1 holds.

We omit the proof because it is similar to that for SDEs.

In what follows we shall establish another criterion. Clearly we need to consider the difference between two solutions of Eq. (4) starting from different initial values, namely

$$\begin{aligned} X^{x,i}(t) - X^{y,i}(t) &= x - y \\ &= \int_0^t [f(X^{x,i}(s), r_i(s)) - f(X^{y,i}, r_i(s))]ds \\ &+ \int_0^t [g(X^{x,i}(s), r_i(s)) - g(X^{y,i}, r_i(s))]dB(s) \\ &+ \int_{[0,t]\times\mathbb{R}^d} [h(X^{x,i}(s-), \rho) - h(X^{y,i}(s-), \rho)]N(dsd\rho). \end{aligned} \tag{26}$$

Let

$$\tilde{\lambda}(x-y) = \lambda \int_{\{z: h(x,z)-h(y,z)\neq 0\}} \Pi(dz),$$

$$\bar{\Pi}(x-y, A) = \int_{\{z: h(x,z)-h(y,z)\in A, h(x,z)-h(y,z)\neq 0\}} \Pi(dz).$$

For a given function $U \in C^2(\mathbb{R}^n \times S; \mathbb{R}_+)$, we define operators by

$$\begin{aligned} \tilde{L}^{(1)}U(x,y,i) &= U_x(x-y,i)[f(x,i) - f(y,i)] \\ &+ \frac{1}{2}\text{trace}\Big([g(x,i) - g(y,i)]^T \\ &\times U_{xx}(x-y,i)[g(x,i) - g(y,i)]\Big), \end{aligned} \tag{27}$$

$$\begin{aligned} \tilde{L}^{(2)}U(x,y,i) &= \tilde{\lambda}(x-y)\int_{\mathbb{R}^d} \big[U(x-y+z) \\ &- U(x-y)\big]\bar{\Pi}(x-y,dz), \end{aligned} \tag{28}$$

$$\begin{aligned} \tilde{\mathcal{L}}U(x,y,i) &= \tilde{L}^{(1)}U(x,y,i) + \tilde{L}^{(2)}U(x,y,i) \\ &+ \sum_{j=1}^N \gamma_{ij} U(x-y,j). \end{aligned} \tag{29}$$

Lemma 4.2: If there exists a function $U \in C^2(\mathbb{R}^n \times S; \mathbb{R}_+)$ and positive numbers c_3, c_4 such that

$$U(0,i) = 0 \quad \forall i \in S, \tag{30}$$

$$c_3|x|^2 \le U(x,i) \quad \forall (x,i) \in \mathbb{R}^n \times S, \tag{31}$$

$$\tilde{\mathcal{L}}U(x,y,i) \le -c_4|x-y|^2 \quad \forall (x,y,i) \in \mathbb{R}^n \times \mathbb{R}^n \times S. \tag{32}$$

Then Assumption 4.2 holds.

Proof We divide the whole proof into three steps.

Step 1. Let $x, y \in \mathbb{R}^n$ and $i \in S$. Let l be a positive number and define the stopping time

$$\tau_l = \inf\{t > 0 : |X^{x,i}(t) - X^{y,i}(t)| \ge l\}.$$

Let $t_l = \tau_l \wedge t$. By the generalized Itô formula

$$\begin{aligned} EU(X^{x,i}(t_l) - X^{y,i}(t_l), r_i(t_l)) &= U(x-y,i) \\ &+ E\int_0^{t_l} \tilde{\mathcal{L}}U(X^{x,i}(s), X^{y,i}(s), r_i(s))ds. \end{aligned}$$

Using (31) and (32) and then letting $l \to \infty$ produce

$$\begin{aligned} c_3 E|X^{x,i}(t) - X^{y,i}(t)|^2 &\le U(x-y,i) \\ &- c_4 E\int_0^t |X^{x,i}(s) - X^{y,i}(s)|^2 ds. \end{aligned} \tag{33}$$

This implies

$$E|X^{x,i}(t) - X^{y,i}(t)|^2 dt \le \frac{1}{c_3} U(x-y,i) \tag{34}$$

and

$$\int_0^\infty E|X^{x,i}(t) - X^{y,i}(t)|^2 dt \le \frac{1}{c_4} U(x-y,i) < \infty. \tag{35}$$

Step 2. We now claim that

$$\lim_{t\to\infty} E|X^{x,i}(t) - X^{y,i}(t)|^2 = 0. \tag{36}$$

If this is not true, then

$$\limsup_{t\to\infty} E|X^{x,i}(t) - X^{y,i}(t)|^2 > 4\varepsilon$$

for some $\varepsilon > 0$. Thus there exists a sequence $\{t_k\}_{k\ge 1}$ with $t_{k+1} > t_k + 1$ such that

$$E|X^{x,i}(t_k) - X^{y,i}(t_k)|^2 \ge 4\varepsilon \quad \forall k \ge 1. \tag{37}$$

For $t_k \le t \le t_k + 1$, it is easy to see from (26) that

$$\begin{aligned} &E|X^{x,i}(t) - X^{y,i}(t)|^2 \\ &\ge \frac{1}{4} E|X^{x,i}(t_k) - X^{y,i}(t_k)|^2 \\ &- E\Big|\int_{t_k}^t [f(X^{x,i}(s), r_i(s)) - f(X^{y,i}(s), r_i(s))]ds\Big|^2 \\ &- E\Big|\int_{t_k}^t [g(X^{x,i}(s), r_i(s)) - g(X^{y,i}(s), r_i(s))]dB(s)\Big|^2 \\ &- E\Big|\int_{[t_k,t]\times\mathbb{R}^d} [h(X^{x,i}(s), \rho) - h(X^{y,i}(s), \rho)]N(dsd\rho)\Big|^2. \end{aligned} \tag{38}$$

However, by the Hölder inequality and hypothesis (H), we derive that

$$\begin{aligned} &E\Big|\int_{t_k}^t [f(X^{x,i}(s), r_i(s)) - f(X^{y,i}(s), r_i(s))]ds\Big|^2 \\ &\le E\int_{t_k}^t |f(X^{x,i}(s), r_i(s)) - f(X^{y,i}(s), r_i(s))|^2 ds \\ &\le C_1 \int_{t_k}^t \Big[1 + E|X^{x,i}(s)|^2 + E|X^{y,i}(s)|^2\Big] ds, \end{aligned} \tag{39}$$

where C_1 is a constant dependent on H (described in (H)). Moreover,

$$E|\int_{t_k}^t [g(X^{x,i}(s), r_i(s)) - g(X^{y,i}(s), r_i(s))]dB(s)|^2$$
$$= E\int_{t_k}^t |g(X^{x,i}(s), r_i(s)) - g(X^{y,i}(s), r_i(s))|^2 ds$$
$$\leq C_1 \int_{t_k}^t \Big[1 + E|X^{x,i}(s)|^2 + E|X^{y,i}(s)|^2\Big] ds. \quad (40)$$

Since h is bounded by M (described in (H)),

$$E|\int_{[t_k,t]\times\mathbb{R}^d} [h(X^{x,i}(s),\rho) - h(X^{y,i}(s),\rho)]N(dsd\rho)|^2$$
$$\leq C_2|t - t_k|, \quad (41)$$

where C_2 depends on M. By Assumption 4.1 we therefore see from (39), (40) and (41) that there is $\delta \in (0,1)$ such that

$$E|\int_{t_k}^t [f(X^{x,i}(s), r_i(s)) - f(X^{y,i}(s), r_i(s))]ds|^2$$
$$+E|\int_{t_k}^t [g(X^{x,i}(s), r_i(s)) - g(X^{y,i}(s), r_i(s))]dB(s)|^2$$
$$+E|\int_{[t_k,t]\times\mathbb{R}^d} [h(X^{x,i}(s),\rho) - h(X^{y,i}(s),\rho)]N(dsd\rho)|^2$$
$$\leq \frac{\varepsilon}{2} \quad \forall t \in [t_k, t_k+\delta],\ k \geq 1.$$

It therefore follows from (38) and (37) that

$$E|X^{x,i}(t) - X^{y,i}(t)|^2 \geq \frac{\varepsilon}{2} \quad \forall t \in [t_k, t_k+\delta],\ k \geq 1. \quad (42)$$

Consequently

$$\int_0^\infty E|X^{x,i}(t) - X^{y,i}(t)|^2 dt \geq \sum_{k=1}^\infty \int_{t_k}^{t_k+\delta} \frac{\varepsilon}{2} dt = \infty,$$

but this contradicts with (35) so (36) must hold.

Step 3. We can now show Assumption 4.2, namely (23). Let $\varepsilon > 0$ be arbitrary. It is easy to observe from (34) that there is a $\delta > 0$ such that

$$E|X^{x,i}(t) - X^{y,i}(t)|^2 < \frac{\varepsilon}{9} \quad \forall t \geq 0 \quad (43)$$

provided $x, y \in \mathbb{R}^n$ with $|x - y| < \delta$.

Now, given any any compact subset K of $\mathbb{R}^n$, we can find finite vectors $x_1, ..., x_u \in K$ such that $\cup_{k=1}^u B(x_k,\delta) \supseteq K$, where $B(x_k,\delta) = \{x \in \mathbb{R}^n : |x - x_k| < \delta\}$. From Step 2 we observe that there is a $T > 0$ such that

$$E|X^{x_k,i}(t) - X^{x_l,i}(t)|^2 < \frac{\varepsilon}{9}, \forall t \geq T,\ 1 \leq k,l \leq u, i \in S. \quad (44)$$

Consequently, for any $(x,y,i) \in K \times K \times S$, find x_l, x_k for $|x - x_l| < \delta$ and $|y - x_k| < \delta$. It then follows from (43) and (44) that

$$E|X^{x,i}(t) - X^{y,i}(t)|^2$$
$$\leq 3\Big(E|X^{x,i}(t) - X^{x_l,i}(t)|^2$$
$$+E|X^{y,i}(t) - X^{x_k,i}(t)|^2 + E|X^{x_k,i}(t) - X^{x_l,i}(t)|^2\Big)$$
$$< \varepsilon \quad \forall t \geq T,$$

as required. □

V. Conclusion

The logic next step is to study the explicit solution and asymptotic properties of the solutions of Kolmogorov-Fokker-Planck equation for the Eq. (4). Moreover, it will be interesting to study the numerical solutions of stochastic hybrid systems.

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