CONTROL OF HYBRID BEHAVIORAL AUTOMATA BY INTERCONNECTION

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Abstract: In this paper we present a structure called *hybrid behavioral automata* (HBA) as a tool to model hybrid systems. One distinct feature of HBA is the classification of the transitions into *active* and *passive* ones. With such a classification, it is possible to model unidirectional discrete synchronizations between HBA. Interconnection operations for HBA, total and partial, are defined. In this paper we also pose a control problem for such structure. A solution and some sufficient conditions, under which the solution solves the problem, are presented.

Keywords: interconnection, behavioral approach, canonical controller

1. INTRODUCTION

In this paper we discuss the hybrid behavioral automata (HBA) model. In particular, we are interested in the interconnections of such structure and its relation with controller design problem.

One of the defining features of the HBA is the introduction of the passive transitions. With the passive transitions in the model, we can model uni-directional discrete synchronizations between automata, in contrast with the common bi-directional synchronizations commonly used in other models.

There are other models that capture the spirit of uni-directional synchronization, for example, I/O automata (Lynch and Tuttle 1989) and its extension, hybrid I/O automata (Lynch *et al.* 2001, Lynch *et al.* 2003). The differences between the these frameworks and our framework are that in our framework, synchronization with multiple active (output) agents is allowed and that *input enabledness* is embedded in the composition semantics rather than imposed on top of the structure as an assumption.

Formulation of control as interconnection has been recently advocated in behavioral system theory (Willems 1997, Polderman and Willems 1998). This had led to some fundamental results, generalizing classical results in the input-output framework; see (Trentelman and Willems 1999, Belur and Trentelman 2002) for the linear case and (van der Schaft and Julius 2002) for extensions to other system classes. The same paradigm also appeared in computer science, for example, in the submodule construction problem (Merlin and Bochmann 1983).

In this paper, a control problem is presented. A solution, derived from the construction of canonical controllers presented in (van der Schaft and Julius 2002) is derived. We also present and dis-

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cuss some conditions under which the proposed controller solves the problem.

The layout of the paper is as follows. In the next section we will present the structure of hybrid behavioral automata. Some notions of interconnection and projection of HBA are given in Section 3. In Section 4, we discuss the control problem and propose a solution, which is designed according to the canonical controller in (van der Schaft and Julius 2002). In Section 5, some concluding remarks are given, including some discussion on some potential directions for further research.

2. DEFINITIONS AND NOTATIONS

We begin by defining hybrid behavioral automata and their behaviors. First, note that throughout this paper we use a general totally ordered set $\mathbb{T}$ as the underlying time axis for the trajectories inside each location. Instances of $\mathbb{T}$ can be $\mathbb{R}$, $\mathbb{R}_+$ or other chosen sets. This time axis is not to be mistaken with the *hybrid time trajectory,* which is defined later.

A hybrid behavioral automaton (HBA) $\mathbb{A}$ is a septuple $(L, W, A, T, P, Inv, \mathfrak{B})$, where

- L is the set of locations or discrete states,
- W is the set of continuous variables taking values in $\mathbf{W}$,
- A is the set of labels,
- T is the set of active jumps/transitions. Each jump is given as a pentuple $(l, \mathbf{a}, l', G, R)$, where l is the origin location, $\mathbf{a}$ is the label of the jump, l' is the target location, $G := (\gamma, g)$ is the guard of the jump, where $\gamma : \mathfrak{B}(l) \times \mathbb{T} \rightarrow$codomain$(\gamma)$ and $g \subset$codomain(γ), and $R : \mathfrak{B}(l) \times \mathbb{T} \rightarrow 2^{\mathfrak{B}(l')}$ is the reset map of the jump.
- P is the set of passive jumps/transitions. We represent each passive jump as a quadruple $(l, \mathbf{a}, l', R)$. Passive jumps are not guarded.
- Inv maps each location $l \in L$ to a pair $Inv(l) := (\nu, V)$, where $\nu : \mathfrak{B}(l) \times \mathbb{T} \rightarrow$codomain$(\nu)$ and $V \subset$codomain(ν).
- $\mathfrak{B}$ maps each location to its continuous behavior. A behavior is a subset of $W^{\mathbb{T}}$.

The guard G and the invariant $Inv(l)$ as introduced above are instances of *dynamic predicates.* In general, a dynamic predicate is a pair $C := (\psi, \Psi)$,

$$\psi : \mathfrak{B} \times \mathbb{T} \rightarrow \text{codomain}(\psi),$$
$$\Psi \subset \text{codomain}(\psi).$$

$\mathfrak{B}$ signifies a behavior over the general (ordered) time axis $\mathbb{T}$. A pair $(w, t) \in \mathfrak{B} \times \mathbb{T}$ is said to satisfy the dynamic predicate C if $\psi(w, t) \in \Psi$. We denote it by $(w, t) \models C$. The negation of this statement is denoted by $(w, t) \not\models C$.

We assume that the maps γ, R, and ν are *causal.* A map $x : \mathfrak{B} \times \mathbb{R} \rightarrow X$ is causal if for any w_1 and w_2 in $\mathfrak{B}$ and $\tau \in \mathbb{R}$, the following implication holds.

$$\left(w_1(t)|_{t \leq \tau} = w_2(t)|_{t \leq \tau} \right) \Longrightarrow (x(w_1, t) = x(w_2, t)) .$$

In order to describe the evolution of such automaton, we need to define a suitable timeline structure. In this case, we use a slightly modified version of *hybrid time trajectory*, introduced in (Tomlin *et al.* 2000). The idea behind the hybrid time trajectory is as follows. We have continuously evolving dynamical system, but punctuated by jumps. Because of this, we choose a timeline that consists of intervals of $\mathbb{T}$. Each interval acts as a timeline for describing the evolution between jumps.

Definition 1. A hybrid time trajectory $\tau = \{I_i\}_{i=0}^{N}$ is a finite or infinite sequence of intervals of $\mathbb{T}$, such that:

- $I_0 = [\tau_0, \tau_0']$ or $(-\infty, \tau_0']$, $\tau_0 \leq \tau_0' \in \mathbb{T}$,
- $I_i = [\tau_i, \tau_i']$ for $i < N$ and, if $N < \infty$, $I_N = [\tau_N, \tau_N']$ or $I_N = [\tau_N, \tau_N')$,
- for all i, $\tau_i \leq \tau_i' = \tau_{i+1}$.

A hybrid time trajectory $\tau' = \{I_i'\}_{i=0}^{N'}$ is said to be a *prefix* of another time trajectory $\tau = \{I_i\}_{i=0}^{N}$ if $N' \leq N$ and $I_i' = I_i$ for all $0 \leq i \leq N'$. A hybrid time trajectory $\tau = \{I_i\}_{i=0}^{N}$ is said to be *infinite* if $N = \infty$ or $\tau_N' = \infty$.

In line with this timeline structure, we describe the evolution of an HBA. A hybrid trajectory is denoted as (τ, ξ). Here τ is a hybrid time trajectory and ξ maps the interval $\{I_n\}_{n \geq 0}$ in τ to a triple (l_n, w_n, j_n), where $l_n \in L$, $w_n : I_n \rightarrow \mathbf{W}$, and $j_n \in (T \cup P)$ or $j_n = \varnothing$. The case where $j_n = \varnothing$ may happen only on the last interval of τ.

A hybrid trajectory (τ', ξ') is said to be a *prefix* of another hybrid trajectory (τ, ξ) if τ' is a prefix of τ and $\xi' = \xi$ on τ'. A hybrid trajectory (τ, ξ) is called *infinite* if τ is infinite.

A hybrid trajectory (τ, ξ) is included in the *hybrid behavior* $\mathcal{A}$ of the automaton $\mathbb{A}$ if the following conditions are satisfied for all $n \geq 0$.

(1) $w_n \in \mathfrak{B}(l_n)$,
(2) $j_n = (l_n, \mathbf{a}, l_{n+1}, G_n, R_n)$, for some $\mathbf{a} \in A$,
(3) $j_n \in T$,
(4) $(w_n, \tau_n') \models G_n$,
(5) $\tau_n' \leq \inf\{t \mid t \geq \tau_n \text{ and } (w_n, t) \not\models Inv(l_n)\}$,
(6) $w_{n+1} \in R_n(w_n, \tau_n')$.

Such trajectory is called a *valid* trajectory. Notice that we do not have passive jumps in a valid trajectory.

A hybrid trajectory (τ,ξ) is included in the *potential behavior* $\bar{\mathcal{A}}$ if it satisfies conditions 1,2,4,5 and 6 above, together with a relaxed version of condition 3, $j_n \in (T\cup P)$. The intuitive idea is that we only include hybrid trajectories without any passive jumps in $\mathcal{A}$, while in $\bar{\mathcal{A}}$ we also allow passive jumps. Obviously $\mathcal{A}\subset\bar{\mathcal{A}}$. Notice that, by the way they are defined, $\mathcal{A}$ and $\bar{\mathcal{A}}$ are *prefix closed*. This means if a hybrid trajectory (τ,ξ) is in $\mathcal{A}$ (or $\bar{\mathcal{A}}$), then any of its prefixes (τ',ξ') is also in $\mathcal{A}$ (or $\bar{\mathcal{A}}$).

Throughout this paper we shall also use the following shorthand notation. We write $l \xrightarrow{\mathbf{a}} l'$ to denote the existence of an active transition going from location $l\in L$ to location $l'\in L$ with label $a\in A$. The existence of a passive transition with the same characteristics is denoted by $l \overset{\mathbf{a}}{\dashrightarrow} l'$. The notations $l\xrightarrow{\mathbf{a}}$ and $l\overset{\mathbf{a}}{\dashrightarrow}$ denote the existence of $l'\in L$ such that $l\xrightarrow{\mathbf{a}} l'$ and $l\overset{\mathbf{a}}{\dashrightarrow} l'$ respectively. The absence of such transitions are denoted as $l \overset{\mathbf{a}}{\not\to} l'$, $l \overset{\mathbf{a}}{\not\dashrightarrow} l'$, $l\overset{\mathbf{a}}{\not\to}$, and $l\overset{\mathbf{a}}{\not\dashrightarrow}$ respectively.

If the information about the guard and the reset map is also included in the notation, we write $l \xrightarrow{(\mathbf{a},G,R)} l'$ to denote $(l,\mathbf{a},l',G,R)\in T$, and $l \overset{(\mathbf{a},R)}{\dashrightarrow} l'$ if $(l,\mathbf{a},l',R)\in P$.

3. COMPOSITION AND PROJECTION OPERATORS

Two HBA, $\mathbb{A}_1$ and $\mathbb{A}_2$ characterized by $(L_i,W,A,T_i,P_i,Inv_i,\mathfrak{B}_i)$, $i=1,2$, can be interconnected to form another HBA $\mathbb{A}=\mathbb{A}_1\parallel\mathbb{A}_2$. The automaton $\mathbb{A}$ is characterized by the septuple $(L,W,A,T,P,Inv,\mathfrak{B})$, where

$$\begin{aligned} L &= L_1\times L_2,\\ \mathfrak{B}((l_1,l_2)) &= \mathfrak{B}_1(l_1)\cap\mathfrak{B}_2(l_2),\\ Inv((l_1,l_2)) &= (\nu,V), \end{aligned}$$

such that $Inv((l_1,l_2))$ is satisfied if and only if both $Inv_1(l_1)$ and $Inv_2(l_2)$ are satisfied.

The set T and P consist of pentuples and quadruples, $((l_1,l_2),\mathbf{a},(l_1',l_2'),G_T,R_T)$ and $((l_1,l_2),\mathbf{a},(l_1',l_2'),R_T)$ respectively, such that the following interconnection semantics are commutatively satisfied.

$$\frac{l_1 \xrightarrow{(\mathbf{a},G,R)} l_1',\ l_2 \overset{\mathbf{a}}{\not\dashrightarrow}}{(l_1,l_2)\xrightarrow{(\mathbf{a},G,R)}(l_1',l_2)} \quad \frac{l_1 \xrightarrow{(\mathbf{a},G,R_1)} l_1',\ l_2 \overset{(\mathbf{a},R_2)}{\dashrightarrow} l_2'}{(l_1,l_2)\xrightarrow{(\mathbf{a},G,R_1\cap R_2)}(l_1',l_2')}$$

$$\frac{l_1 \overset{\mathbf{a}}{\not\dashrightarrow},\ l_2 \overset{(\mathbf{a},R)}{\dashrightarrow} l_2'}{(l_1,l_2)\overset{(\mathbf{a},R)}{\dashrightarrow}(l_1,l_2')} \quad \frac{l_1 \overset{(\mathbf{a},R_1)}{\dashrightarrow} l_1',\ l_2 \overset{(\mathbf{a},R_2)}{\dashrightarrow} l_2'}{(l_1,l_2)\overset{(\mathbf{a},R_1\cap R_2)}{\dashrightarrow}(l_1',l_2')}$$

By taking intersection of reset maps, we mean intersecting their respective images.

The interconnection operation described here possesses some ideal properties that make it suitable for establishing modularity for hybrid behaviors, namely *commutativity* and *associativity* (van der Schaft and Schumacher 2001).

Notice that all (continuous) variables are involved in the synchronization. This type of interconnections is called *total interconnections*. It is also possible to define *partial interconnections*, where only a part of the variables are synchronized.

Two HBA, $\mathbb{A}_1$ and $\mathbb{A}_2$ characterized by $(L_i,W_i\cup Z,A,T_i,P_i,Inv_i,\mathfrak{B}_i)$, $i=1,2$, are partially interconnected over Z to form another HBA $\mathbb{A}=\mathbb{A}_1\parallel_Z\mathbb{A}_2$. The automaton $\mathbb{A}$ is characterized by the septuple $(L,W_1\cup W_2\cup Z,A,T,P,Inv,\mathfrak{B})$, where

$$\begin{aligned} L &= L_1\times L_2,\\ \mathfrak{B}((l_1,l_2)) &= \mathfrak{B}_1(l_1)\parallel_Z\mathfrak{B}_2(l_2),\\ Inv((l_1,l_2)) &= (\nu,V),\ \text{where} \end{aligned}$$

$Inv_i(l_i)=:(\nu_i,V_i),\ i=1,2,$
$\nu((w_1,w_2,z),t)=(\nu_1((w_1,z),t),\nu_2((w_2,z),t))$, and
$V=V_1\times V_2$.

The set T and P consist of pentuples and quadruples, $((l_1,l_2),\mathbf{a},(l_1',l_2'),G_T,R_T)$ and $((l_1,l_2),\mathbf{a},(l_1',l_2'),R_T)$ respectively, such that the following interconnection semantics are commutatively satisfied.

$$\frac{l_1 \xrightarrow{(\mathbf{a},G,R)} l_1',\ l_2 \overset{\mathbf{a}}{\not\dashrightarrow}}{(l_1,l_2)\xrightarrow{(\mathbf{a},G,R)}(l_1',l_2)} \quad \frac{l_1 \xrightarrow{(\mathbf{a},G,R_1)} l_1',\ l_2 \overset{(\mathbf{a},R_2)}{\dashrightarrow} l_2'}{(l_1,l_2)\xrightarrow{(\mathbf{a},G,R_1\parallel_Z R_2)}(l_1',l_2')}$$

$$\frac{l_1 \overset{\mathbf{a}}{\not\dashrightarrow},\ l_2 \overset{(\mathbf{a},R)}{\dashrightarrow} l_2'}{(l_1,l_2)\overset{(\mathbf{a},R)}{\dashrightarrow}(l_1,l_2')} \quad \frac{l_1 \overset{(\mathbf{a},R_1)}{\dashrightarrow} l_1',\ l_2 \overset{(\mathbf{a},R_2)}{\dashrightarrow} l_2'}{(l_1,l_2)\overset{(\mathbf{a},R_1\parallel_Z R_2)}{\dashrightarrow}(l_1',l_2')}$$

By performing partial interconnection on the reset maps we mean partially interconnecting their respective images in the behavioral sense [3].

Let $\mathbb{A}=(L,W\cup Z,A,T,P,Inv,\mathfrak{B})$. We can project the automaton to the set of variables W, written as $\pi_W\mathbb{A}$, by defining $\pi_W\mathbb{A}=(L,W,A,\pi_W T,\pi_W P,\pi_W Inv,\pi_W\mathfrak{B})$, where

$$\pi_W\mathfrak{B}:=\{w\mid\exists z \text{ such that } (w,z)\in\mathfrak{B}\}.$$

The projected set of active transitions $\pi_W T$ consists of pentuples $(l,\mathbf{a},l',\pi_W G,\pi_W R)$, with $(l,\mathbf{a},l',G,R)\in T$. The projected guard and reset map are defined as follows. For any $w\in\pi_W\mathfrak{B}(l)$ and $t\in\mathbb{T}$, the pair (w,t) satisfies $\pi_W G$ if and only if there is a $z\in Z^{\mathbb{T}}$ such that $(w,z)\in\mathfrak{B}(l)$ and (w,z,t) satisfies G. For any $w\in\pi_W\mathfrak{B}(l)$ and $t\in\mathbb{T}$, the trajectory $w'\in\mathfrak{B}'(l')$ is included in $R'(w,r)$ if and only if there are z and z' such that

$$\begin{aligned} (w,z)&\in\mathfrak{B}(l),\\ (w',z')&\in\mathfrak{B}(l'),\\ (w',z')&\in R(w,z,t). \end{aligned}$$

[3] $(R_1\parallel_Z R_2):=\{(w_1,w_2,z)\mid(w_1,z)\in R_1 \text{ and } (w_2,z)\in R_2\}$

The projected set of passive transitions $\pi_W P$ consists of quadruples $(l, \mathbf{a}, l', \pi_W R)$, with $(l, \mathbf{a}, l', R) \in P$. The projected invariant $\pi_W Inv$ is such that for any $l \in L$, $w \in \pi_W \mathfrak{B}(l)$ and $t \in \mathbb{R}$, the pair (w, t) satisfies $\pi_W Inv(l)$ if and only if there exists a z such that $(w, z) \in \mathfrak{B}(l)$ and (w, z, t) satisfies $Inv(l)$.

We also define another notion of projection, which we call *factorization*. The idea behind it is as follows.

Interconnecting two automata results in an automaton whose discrete dynamics is somewhat larger (i.e. more locations and more transitions) than those of the components. By factorizing the interconnected automata, we aim to see the 'effect' of the interconnection on the individual component.

Take two HBA $\mathbb{A}_i = (L_i, W, A, T_i, P_i, Inv_i, \mathfrak{B}_i)$, $i = 1, 2$. Let $\mathbb{A} = (L_1 \times L_2, W, A, T, P, Inv, \mathfrak{B}) = \mathbb{A}_1 \parallel \mathbb{A}_2$. Factorizing $\mathbb{A}$ with respect to its component $\mathbb{A}_1$, denoted as $\pi_{\mathbb{A}_1}\mathbb{A}$ can be done as follows. First, we construct the following equivalent relation. For any (l_{1i}, l_{2i}) and (l_{1j}, l_{2j}) in $L_1 \times L_2$,

$$(l_{1i}, l_{2i}) \approx (l_{1j}, l_{2j}) \text{ iff } (l_{1i} = l_{1j}).$$

Each equivalent class of $\approx$ represents a location in L_1 and is named accordingly.

The action of the factorization $\pi_{\mathbb{A}_1}$ to $\mathbb{A}$ results in $\pi_{\mathbb{A}_1}\mathbb{A} = (L_1, W, A, T', P', Inv', \mathfrak{B}')$, where

$$T' = \{(l_i, \mathbf{a}, l_j, G, R) \mid \exists l_i' \in l_i, l_j' \in l_j, \text{ such that } (l_i', \mathbf{a}, l_j', G, R) \in T\},$$

$$P' = \{(l_i, \mathbf{a}, l_j, R) \mid \exists l_i' \in l_i, l_j' \in l_j, \text{ such that } (l_i', \mathbf{a}, l_j', R) \in P\},$$

$$\mathfrak{B}'(l_i) = \bigcup_{l \in l_i} \mathfrak{B}(l),$$

and the invariant $Inv'(l_i)$ is such that any pair $(w, t) \in \mathfrak{B}'(l_i) \times \mathbb{T}$ satisfies it iff (w, t) satisfies at $Inv(l)$ for at least one $l \in l_i$.

Factorizing $\mathbb{A}$ with respect to $\mathbb{A}_1$ gives us information about the effect of the interconnection to $\mathbb{A}_1$. This is particularly useful when interconnection is seen as controlling (van der Schaft and Julius 2002). In this point of view, a plant model (in this case it is $\mathbb{A}_1$) and a desired specification $\mathbb{S}$ are given. The problem is to find a controller, in this case $\mathbb{A}_2$, such that $\pi_{\mathbb{A}_1}(\mathbb{A}_1 \parallel \mathbb{A}_2) = \mathbb{S}$. This formulation is very closely related to control as seen from behavioral point of view (Willems 1997).

4. THE CONTROL PROBLEM AND PASSIVE CANONICAL CONTROLLER

The behavioral approach to control theory sees control as interconnection. Given a plant (in term of behavior) $\mathcal{P}$ and a specification $\mathcal{S}$ (also in term of behavior), the problem of finding a controller that achieves the desired closed-loop behavior is translated to the problem of finding a controller behavior $\mathcal{C}$, such that $\mathcal{P} \parallel \mathcal{C} = \mathcal{S}$ (Willems 1997, van der Schaft and Julius 2002). The symbol $\parallel$ signifies behavior interconnection (Willems 1997, Polderman and Willems 1998). This formulation is very closely related to the submodule construction problem in computer science (Merlin and Bochmann 1983).

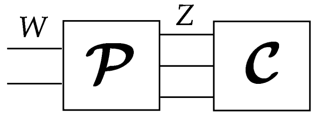

Fig. 1. Control with partial interconnection.

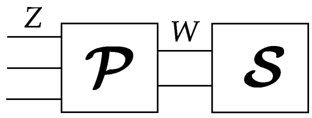

Fig. 2. The controller $\mathcal{C}_{\text{can}} = \pi_Z(\mathcal{P} \parallel_W \mathcal{S})$.

In most of the problems, however, not all variables of the plant are available for interconnection with the controller. Most problems deal with *partial interconnections*. This type of interconnection can be described as in Figure 1. In this figure, W represents the set of variables on which the specification is expressed, while Z represents those used in the interconnection with the controller. Note that these two sets are not necessarily disjunct. Partial interconnections are denoted by adding a subscript to the composition operator. Thus, the structure in Figure 1 is $\mathcal{P} \parallel_Z \mathcal{C}$.

Such problems for general behaviors have been treated in (van der Schaft and Julius 2002), where a construction for *canonical controllers* is given. This construction is shown in Figure 2. A canonical controller $\mathcal{C}_{\text{can}}$ constructed in this way solves the problem provided that the plant $\mathcal{P}$ and the specification $\mathcal{S}$ satisfy a couple of conditions, which can be thought of as some generalized controllability and observability conditions. In this case, we then have

$$\pi_W\left(\mathcal{P} \parallel_Z \mathcal{C}_{\text{can}}\right) = \mathcal{S}.$$

In the following, we introduce the notion of *rooted hybrid behavioral automata*. Simply speaking, the root of an automata is the location in which all trajectories are assumed to start. Thus, the root acts as discrete initial condition for the evolution. An HBA $\mathbb{A}$ that has a root l is denoted as $\mathbb{A}(l)$. To this rooted automaton we can associate a *rooted hybrid behavior* $\mathcal{A}(l)$, such that a hybrid trajectory $(\tau, \xi) \in \mathcal{A}(l)$ if $(\tau, \xi) \in \mathcal{A}$ and [4] $l_0 = l$.

[4] Recall that l_0 is the location of the first interval of τ.

The *rooted potential behavior* $\bar{\mathcal{A}}(l)$ is defined in a similar fashion.

We are going to treat the following control problem.

Problem. Given a plant in terms of a rooted HBA $\mathbb{P}(l) = (L, W \cup Z, A, T_p, \varnothing, Inv_p, \mathfrak{B}_p)$, and a specification $\mathbb{S}(l) = (L, W, A, T_s, \varnothing, Inv_s, \mathfrak{B}_s)$. Notice that we assume that both the plant and the specification do not have any passive transition. The problem is to find a controller $\mathbb{C}(l)$, which is also expressed in terms of rooted HBA, such that

$$(\pi_W \circ \pi_{\mathbb{P}}) (\mathcal{P} \parallel_Z \mathcal{C})(l, l) = \mathcal{S}(l).$$

The operators $\pi_{\mathbb{P}}$ and π_W denote factorization of the interconnected automaton with respect to $\mathbb{P}$ and projection of the continuous dynamics to the W variables.

A solution, which is adopted from (van der Schaft and Julius 2002), is proposed. The idea is as follows. We shall use a controller that has no active transitions. Such controller is called a *passive controller*. The controller has the same set of locations as the plant, and the set of passive transitions in the controller is the same as the set of active transitions in the plant less the information about the guard. The controller is a rooted HBA $\mathbb{C}(l) = (L, Z, A, \varnothing, P_c, Inv_c, \mathfrak{B}_c)$. Since we assume no active transitions in the controller, we also assume the invariant Inv_c is a dynamic predicate that is always satisfied. Moreover, the behavior $\mathfrak{B}_c$ is defined as follows.

$$\mathfrak{B}_c(l) = \pi_Z(\mathfrak{B}_p(l) \parallel_W \mathfrak{B}_s(l)).$$

Notice that this construction is similar to that in (van der Schaft and Julius 2002).

Theorem 2. The proposed controller $\mathbb{C}(l)$ solves the control problem,

$$(\pi_W \circ \pi_{\mathbb{P}}) (\mathcal{P} \parallel_Z \mathcal{C})(l, l) = \mathcal{S}(l),$$

if the following conditions hold.

(c1) For any $l \in L$, $\mathfrak{B}_s(l) \subset \pi_W (\mathfrak{B}_p(l))$.

(c2) For any $l \in L$ and any pairs $(w, z), (\tilde{w}, z) \in \mathfrak{B}_p(l)$, the following implication holds.

$$(w \in \mathfrak{B}_s(l)) \Rightarrow (\tilde{w} \in \mathfrak{B}_s(l))$$

(c3) The set of active transitions $T_s = \pi_W T_p$.

(c4) The invariant $Inv_s = \pi_W Inv_p$.

(c5) For any $l \in L$, $\mathbf{a} \in A$, there can be at most one transition in T_p that starts in location l with label $\mathbf{a}$.

(c6) For any $l \in L$, $t \in \mathbb{T}$, $(l, \mathbf{a}, l', G, R) \in T_p$ and any pairs $(w, z), (w, \tilde{z}) \in \mathfrak{B}_p(l)$, the following implications hold

$$(w, z, t) \models G \Leftrightarrow (w, \tilde{z}, t) \models G, \quad (2a)$$

$$(w, z, t) \models Inv_p(l) \Leftrightarrow (w, \tilde{z}, t) \models Inv_p(l), \quad (2b)$$

$$(w', z') \in R(w, z, t) \Leftrightarrow (w', \tilde{z}') \in R(w, z, t) \text{ for all } (w', z'), (w', \tilde{z}') \in \mathfrak{B}_p(l'). \quad (2c)$$

Proof. In this proof we shall denote $\mathbb{P} \parallel_Z \mathbb{C} := \mathbb{Q}$ for brevity. Consequently, its behavior is denoted as $\mathcal{Q}$. Assume that all the conditions above hold. Based on (c1) and (c2), we can infer [5]

$$\mathfrak{B}_s(l) = \pi_W (\mathfrak{B}_p(l) \parallel_Z \mathfrak{B}_c(l)), \forall l \in L. \quad (3)$$

Take any hybrid trajectory $(\tau, \xi) \in \mathcal{S}(l)$. We shall prove that $(\tau, \xi) \in (\pi_W \circ \pi_{\mathbb{P}}) \mathcal{Q}(l, l)$. Let $\tau = \{I_i\}_{i=0}^N$. Since $(\tau, \xi) \in \mathcal{S}(l)$, the following conditions must be satisfied.for all $N \geq n \geq 0$

$$l_0 = l, \quad (4a)$$

$$w_n \in \mathfrak{B}_s(l_n), \quad (4b)$$

$$j_n = (l_n, \mathbf{a}_n, l_{n+1}, G_n, R_n) \in T_s, \quad (4c)$$

$$(w_n, \tau_n') \models G_n, \quad (4d)$$

$$\tau_n' \leq \inf\{t \mid t \geq \tau_n \text{ and } (w_n, t) \not\models Inv_s(l_n)\}, \quad (4e)$$

$$w_{n+1} \in R_n(w_n, \tau_n'). \quad (4f)$$

We argue that there is a trajectory $(\tau, \tilde{\xi}) \in \mathcal{Q}(l, l)$ such that for all $N \geq n \geq 0$

$$\tilde{l}_0 = (l, l), \quad (5a)$$

$$\tilde{l}_n = (l_n, l_n), \quad (5b)$$

$$\tilde{w}_n = (w_n, z_n) \in \mathfrak{B}_p(l_n) \parallel_Z \mathfrak{B}_c(l_n), \quad (5c)$$

$$\tilde{j}_n = (\tilde{l}_n, \mathbf{a}_n, \tilde{l}_{n+1}, \tilde{G}_n, \tilde{R}_n) \in T_q, \text{ where} \quad (5d)$$

$$(l_n, \mathbf{a}_n, l_{n+1}, \tilde{G}_n, \tilde{R}_n) \in T_p, G_n = \pi_W \tilde{G}_n, R_n = \pi_W \tilde{R}_n, \quad (5e)$$

$$(\tilde{w}_n, \tau_n') \models \tilde{G}_n, \quad (5f)$$

$$\tau_n' \leq \inf\{t \mid t \geq \tau_n \text{ and } (\tilde{w}_n, t) \not\models Inv_p(l_n)\}, \quad (5g)$$

$$\tilde{w}_{n+1} = (w_{n+1}, z_{n+1}) \in \tilde{R}_n(w_n, z_n, \tau_n'). \quad (5h)$$

These relations are obtained by the following. Equation (5e) is obtained through (4c) and (c3), then (5d) is obtained using the definition of partial interconnection. This implies (5b). Equation (5c) is obtained using (3). Next, (5f) is implied by $G_n = \pi_W \tilde{G}_n$, while (5g) is implied by (c4) and (c6). Finally, (5h) is implied by (c3), and due to (2c) z_{n+1} is not restricted, since any z_{n+1} such that $(w_{n+1}, z_{n+1}) \in \mathfrak{B}_p(l_{n+1})$ will satisfy (5h). This guarantees that the whole argument can be established iteratively, starting with $n = 0$. Given the existence of such $(\tau, \tilde{\xi}) \in \mathcal{Q}(l, l)$, it follows that $(\tau, \xi) \in (\pi_W \circ \pi_{\mathbb{P}}) \mathcal{Q}(l, l)$. Hence we have that

$$\mathcal{S}(l) \subset (\pi_W \circ \pi_{\mathbb{P}}) \mathcal{Q}(l, l). \quad (6)$$

To show the converse, take any $(\tau, \xi) \in (\pi_W \circ \pi_{\mathbb{P}}) \mathcal{Q}(l, l)$. We shall show that $(\tau, \xi) \in \mathcal{S}(l)$. Again, we let $\tau = \{I_i\}_{j=0}^N$. Since $(\tau, \xi) \in (\pi_W \circ \pi_{\mathbb{P}}) \mathcal{Q}(l, l)$, there is a $(\tau, \tilde{\xi}) \in \mathcal{Q}(l, l)$ such that for all $N \geq n \geq 0$, (5a)-(5h) hold. The reasoning is as follows. The controller $\mathbb{C}(l)$ has only passive transitions, therefore any transition in $(\mathbb{P} \parallel_Z \mathbb{C}) (l, l)$ must be a synchronization between a passive transition of $\mathbb{C}$

[5] Please see (van der Schaft and Julius 2002) for a proof.

and an active transition of $\mathbb{P}$. Using (c5) and the definition of partial interconnection, we can infer (5b). This equation then implies (5c). Further, (5b) and the definition of π_W imply (5d), (5e), (5f), (5g), and (5h). Again, due to (2c) z_{n+1} is not restricted, therefore ensuring that the whole argument can be established iteratively. Now we shall prove that $(\tau, \xi) \in \mathcal{S}(l)$ by showing that (4a) - (4f) hold. First, (4b) is implied by (3), then (c3) and (5e) imply (4c), (4d) and (4f). Finally, (4e) is implied by (c4). Now, we have established that

$$\mathcal{S}(l) \supset (\pi_W \circ \pi_{\mathbb{P}})\, \mathcal{Q}(l, l), \tag{7}$$

and hence completed the proof.

Let us now discuss the conditions posed in the theorem, i.e. (c1) - (c6). The first two conditions arise as sufficient conditions to guarantee that we can establish (3). These conditions are adopted from (van der Schaft and Julius 2002). In the same reference, some variants of the conditions are also presented. Conditions (c3) - (c5) are essential, because of the passive nature of the controller. Since the controller only has passive transitions, it cannot be used to influence the active transitions in the plant, hence (c3) and (c4). Condition (c5) is there to guarantee that the location of the controller can accurately follow that of the plant, without any possibility of being misleaded due to some nondeterminism. Finally, condition (c6) is there to guarantee that the choice for z_n does not influence the choice for z_{n+1}. As explained above, this in turn guarantees that the whole reasoning can be done iteratively (interval wise).

5. CONCLUDING REMARKS

In this paper we present a formalism for modelling hybrid systems, namely the hybrid behavioral automata. We also discuss the interconnection properties of such structure, together with the important notions of projections and partial interconnection. These notions play an important role when we discuss control problems.

We have proposed a solution for the control problem posed in the previous section. Sufficient conditions, under which the proposed controller solves the problem, are also given. It is interesting to notice that the proposed controller so far, does not affect the discrete dynamics of plant. The controller is passive, and hence any (discrete) synchronization with the plant is done unidirectionally. The design of a controller whose discrete dynamics interact with the plant as well as its continuous part is an interesting topic for future research.

Another important point to realize is that so far we do not consider the issue of compatibility for the interconnections. The concept of compatibility in behavioral system theory is treated for example in (Willems 1997) where the notions of regular and regular feedback interconnections are introduced. A discussion about the notion of compatibility for general behavior interconnection can be seen in (Julius and van der Schaft 2003). Incorporating this idea in the controller design problem will give rise to a problem of compatible controller design. The solution to this problem will be a controller whose interconnection with the plant is realizable in a strict sense. We also identify this topic as a potentially fruitful research direction.

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