Balancing dwell times for switching linear systems

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Abstract

Switching Linear Systems (SLSs) are a subclass of hybrid systems characterized by a Finite State Machine (FSM) and a set of linear dynamical systems, each corresponding to a state of the FSM. The transition between two different states of the FSM is caused by external uncontrollable events that act as discrete disturbances. In the past few years structural properties of SLSs have been the topic of intensive study and in particular much work has been devoted to the attempt of characterizing their stability and/or stabilizability properties. We focus on the class of uncontrolled SLSs with a dwell time associated to each transition. Loosely, a dwell time function assigns to each transition a dwell time that serves as a minimal delay for the transitions. Notice that in our setting the dwell time is associated with transitions rather than with locations. The motivation to use the notion of dwell time function lies in the possibility to quantify the balance between long delays for some transitions and short delays for others. For instance, in a cycle of transitions, instantaneous transitions could be compensated by long delays elsewhere in the cycle. A recent result (see "Can linear stabilizability analysis be generalized to switching systems?" by E. De Santis, M.D. Di Benedetto, G. Pola), which extends Kalman decomposition to the class of controlled SLSs, shows that a controlled SLS is asymptotically stabilizable if and only if an uncontrolled SLS, appropriately associated to the controlled SLS, is asymptotically stable. Then, the stabilizability problem for the class of controlled SLSs directly translates to the stability analysis of uncontrolled SLSs. Therefore, we focus on stability problems for uncontrolled SLSs. It is well-known that if transitions are sufficiently delayed and if the dynamics in each location is asymptotically stable, then the uncontrolled SLS is asymptotically stable. The case of stability under quadratic Lyapunov function analysis is investigated. In this special case, an explicit condition on the dwell time function that ensures asymptotic stability is provided. Although this condition is based on estimates and is therefore conservative, it yields the possibility of instantaneous transitions when applied to the case where a common quadratic Lyapunov function exists. With respect to previous work in this research area, our approach gives less conservative conditions.

Switching linear systems are an important subclass of hybrid systems and they have been extensively studied in the literature. Considerable attention has been payed to the characterization of their stability properties (e.g. [1], [2], [6], [8], [10], [9], [13], [12] and [14]). Despite the simple dynamics within each location, due to the interaction among the locations a satisfactory stability theory for switching linear systems is still lacking. This paper, following the approaches developed in [6, 13], offers new sufficient conditions ensuring asymptotic stability. In specific cases these yield tighter results than those in [6, 13].

Switching linear systems are dynamical systems characterized by a hybrid state composed by a discrete component ℓ, called location and a continuous component x. The evolution of the discrete

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component ℓ is may be thought of as being caused by unknown and external discrete disturbance, and follows a discrete event system [7]. The evolution of the continuous component x is governed by linear differential equations depending on the current discrete state ℓ. In our setting there is no state reset, that is, whenever a discrete transition occurs, the continuous state does not jump. This implies that the continuous state evolution is a continuous function of the time. A formal definition of switching linear systems based on [3], is given below.

Definition 1 *A Switching Linear System (SLS) $\mathcal{S}$ is a tuple $(L, \Delta, E, \mathbb{R}^n, A)$ where $L = \{l_1, l_2, ..., l_N\}$ is a finite set of locations, $N \in \mathbb{N}$; $E \subset L \times L$ is a collection of discrete transitions; $\Delta : E \to \mathbb{R}^+$ is the dwell time function; $\mathbb{R}^n$ is the continuous state space; $A : L \to \mathbb{R}^{n\times n}$ is a family of square matrices associated with each location.*

Loosely, the dwell time function Δ enables transitions with a delay. The transition e can take place $\Delta(e)$ units of time after the previous transition. The precise effect of Δ can be formalized using the language of timed automata. In this framework Δ specifies a family of *guards*. Notice that there are no *invariants*, i.e., transitions are not forced. Given a SLS $\mathcal{S} = (L, \Delta, E, \mathbb{R}^n, A)$, the tuple $\mathcal{D}_{\mathcal{S}} = (L, \Delta, E)$ can be viewed as a *Timed Automaton* (TA), having state set L, transition relation E, and guards defined by Δ. TA $\mathcal{D}_{\mathcal{S}}$ characterizes the structure of discrete variables of $\mathcal{S}$. Moreover $\mathcal{D}_{\mathcal{S}}$ can be decomposed into its *strongly connected components* [7], i.e. the maximal sets of mutually reachable states, and all strongly connected components determine a directed acyclic graph.
In principle we would like to allow a instantaneous jumps from one location to the other. To avoid technical complications concerning Zeno behavior, however, we assume a uniform minimal dwell time. To that effect we choose $\delta_{\min} > 0$ and we require that for all $e \in E$: $\Delta(e) \geq \delta_{\min}$. It should be emphasized that $\delta_{\min}$ is a design parameter and can be chosen arbitrarily small.
Summarizing, in our setting a switching linear systems consists of a finite number of locations $\ell \in L$. In location ℓ the continuous state x evolves according to $\frac{d}{dt}x = A(\ell)x$. A transition $e \in E$ from location ℓ_1 to ℓ_2 can occur only $\Delta(e)$ times units after location ℓ_1 was entered. Recall that a hybrid time basis τ is an infinite or finite sequence of intervals I_j satisfying the following conditions: $I_j = \{t \in \mathbb{R}_0^+ : t_j \leq t \leq t'_j\}$; if $\text{card}(\tau) = m + 1 < \infty$, then I_m is of the form $I_m = \left\{t \in \mathbb{R}_0^+ : t_m \leq t < \infty\right\}$ and $t'_m = \infty$; for all j, $t_j \leq t'_j$ and for $j > 0$, $t_j = t'_{j-1}$. Denote by $\mathcal{T}$ the set of all hybrid time bases. The definition of SLSs temporal evolution can be then formalized by means of the notion of *execution*.

Definition 2 *An execution χ of a SLS $\mathcal{S}$ is a collection $(\ell_0, x_0, \tau, \ell, x)$ with $(\ell_0, x_0) \in L \times \mathbb{R}^n$, $\tau \in \mathcal{T}$, $\ell : \tau \to L$, $x : \mathbb{R}_0^+ \times \mathbb{N} \to \mathbb{R}^n$ satisfying:*

- *Discrete evolution: For all j: $(\ell(I_j), \ell(I_{j+1})) \in E$ and $\ell(I_0) = \ell_0$.*
- *Continuous evolution: $x(0, \ell(I_0)) = x_0$, $\forall t \in I_j$ x satisfies $\frac{d}{dt}x(t) = A_{\ell(I_j)}x(t)$, and $x(t'_j, \ell(I_j)) = x(t_{j+1}, \ell(I_{j+1}))$.*

The problem we address in this paper is the analysis of asymptotic stability of SLSs according to the following definition.

Definition 3 *[3] A Switching Linear System $\mathcal{S}$ is stable if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for any continuous initial state x_0 with $\|x_0\| < \delta$, $\|x(t, j)\| < \varepsilon$, $\forall t \geq 0$, $\forall j \geq 0$, for all executions with continuous initial state x_0. An SLS $\mathcal{S}$ is (globally) asymptotically stable if it is stable and for any $\varepsilon > 0$ and for any $x_0 \in \mathbb{R}^n$, there exists time $\mathbf{t}$ such that $x(t, j) \in \varepsilon\mathcal{B}$, $\forall t \geq \mathbf{t}$, $\forall j \geq \mathbf{j}$, for all executions with initial continuous state x_0, where $\mathbf{j} = \inf\{j : \mathbf{t} \in I_j\}$.*

Stability analysis of switching linear systems has been addressed in the literature in several papers, for a survey see [2] and [14]. A recent result [5], which extends Kalman decomposition to the class of controlled SLSs, shows that a controlled SLS is asymptotically stabilizable if and only if an uncontrolled SLS, appropriately associated to the controlled SLS, is asymptotically stable. Then,

the stabilizability problem for the class of controlled SLSs directly translates to the stability analysis of uncontrolled SLSs.
As shown by Liberzon et al. [10] and Johansson et al. [8], existence of a common Lyapunov function for all subsystems $\dot{x} = A(\ell)x$, $\ell \in L$ ensures asymptotic stability for all dwell time function Δ. In general, such function can be searched by using LMI techniques. In particular, its existence is ensured in the case of subsystems with asymptotically stable matrices either commuting pairwise (see [12]) or generating a solvable Lie algebra (see [9] and [1]). Moreover, stability results for SLSs exhibiting a minimum time separation between consecutive switchings are given by Morse [13] and Hesphana et al. [6]. In [13] Morse introduced the notion of *minimum dwell time*. In our notation this amounts to the requirement that $\Delta(\ell_1, \ell_2) \geq \delta_1$ for an appropriately chosen set of positive numbers δ_j called minimum dwell times. Notice that the minimum dwell time for a particular transitions only depends on the 'departure location' and not on the destination. In [6], Hesphana et al. propose a more flexible notion, that of *average dwell time*. Asymptotic stability is ensured if the number of jumps in a time unit is less than a prescribed parameter.
In the present paper we analyse the dwell time function Δ, explicitly incorporating the possibility that the minimum dwell time may depend on the transition rather than the location. Also, our analysis is concerned with the interaction among the dwell times corresponding to different transitions. Regarding the latter, this means that large dwell times for one transition allows for small dwell times of subsequent transitions.
The problem we address in this paper is formalized in the following.

Problem 4 *Given a SLS $\mathcal{S} = (L, \Delta, E, \mathbb{R}^n, A)$, find conditions on the dwell time function Δ that ensures the asymptotic stability of $\mathcal{S}$.*

Since the dwell time induces guards only, it is easily seen that all locations should be Hurwitz.

Lemma 5 *A necessary condition for a SLS $\mathcal{S} = (L, \Delta, E, \mathbb{R}^n, A)$ to be asymptotically stable is that $A(\ell)$ is Hurwitz for any $\ell \in L$.*

In the sequel we therefore assume that SLSs under consideration satisfy the necessary condition above. Moreover given the topological properties of discrete variables acting on SLSs the following result holds.

Lemma 6 *An SLS $\mathcal{S}$ is asymptotically stable if and only if each strongly connected component of $\mathcal{S}$ is asymptotically stable.*

Let $\mathcal{S} = (L, \Delta, E, \mathbb{R}^n, A)$ be a SLS such that $A(\ell)$ is Hurwitz for any $\ell \in L$. Consider a family of symmetric and positive definite matrices $\{Q_\ell\}_{\ell \in L}$. There exists a unique family of symmetric and positive definite matrices $\{P_\ell\}_{\ell \in L}$ such that:

$$A^T(\ell)P_\ell + P_\ell A(\ell) = -Q_\ell, \quad \forall \ell \in L. \tag{1}$$

Functions $V_\ell(x) = x^T P_\ell x$ are Lyapunov functions for systems $\dot{x} = A(\ell)x$, $\ell \in L$. Given positive real numbers $c_\ell > 0$, $\ell \in L$, let us define the sets

$$\Omega(\ell, c_\ell) = \left\{ x \in \mathbb{R}^n \mid x^T P_\ell x \leq c_\ell \right\}.$$

By analyzing mutual position of sets $\Omega(\ell, c_\ell)$, in the continuous state space, a condition on the dwell time function may be found, ensuring asymptotic stability of the SLS under consideration.

Proposition 7 *Let $\mathcal{S} = (L, \Delta, E, \mathbb{R}^n, A)$ be a SLS such that $A(\ell)$ is Hurwitz for all $\ell \in L$. For any $e = (\ell_1, \sigma, \ell_2) \in E$ define*

$$\Delta_1(e) := \max\left(\delta_0, \max_{x_0 \in \Omega(\ell_1, c_{\ell_1})} \min\left\{ t \geq 0 : e^{A(\ell_1)t} x_0 \in \lambda_e\ \Omega(\ell_1, c_{\ell_1}) \right\}\right), \tag{2}$$

where $\lambda_e = \max\{\lambda > 0 : \lambda\ \Omega(\ell_1, c_{\ell_1}) \subset \Omega(\ell_2, c_{\ell_2})\}$. If $\Delta(e) \geq \Delta_1(e)$ for any $e \in E$, the SLS $\mathcal{S}$ is asymptotically stable.

Remark 8 *Sets $\Omega(\ell, c_\ell)$, $\ell \in L$ are domains of attraction [4], induced by quadratic Lyapunov functions. By replacing in Proposition 7 sets $\Omega(\ell, c_\ell)$, $\ell \in L$ with general domains of attraction, results still hold. However we restrict our attention to sets $\Omega(\ell, c_\ell)$, since they allow algebraic estimates of function $\Delta_1(.)$.*

The computation of (2) may be done numerically. In the following we derive an estimate of the function $\Delta_1(.)$ defined in (2), which can be easily computed. Given a square matrix M, $\sigma^+(M)$ and $\sigma^-(M)$ denote the largest and the smallest eigenvalue of M respectively. Finally for any $\ell \in L$, let us write $P_\ell = R_\ell R_\ell$ where R_ℓ is the unique positive definite symmetric square root of P_ℓ. The following result holds.

Proposition 9 *Let $\mathcal{S} = (L, \Delta, E, \mathbb{R}^n, A)$ be a SLS such that $A(\ell)$ is Hurwitz for any $\ell \in L$. Define for any $e = (\ell_1, \sigma, \ell_2) \in E$,*

$$\Delta_2(e) := \max\left(\delta_0, -\frac{}{}- \frac{\sigma^-(Q_{\ell_1})}{\sigma^+(P_{\ell_1})} \ln\left(\frac{c_{\ell_2}}{c_{\ell_1}}\sigma^-\left(R_{\ell_2}^{-1} P_{\ell_1} R_{\ell_2}^{-1}\right)\right)\right). \tag{3}$$

If $\Delta(e) \geq \Delta_2(e)$ for any $e \in E$, the SLS $\mathcal{S}$ is asymptotically stable.

Proof. Let λ_1 be defined as

$$\lambda_1 = \max\{\lambda \geq 0 \mid \lambda\Omega(\ell_1, c_{\ell_1}) \subset \Omega(\ell_2, c_{\ell_2})\}.$$

Let $x_0 \in \Omega(\ell_1, c_{\ell_1})$. We want to find an estimate for the smallest time instant τ_1 such that

$$\mathrm{e}^{A(\ell_1)t} x_0 \in \lambda_1 \Omega(\ell_1, c_{\ell_1}) \quad t \geq \tau_1.$$

Notice that $\mathrm{e}^{A(\ell_1)t} x_0 \in \lambda_1 \Omega_1$ if and only if

$$(\mathrm{e}^{A(\ell_1)t} x_0)^{\mathrm{T}} P_{\ell_1} \mathrm{e}^{A(\ell_1)t} x_0 \leq c_1 \lambda_1^2. \tag{4}$$

Define $x(t) = \mathrm{e}^{A(\ell_1)t} x_0$ and $z(t) = x(t)^T P_{\ell_1} x(t)$. By (1) if follows that

$$\begin{aligned} \frac{d}{dt} z(t) &= -x(t)^{\mathrm{T}} Q_{\ell_1} x(t) \\ &\leq -\sigma^{-1}(Q_{\ell_1}) x(t)^{\mathrm{T}} x(t) \\ &\leq -\underbrace{\frac{\sigma^-(Q_{\ell_1})}{\sigma^+(P_{\ell_1})}}_{\mu} z(t). \end{aligned}$$

It follows that

$$z(t) \leq \mathrm{e}^{-\mu t} z(0).$$

Now, take for $z(0) = c_1$, the 'worst' possible case and combining this with (4), then it follows that we can take

$$\tau_1 = -\frac{1}{\mu} \ln \lambda_1^2 = -\frac{\sigma^-(Q_{\ell_1})}{\sigma^+(P_{\ell_1})} \ln \lambda_1^2. \tag{5}$$

Finally, we derive an expression for λ_1^2. We want to find the largest λ_1 such that $x \in \lambda_1 \Omega(\ell_1, c_{\ell_1})$ implies $x \in \Omega(\ell_2, c_{\ell_2})$. In other words there should hold

$$x^{\mathrm{T}} P_{\ell_1} x \leq c_{\ell_1} \lambda_1^2 \Rightarrow x^{\mathrm{T}} P_{\ell_2} x \leq c_{\ell_2}.$$

To find λ_1 we observe that

$$\max_{x^{\mathrm{T}} P_{\ell_1} x \leq c_1 \lambda_1^2} x^{\mathrm{T}} P_{\ell_2} x = \max_{x^{\mathrm{T}} P_{\ell_1} x = c_1 \lambda_1^2} x^{\mathrm{T}} P_{\ell_2} x.$$

Decompose $P_2 = R_2 R_2$ where $R_2 = R_2^{\mathrm{T}} > 0$. Then

$$\max_{x^{\mathrm{T}} P_1 x = c_1 \lambda_1^2} x^{\mathrm{T}} P_2 x = \max_{y^{\mathrm{T}} R_2^{-1} P_1 R_2 y = c_1 \lambda_1^2} y^{\mathrm{T}} y = \frac{c_1 \lambda_1^2}{\sigma^-(R_2^{-1} P_1 R_2)}.$$

We want this maximum not exceed c_2 and therefore we take

$$\lambda_1^2 = \frac{c_{\ell_2}}{c_{\ell_1}} \sigma^- \left(R_{\ell_2}^{-1} P_{\ell_1} R_{\ell_2}^{-1} \right).$$

Substituting this in (5) yields

$$\tau_1 = - - \frac{\sigma^-(Q_{\ell_1})}{\sigma^+(P_{\ell_1})} \ln \left(\frac{c_{\ell_2}}{c_{\ell_1}} \sigma^- \left(R_{\ell_2}^{-1} P_{\ell_1} R_{\ell_2}^{-1} \right) \right).$$

Therefore by defining

$$\Delta_2(e) := \max \left(\delta_0, -\frac{\sigma^-(Q_{\ell_1})}{\sigma^+(P_{\ell_1})} \ln \left(\frac{c_{\ell_2}}{c_{\ell_1}} \sigma^- \left(R_{\ell_2}^{-1} P_{\ell_1} R_{\ell_2}^{-1} \right) \right) \right),$$

the statement follows. ■

It is easy to see that $\Delta_2(e) \geq \Delta_1(e)$ for all $e \in E$. Note that $\Delta_2(.)$ depends on matrices Q_{ℓ_1}, Q_{ℓ_2} and constants c_{ℓ_2} and c_{ℓ_1}.

Consider a SLS $\mathcal{S} = (L, \Delta, E, \mathbb{R}^n, A)$ and assume that there exists a symmetric and positive definite matrix $P = P^T > 0$ satisfying Lyapunov equations (1) for all $\ell \in L$ with $P_\ell = P$ and for some matrices Q_ℓ. In this case a common Lyapunov function $V(x) = x^T P x$ exists and the SLS is asymptotically stable for all dwell time functions Δ. This actually means that even if $\Delta(e) = \delta_0$ for all $e \in E$, the SLS $(L, \Delta, E, \mathbb{R}^n, A)$ is asymptotically stable. By considering the same case in the estimate (3), by choosing $c_{\ell_1} = c_{\ell_2}$, it is easily seen that $\Delta_2(e) = \delta_0$ for all $e \in E$: thus, estimate (3) of Δ and Δ coincide if a common quadratic Lyapunov function exists.

Propositions 7 and 9 provide sufficient conditions on dwell time function $\Delta(.)$ that guarantee asymptotic stability of SLSs. Then one might think to find the smallest values for $\Delta(.)$, ensuring asymptotic stability of the SLS under consideration; however an optimization problem is not well posed in this context since more than one function should be minimized. Indeed, the optimization problem can be formulated by defining a function of dwell times associated to any discrete transition. Let $\mathcal{F} : \left(\mathbb{R}_0^+\right)^{\mathrm{card}(E)} \to \mathbb{R}$ be a function associating to dwell times $\Delta(e_1), \Delta(e_2), ...$ of discrete transitions $e \in E$, a real number $\mathcal{F}(\Delta(e_1), \Delta(e_2), ...)$ and consider the following optimization problem:

$$\min_{\{\Omega(\ell, c_\ell)\}_{\ell \in L}} \mathcal{F}(\Delta(e_1), \Delta(e_2), ...) \tag{6}$$

where for any $\ell \in L$, $\Omega(\ell, c_\ell)$ depends itself on P_ℓ and c_ℓ. The existence of solution to Problem (6) actually depends on function $\mathcal{F}(\Delta(e_1), \Delta(e_2), ...)$. Further investigations will concentrate on studying conditions on $\mathcal{F}$, ensuring existence of the solution. Moreover some computable procedure solving Problem (6) will be investigated, taking particular attention to the rule of constants c_ℓ, $\ell \in L$ and matrices P_ℓ, $\ell \in L$.

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