Cautious hierarchical switching control of stochastic linear systems

M.C. Campi[1,*], João P. Hespanha[2], and Maria Prandini[3]

[1] *Dipartimento di Elettronica per l'Automazione, Università degli Studi di Brescia, Italy*
[2] *Department of Electrical and Computer Engineering, University of California, Santa Barbara, USA*
[3] *Dipartimento di Elettronica e Informazione, Politecnico di Milano, Italy*

SUMMARY

Standard switching control methods are based on the certainty equivalence philosophy in that, at each switching time, the supervisor selects the candidate controller that is better tuned to the currently estimated process model. In this paper, we propose a new supervisory switching logic that takes into account the uncertainty on the process description when performing the controller selection. Specifically, a probability measure describing the likelihood of the different models is computed on-line based on the collected data and, at each switching time, the supervisor selects the candidate controller that, according to this probability measure, performs the best on the average. If the candidate controller class is hierarchically structured so that for each model one has available several controllers with distinct levels of robustness, the supervisor automatically selects the controller that suitably compromises robustness versus performance, given the current level of model uncertainty. The use of randomized algorithms makes the supervisor implementation computationally tractable.

KEY WORDS: Switching control; cautious control; randomized methods; stochastic systems; stability.

1. Introduction

Suppose that a process with transfer function G° has to be regulated by choosing a controller in some candidate controller class $\{K(\gamma), \gamma \in \Gamma\}$. In a standard optimal control setting, the control performance achieved by applying controller $K(\gamma)$ to the process G° is measured by some (positive) cost criterion $J(G^\circ; K(\gamma))$: the lower the value of $J(G^\circ; K(\gamma))$, the more satisfactory the control performance. Here, J can represent any cost, e.g., of the H_2 or H_∞ type. If the process is known, an optimal controller is computed by minimizing J over the candidate controller class.

Consider now the case when the process is not known, and suppose that a parametric class of admissible process models is introduced, $\{G(\vartheta), \vartheta \in \Theta\}$. Then, the problem of selecting the best controller according to J can be addressed along a dual control approach by introducing a state variable representing the unknown parameter vector, and solving the resulting optimal control problem on the augmented state-space representation of the process. The optimal controller incorporates a self-adjusting mechanism, in that it selects a control input that compromises the control objective versus estimation needs (dual action, see e.g. [1]). However, such an optimal dual control approach is generally difficult to implement because it is computationally excessive.

A computationally feasible – though sub-optimal – approach to the design of self-adjusting controllers is the so-called switching control design method originally introduced in [2] and further developed in e.g. [3]- [7]. The switching control scheme consists of an inner loop where a candidate controller is connected in closed-loop with the process, and an outer loop where a supervisor decides which controller to select and when to switch to a different one, based on the input-output data.

The switching times are chosen so as to avoid switching that is too fast with respect to the system's settling time, thus causing instability. As for the controller selection, it is typically based on an "estimator-based" procedure ([3, 4]). Specifically, at any switching time, a performance signal – given, e.g., by the integral norm of an estimation error – is computed for each admissible model parameter. The supervisor then selects the candidate controller associated with the model that minimizes the performance signal (certainty equivalence approach). Implementation and analysis of the switching control scheme are typically simplified by considering a finite set of candidate controllers. This set is called a *finite controller cover* ([8, 9]).

In a standard switching control scheme, the compromise between robustness and performance is made offline when the controller cover is designed. If the controller cover consists of a small number of controllers, each one stabilizing a wide set of models, then stability is generally rapidly achieved, even before a large amount of information has been accrued, but in the long run the resulting performance is typically low. In contrast, if the controller cover consists of a large number of controllers, each one tailored to a narrow set of models, a highly performing control system is potentially achieved, but poor performance will most likely occur until there is sufficient data to obtain an accurate estimate of the process model.

In this paper, we propose a *cautious switching logic* that still relies on a parameterized class of admissible process models but, differently from the certainty equivalence-based logic, also takes into account the uncertainty in the process description when performing the controller selection.

The controller choice is based on a probability measure $\mathcal{P}_t$ computed on-line, which describes the likelihood of the different process models. At any switching time t, the supervisor selects the controller that minimizes the average control cost $c_t(\gamma) := \mathrm{E}_{\mathcal{P}_t}[J(\vartheta,\gamma)]$, $\gamma \in \Gamma$, where $J(\vartheta,\gamma)$ is the short-hand-notation for $J(G(\vartheta);K(\gamma))$ and $\mathrm{E}_{\mathcal{P}_t}[J(\vartheta,\gamma)]$ is the expectation of $J(\vartheta,\gamma)$ with respect to the measure $\mathcal{P}_t$ for ϑ. Minimizing $c_t(\cdot)$ corresponds to optimizing the average control system behavior where different models are given different weights according to their likelihood at time t (cautious control, [10], [1, page 438]).

With cautious switching, we overcome the difficulty in standard switching control that arises from being forced to establish an a-priori compromise between robust stability and performance by associating to each model a single candidate controller. Thanks to the cautious switching logic, this association is no longer necessary. To be specific, we propose to integrate in the cautious switching scheme a hierarchically-structured class of candidate controllers composed of different finite controller covers: the lower level cover contains controllers with a high level of robustness (with respect to uncertainty in the system parameter ϑ), but low performance guarantees, and, as we go up in the hierarchical structure, we have controller covers with increasing performance, while progressively penalizing robustness. When the distribution $\mathcal{P}_t$ is spread over the set Θ, it is expected that the cautious supervisor will select a controller that is robust, though low performing. As time goes by, more and more information is accumulated and the distribution $\mathcal{P}_t$ is expected to become more sharply peaked around the model that better describes the actual process. Consequently, the cautious supervisor will select controllers better tailored to the true process, ultimately resulting in an improvement of performance. Thus, in finite time the control scheme is robust, and it progressively becomes better performing.

The use of average control cost criteria was originally proposed in [11, 12] in the context of robust control and then extended to an adaptive scenario in [13]. In [13], a general adaptive control set-up is considered and asymptotic tuning properties are proved under certain stability conditions. No stability analysis is performed however. The contribution of this paper is twofold:

i) proposing a hierarchical structure within the framework of switching control;

ii) providing a stability analysis for the corresponding cautious scheme.

In the light of the stability analysis developed in the present paper, the results in [13] can be used to further prove tuning properties of the control scheme proposed herein.

The paper is structured as follows. The cautious switching scheme is described in Section 2. The stability analysis is dealt with in Section 3. Section 4 concludes the paper with final remarks and a discussion of open issues.

2. The cautious switching scheme

Process: Consider the stochastic linear process

$$\mathcal{A}(\vartheta^\circ, z^{-1})y_{t+1} = \mathcal{B}(\vartheta^\circ, z^{-1})u_t + w_{t+1}, \tag{1}$$

where the polynomials $\mathcal{A}(\vartheta^\circ, z^{-1}) = 1 - \sum_{i=1}^{n_p} a_i^\circ z^{-i}$ and $\mathcal{B}(\vartheta^\circ, z^{-1}) = \sum_{i=1}^{m_p} b_i^\circ z^{-(i-1)}$ depend on the *unknown* parameter vector $\vartheta^\circ = [a_1^\circ, \ldots, a_{n_p}^\circ,\ b_1^\circ, \ldots, b_{m_p}^\circ]^T$ with $n_p, m_p > 0$, and $\{w_t\}$ is a sequence of independent and identically distributed Gaussian random variables with zero mean and variance $\sigma^2 > 0$.

We suppose that some a-priori knowledge on ϑ° is available. Specifically, we assume that:

Assumption 1. *ϑ° is an interior point of a known compact set $\Theta \subset \mathbb{R}^{n_p+m_p}$, such that for all $\vartheta \in \Theta$ the system (1) with ϑ in place of ϑ° is λ-stabilizable ($0 < \lambda < 1$), i.e. $\mathcal{A}(\vartheta^\circ, z^{-1})$ and $\mathcal{B}(\vartheta^\circ, z^{-1})$ have no pole-zero cancellations in $\{z \in \mathbb{C} : |z| > \lambda\}$.*

Model class: The model class is obtained from (1) by replacing ϑ° with a generic parameter $\vartheta \in \Theta$:

$$\mathcal{A}(\vartheta, z^{-1})y_{t+1} = \mathcal{B}(\vartheta, z^{-1})u_t + w_{t+1}, \quad \vartheta \in \Theta. \tag{2}$$

Candidate controller class: As for the controller class, we consider a finite set of candidate controllers with structure

$$\mathcal{C}(\gamma, z^{-1})u_t = \mathcal{D}(\gamma, z^{-1})y_t,$$

where the polynomials $\mathcal{C}(\gamma, z^{-1}) = 1 - \sum_{i=1}^{m_c} c_i z^{-i}$ and $\mathcal{D}(\gamma, z^{-1}) = \sum_{i=0}^{n_c} d_i z^{-i}$ are parameterized by $\gamma = [d_0, \ldots, d_{n_c}, c_1, \ldots, c_{m_c}]^T \in \Gamma \subseteq \mathbb{R}^{n_c+m_c+1}$. We assume that to each

$\vartheta \in \Theta$ there corresponds at least one $\gamma \in \Gamma$ such that the closed-loop system formed by the model with parameter ϑ and the controller with parameter γ is λ-stable, namely all the closed-loop eigenvalues are less than λ in absolute value. The way such a controller class is generated is immaterial for the analysis to follow. However, one can think that it has been formed by putting together all controllers belonging to a hierarchical structure of covers as discussed in the introduction. Details on how such structure can be constructed in a specific context is beyond the scope of the present paper and the reader is referred to [8, 9] for more discussion.

Control cost criterion: The control cost of the closed-loop system formed by the model with parameter ϑ and the controller with parameter γ is evaluated by

$$J(\vartheta, \gamma) := \begin{cases} \dfrac{\alpha J'(\vartheta,\gamma)}{1+\alpha J'(\vartheta,\gamma)} & \text{if the closed-loop system is } \lambda\text{-stable} \\ 1 & \text{otherwise,} \end{cases} \tag{3}$$

where $J'(\vartheta, \gamma)$ is some positive performance criterion (e.g. an H_2 or H_∞ cost), and α is a positive constant. The criterion J thus combines both stability and performance requirements and penalizes those controllers unable to meet the robust λ-stability requirement. J is normalized so that it takes values in $[0, 1]$. This is done for technical reasons related to the cautious controller selection through randomized methods as discussed in Section 2.1.

Switching logic: The tasks of the switching logic are to generate a switching signal that triggers the replacement of the controller in the loop with a new controller and to select the new controller. The controller selection is described in detail in Section 2.1. As for the switching signal, we adopt the so-called *dwell-time switching logic* where a dwell-time is forced between consecutive switching instants ([4, 14, 15, 16]). The actual dwell-time selection is discussed in Section 2.2.

2.1. Cautious controller selection

At each switching time t, the supervisor selects the next candidate controller by minimizing the average cost $c_t(\gamma) = \mathrm{E}_{\mathcal{P}_t}[J(\vartheta, \gamma)]$ over the controller parameter set Γ. The controller selection procedure involves two steps: i) computing the probability measure $\mathcal{P}_t$, and ii) computing and minimizing the average cost $c_t(\cdot)$. These two steps are discussed separately.

Computing and minimizing $c_t(\cdot)$: An exact computation of $c_t(\cdot)$ (the integral of $J(\vartheta, \gamma)$ over Θ with respect to measure $\mathcal{P}_t$) is in general hard. As a matter of fact, for many control objectives the integrand function $J(\vartheta, \gamma)$ cannot be computed in a closed-form, so that even the evaluation of $J(\vartheta, \gamma)$ for a given pair (ϑ, γ) may be time consuming. Additional difficulties arise in connection with the computation of the integral. The approach adopted here to overcome this difficulty follows to a large extent the ideas in [12, 13] and is based on the use of randomized methods. The resulting minimizer is only approximately optimal.

First we state a procedure that describes how $c_t(\cdot)$ is minimized along the randomized approach. The properties of the so-obtained minimizer are discussed in turn.

Algorithm 1. *Given $\epsilon \in (0, 1)$ and $\delta \in (0, 1)$, do the following:*

1. *extract at random* $M(\epsilon,\delta) \geq \frac{2}{\epsilon^2}\ln\frac{2|\Gamma|}{\delta}$ *independent model parameters* $\vartheta_{1,t}$, $\vartheta_{2,t}, \ldots, \vartheta_{M(\epsilon,\delta),t}$ *according to the probability distribution* $\mathcal{P}_t$*;*
2. *compute* $\hat{\mathrm{E}}_{\mathcal{P}_t}[J(\vartheta,\gamma)] := \dfrac{1}{M(\epsilon,\delta)} \displaystyle\sum_{i=1}^{M(\epsilon,\delta)} J(\vartheta_{i,t},\gamma)$*;*
3. *choose* $\gamma_t := \arg\min_{\gamma\in\Gamma} \hat{\mathrm{E}}_{\mathcal{P}_t}[J(\vartheta,\gamma)]$.

We prove next that the controller parameter γ_t obtained through Algorithm 1 is an approximate minimizer of $\mathrm{E}_{\mathcal{P}_t}[J(\vartheta,\gamma)]$ over Γ. In the following proposition, the phrase "with probability not less than $1-\delta$" makes reference to the probability involved in the random extractions of $\vartheta_{i,t}$'s, once the past up to time t has been fixed.

Proposition 1. *The controller parameter* γ_t *computed via Algorithm 1 is an approximate minimizer of* $\mathrm{E}_{\mathcal{P}_t}[J(\vartheta,\gamma)]$ *to accuracy* ϵ *with confidence* $1-\delta$*, i.e.,* $\mathrm{E}_{\mathcal{P}_t}[J(\vartheta,\gamma_t)] \leq \min_{\gamma\in\Gamma}\mathrm{E}_{\mathcal{P}_t}[J(\vartheta,\gamma)]+\epsilon$*, with probability not less than* $1-\delta$*.*

Proof. γ_t is the minimizer of the sampling estimate $\hat{\mathrm{E}}_{\mathcal{P}_t}[J(\vartheta,\gamma)]$, which is based on a random selection of parameters $\vartheta_{i,t} \in \Theta$ and, as such, it is a random variable over the space $\Theta^{M(\epsilon,\delta)} := \Theta \times \Theta \times \ldots \times \Theta$, $M(\epsilon,\delta)$ times. Consider the set Q_t of multi-samples $\xi = (\vartheta_1,\ldots,\vartheta_{M(\epsilon,\delta)}) \in \Theta^{M(\epsilon,\delta)}$ such that $\hat{\mathrm{E}}_{\mathcal{P}_t}[J(\vartheta,\gamma)] = \frac{1}{M(\epsilon,\delta)}\sum_{i=1}^{M(\epsilon,\delta)} J(\vartheta_i,\gamma)$ is an uniformly good approximation to $\mathrm{E}_{\mathcal{P}_t}[J(\vartheta,\gamma)]$ over the set Γ to accuracy $\epsilon/2$, namely

$$Q_t := \big\{\xi \in \Theta^{M(\epsilon,\delta)} : \max_{\gamma\in\Gamma} |\hat{\mathrm{E}}_{\mathcal{P}_t}[J(\vartheta,\gamma)] - \mathrm{E}_{\mathcal{P}_t}[J(\vartheta,\gamma)]| \leq \frac{\epsilon}{2}\big\}.$$

Then, if $\xi \in Q_t$, letting $\gamma_t := \arg\min_{\gamma\in\Gamma}\hat{\mathrm{E}}_{\mathcal{P}_t}[J(\vartheta,\gamma)]$, and $\gamma_t^\circ := \arg\min_{\gamma\in\Gamma}\mathrm{E}_{\mathcal{P}_t}[J(\vartheta,\gamma)]$, we have

$$\begin{aligned}\mathrm{E}_{\mathcal{P}_t}[J(\vartheta,\gamma_t)] &\leq \hat{\mathrm{E}}_{\mathcal{P}_t}[J(\vartheta,\gamma_t)] + \frac{\epsilon}{2} \leq \hat{\mathrm{E}}_{\mathcal{P}_t}[J(\vartheta,\gamma_t^\circ)] + \frac{\epsilon}{2}\\ &\leq \mathrm{E}_{\mathcal{P}_t}[J(\vartheta,\gamma_t^\circ)] + \frac{\epsilon}{2} + \frac{\epsilon}{2} = \min_{\gamma\in\Gamma}\mathrm{E}_{\mathcal{P}_t}[J(\vartheta,\gamma)] + \epsilon. \end{aligned} \tag{4}$$

The proof is concluded by showing that $\xi_t = (\vartheta_{1,t},\ldots,\vartheta_{M(\epsilon,\delta),t})$ belongs to Q_t with probability no smaller than $1-\delta$.

Since the parameters $\vartheta_{1,t},\ldots,\vartheta_{M(\epsilon,\delta),t}$ are independently extracted according to $\mathcal{P}_t$, an application of Hoeffding's inequality ([23]) yields

$$\begin{aligned}\Pr\{\xi_t \in Q_t\} &= 1 - \mathcal{P}_t^{M(\epsilon,\delta)}\big\{\xi_t \in \Theta^{M(\epsilon,\delta)} : \max_{\gamma\in\Gamma}|\hat{\mathrm{E}}_{\mathcal{P}_t}[J(\vartheta,\gamma)] - \mathrm{E}_{\mathcal{P}_t}[J(\vartheta,\gamma)]| > \frac{\epsilon}{2}\big\}\\ &\geq 1 - 2|\Gamma|e^{-M(\epsilon,\delta)\epsilon^2/2}.\end{aligned}$$

Recalling that $M(\epsilon,\delta) \geq \frac{2}{\epsilon^2}\ln\frac{2|\Gamma|}{\delta}$, it is straightforward to conclude that $\Pr\{\xi_t \in Q_t\} \geq 1-\delta$. ■

Computing $\mathcal{P}_t$: In general, $\mathcal{P}_t$ describes the likelihood of different parameters in Θ. Depending on the situation at hand, $\mathcal{P}_t$ can be given different mathematical formalizations. Here, we discuss in detail the case in which $\mathcal{P}_t$ is the exact a-posteriori probability density of ϑ° given data, where ϑ° is assumed to be a stochastic variable.

It is important to emphasize that regarding ϑ° as a random variable simply allows us to motivate the algorithm used to compute $\mathcal{P}_t$. Once this algorithm had been derived, the stochastic nature of ϑ° is immaterial and, in fact, the stability proof in Section 3 will be valid for any value of ϑ° in Θ (regardless of whether or not there is an underlying distribution).

Assume then that ϑ° is stochastic and randomly chosen according to a distribution $\mathcal{P}$. Having a stochastic ϑ° is a common assumption in adaptive control and is known under the name of Bayesian embedding ([17]-[22]). For instance, in [19]-[21] ϑ° is supposed to be Gaussian and independent of $\{w_t\}$. Then, under the assumption that u_t is a Borel measurable function of the observations $y_i, i \leq t$, the a-posteriori distribution $\mathcal{P}_t$ of ϑ° given the observations up to time t is still Gaussian with mean and variance that can be computed using the Kalman filter equations ([22]).

As in [19]-[21], we assume that ϑ° is independent of the noise process $\{w_t\}$. However, here ϑ° takes values in the compact set Θ according to a Gaussian distribution truncated to Θ. Specifically, we set $\mathcal{P} \sim \mathcal{N}_\Theta(M, V)$, where $\mathcal{N}_\Theta(M, V)$ denotes the rescaled Gaussian probability density with mean M and variance $V > 0$, whose support is restricted to the set Θ. Thus, our framework differs from the standard Bayesian embedding setting of [22] in two respects: (1) the selection of u_t ultimately depends on the randomized Algorithm 1 and, as a consequence, the assumption made in [22] that u_t is a Borel measurable function of the observations y_i, $i \leq t$, is no longer satisfied; and (2) we do not assume that ϑ° is Gaussian, but a truncated Gaussian.

Due to the above two differences, the results in [22] are not directly applicable to our context. On the other hand, as for (1), it is just a matter of technical details to show that the results in [22] extend to our context, provided that $\mathcal{P}_t$ is interpreted as the a-posteriori distribution of ϑ° given the observations y_i, $i \leq t$, and the parameters $(\vartheta_{1,i}, \ldots, \vartheta_{M(\epsilon,\delta),i})$, $i \leq t$, extracted at step 1 of Algorithm 1. As for difference (2), we can take advantage of the special structure of $\mathcal{P} \sim \mathcal{N}_\Theta(M, V)$, to derive a recursive expression for $\mathcal{P}_t$ still based on the Kalman filter equations. Indeed, the fact that $\vartheta^\circ \in \Theta$ can be recast as an additional observation. When computing the a-posteriori distribution of ϑ°, one can first compute it as though ϑ° were not restricted to Θ (and therefore use a Kalman filter), and then complement the resulting distribution with the additional observation that $\vartheta^\circ \in \Theta$. This final step simply amounts to truncating (and rescaling) the distribution achieved through the Kalman filter. This leads to the following algorithm.

Algorithm 2.

1. *compute M_t and V_t through the Kalman filter equations*

$$\begin{aligned}
K_{t-1} &= V_{t-1}\varphi_{t-1}/(\varphi_{t-1}^T V_{t-1}\varphi_{t-1} + \sigma^2) \\
M_t &= M_{t-1} + K_{t-1}(y_t - \varphi_{t-1}^T M_{t-1}) \\
V_t &= V_{t-1} - V_{t-1}\varphi_{t-1}\varphi_{t-1}^T V_{t-1}/(\varphi_{t-1}^T V_{t-1}\varphi_{t-1} + \sigma^2),
\end{aligned}$$

   initialized with $M_0 = M$ and $V_0 = V$;
2. *set $\mathcal{P}_t \sim \mathcal{N}_\Theta(M_t, V_t)$.*

In the algorithm $\varphi_t = [\, y_t,\ \ldots,\ y_{t-n_p+1},\ u_t,\ \ldots,\ u_{t-m_p+1}\,]^T$ is the regression vector associated with the model (2) rewritten as

$$y_{t+1} = \varphi_t^T \vartheta + w_{t+1}.$$

2.2. Dwell-time selection

We adaptively select the dwell-time interval between consecutive switchings based on the model parameters extracted in step 1 of Algorithm 1. Specifically, we compute the switching time sequence $\{t_i\}$ by the recursive equation

$$t_{i+1} = t_i + \tau_D(\xi_{t_i}, \gamma_{t_i}), \quad i = 0, 1, \ldots \tag{5}$$

initialized with $t_0 = 0$, where $\tau_D:\ \Theta^{M(\epsilon,\delta)} \times \Gamma \to \mathbb{N}$ is the *dwell-time function*, and $\xi_{t_i} = (\vartheta_{1,t_i}, \ldots, \vartheta_{M(\epsilon,\delta),t_i})$ is the set of $M(\epsilon,\delta)$ model parameters extracted at step 1 of Algorithm 1 at time $t = t_i$.

To define the dwell-time function, we first introduce some notation. Consider the closed-loop system

$$\begin{cases} \mathcal{A}(\vartheta, z^{-1})\, y_{t+1} = \mathcal{B}(\vartheta, z^{-1})\, u_t + w_{t+1} \\ \mathcal{C}(\gamma, z^{-1}) u_t = \mathcal{D}(\gamma, z^{-1}) y_t. \end{cases} \tag{6}$$

By letting $x_t := [y_t \ldots y_{t-(n-1)}\, u_{t-1} \ldots u_{t-(m-1)}]^T$ where $n := \max\{n_p, n_c+1\}$ and $m := \max\{m_p, m_c+1\}$, system (6) can be rewritten as

$$\begin{cases} x_{t+1} = A(\vartheta)\, x_t + B(\vartheta) u_t + C w_{t+1}, \\ u_t = L(\gamma) x_t, \end{cases}$$

where

$$A(\vartheta) = \left[\begin{array}{cccc|cccc} a_1 & \ldots & a_{n-1} & a_n & b_2 & \ldots & b_{m-1} & b_m \\ 1 & 0 & \ldots & & 0 & \ldots & & 0 \\ & \ddots & \ddots & & & \ddots & & 0 \\ & & 1 & 0 & & & & 0 \\ \hline 0 & \ldots & \ldots & 0 & 0 & \ldots & \ldots & 0 \\ 0 & \ldots & \ldots & 0 & 1 & 0 & & \\ & \ddots & \ddots & & & \ddots & \ddots & \\ & & 0 & 0 & & & 1 & 0 \end{array}\right],\ B(\vartheta) = \left[\begin{array}{c} b_1 \\ 0 \\ \vdots \\ 0 \\ \hline 1 \\ 0 \\ \vdots \\ 0 \end{array}\right],\ C = \left[\begin{array}{c} 1 \\ 0 \\ \vdots \\ 0 \\ \hline 0 \\ 0 \\ \vdots \\ 0 \end{array}\right],$$

$$L(\gamma) = \left[\begin{array}{cccc|cccc} d_0 & \cdots & d_{n-2} & d_{n-1} & c_1 & \cdots & c_{m-2} & c_{m-1} \end{array}\right],$$

with $a_i = 0$ if $i > n_p$, $d_i = 0$ if $i > n_c$, $b_i = 0$ if $i > m_p$, $c_i = 0$ if $i > m_c$, thus leading to the state-space representation $x_{t+1} = F(\vartheta, \gamma)\, x_t + C w_{t+1}$, where $F(\vartheta, \gamma) = A(\vartheta) + B(\vartheta) L(\gamma)$.

Fix now a *contraction constant* $\mu \in (0, 1)$. Then, $\tau_D(\xi_{t_i}, \gamma_{t_i})$ is defined as

$$\tau_D(\xi_{t_i}, \gamma_{t_i}) = \min\left\{k \geq 1 :\ \|F(\vartheta_{j,t_i}, \gamma_{t_i})^k\| \leq \mu \text{ for at least one } j \in \{1, \ldots, M(\epsilon,\delta)\}\right\}. \tag{7}$$

It is a fact proven in Theorem 1 that such ϑ_{j,t_i} actually exists. The intuitive idea is that the controller with parameter γ_{t_i} is kept fixed until the k-step transition matrix $F(\vartheta, \gamma_{t_i})^k$ becomes a contraction at least for one selected model parameter.

3. Stability Analysis

In this section, we analyze the cautious switching control scheme:

$$\begin{cases} y_{t+1} = [1 - \mathcal{A}(\vartheta^\circ, z^{-1})]\, y_{t+1} + \mathcal{B}(\vartheta^\circ, z^{-1})\, u_t + w_{t+1} \\ u_t = \mathcal{D}(\gamma_t, z^{-1})\, y_t + [1 - \mathcal{C}(\gamma_t, z^{-1})]\, u_t, \end{cases} \tag{8}$$

where

$$\gamma_t := \begin{cases} \gamma_{t_i} & \text{if } t = t_i \\ \gamma_{t-1} & \text{otherwise.} \end{cases}$$

The analysis is performed under the following assumption:

Assumption 2. *For any admissible model with parameter $\vartheta \in \Theta$, there exists a candidate controller (identified here by parameter $\gamma(\vartheta)$) attaining closed-loop λ-stability and $\sup_{\vartheta\in\Theta} J'(\vartheta,\gamma(\vartheta)) < \infty$ (J' appears in definition (3)).*

The first part of this assumption simply requires that the controller class be selected in a 'wise' way so that at least one controller is able to λ-stabilize any potential true process. If the controller with parameter $\gamma(\vartheta)$ λ-stabilizes the model with parameter ϑ, then $J'(\vartheta,\gamma(\vartheta)) < \infty$ for any common performance criterion J'. Since Θ is compact, $\sup_{\vartheta\in\Theta} J'(\vartheta,\gamma(\vartheta)) < \infty$ is then met under a continuity condition satisfied by any commonly used control method.

Under Assumption 2, we shall prove that the closed-loop system (8) is L^2-stable in the following usual sense

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N-1} [u_t^2 + y_t^2] < \infty, \text{ a.s. (almost surely).} \tag{9}$$

We start with the following preliminary result.

Proposition 2. *The model parameters $\vartheta_{1,t}, \vartheta_{2,t}, \dots, \vartheta_{M(\epsilon,\delta),t}$ are such that*

$$(\vartheta_{i,t} - \vartheta^\circ)^T \sum_{s=1}^{t} \varphi_{s-1}\varphi_{s-1}^T(\vartheta_{i,t} - \vartheta^\circ) = o\Big(\sum_{s=1}^{t} \|\varphi_{s-1}\|^2\Big), \text{ a.s.,} \quad i = 1,\dots,M(\epsilon,\delta).$$

Proof. Fix a real constant $\beta > 0$ and define

$$v_t = \log^{1+\beta}\big(\sum_{s=1}^{t} \|\varphi_{s-1}\|^2\big) \tag{10}$$

and $S_t = \Big\{\vartheta \in \mathbb{R}^{n_p+m_p} : (\vartheta - M_t)^T V_t^{-1}(\vartheta - M_t) > v_t\Big\}$.

We prove the intermediate result that

$$\frac{\displaystyle\int_{S_t} f_g(\vartheta; M_t, V_t)d\vartheta}{\displaystyle\int_{\Theta} f_g(\vartheta; M_t, V_t)d\vartheta} = o\left(\frac{1}{t^2}\right), \quad \text{a.s.,} \tag{11}$$

where $f_g(\cdot; M_t, V_t)$ denotes the Gaussian density function with mean M_t and variance V_t, from which the proposition thesis will then be derived.

To determine the asymptotic behavior of the left-hand-side of (11), we need first to prove that v_t grows unbounded as t tends to infinity at the following rate

$$\log^{1+\beta}(t) = O(v_t), \quad \text{a.s..} \tag{12}$$

From the equality $y_t = \varphi_{t-1}^T \vartheta^\circ + w_t$, we have that $y_t^2 + \|\varphi_{t-1}\|^2 \geq h w_t^2$, $t \geq 0$, where h is a suitable constant. Since $n_p > 0$ so that y_t is one entry of φ_t, this in turn implies that $\|\varphi_t\|^2 + \|\varphi_{t-1}\|^2 \geq h w_t^2$, $t \geq 0$ and, hence, $\sum_{k=1}^{t} \|\varphi_{k-1}\|^2 \geq h \sum_{k=1}^{t-1} w_k^2$. Due to the fact that $t = O(\sum_{k=1}^{t-1} w_k^2)$, a.s., this entails $t = O(\sum_{k=1}^{t} \|\varphi_{k-1}\|^2)$, a.s., which yields (12) in view of the definition (10)

We now bound separately the numerator and the denominator in (11).

Let $f_{\chi^2}(\cdot; n_p + m_p)$ be the χ^2 density with $n_p + m_p$ degrees of freedom and define $\eta : \mathbb{R}^{n_p+m_p} \to [0, 1]$ to be the integral of the tail of the χ^2 distribution f_{χ^2}, i.e.

$$\eta(v) = \int_{z>v} f_{\chi^2}(z; n_p + m_p) dz. \tag{13}$$

Based on the definition of S_t, it is easily seen that the numerator of the left-hand-side of equation (11) can be expressed in terms of η as follows

$$\int_{S_t} f_g(\vartheta; M_t, V_t) d\vartheta = \eta(v_t). \tag{14}$$

Thus, recalling that $f_{\chi^2}(z; n_p + m_p) = c z^{[(n_p+m_p)/2-1]} e^{-z/2} = c z^{[(n_p+m_p)/2-1]} e^{-z/4} e^{-z/4}$, where c is a normalizing constant, we conclude that

$$\int_{S_t} f_g(\vartheta; M_t, V_t) d\vartheta = \int_{z>v_t} c z^{[(n_p+m_p)/2-1]} e^{-z/4} e^{-z/4} dz.$$

We want to bound the expression $c z^{[(n_p+m_p)/2-1]} e^{-z/4}$ under the sign of integral. The function $c z^{[(n_p+m_p)/2-1]} e^{-z/4}$ is decreasing for any z large enough. Hence, as $v_t \to \infty$ a.s. (see (12)), $c z^{[(n_p+m_p)/2-1]} e^{-z/4} \leq c v_t^{[(n_p+m_p)/2-1]} e^{-v_t/4}$, $\forall z > v_t$, for any t large enough, say $t \geq t'$, almost surely. For $t \geq t'$, we then conclude that:

$$\int_{S_t} f_g(\vartheta; M_t, V_t) d\vartheta \leq c v_t^{[(n_p+m_p)/2-1]} e^{-v_t/4} \int_{z>v_t} e^{-z/4} dz = c v_t^{[(n_p+m_p)/2-1]} 4 e^{-v_t/2} = o(e^{-v_t/4}). \tag{15}$$

Consider now the denominator in equation (11). It can be bounded as follows

$$\int_{\Theta} f_g(\vartheta; M_t, V_t) d\vartheta \geq \int_{\mathcal{D}_t} f_g(\vartheta; M_t, V_t) d\vartheta, \quad t \geq 0, \tag{16}$$

where $D_t := \{\vartheta \in \mathbb{R}^{n_p+m_p} : (\vartheta - \vartheta^\circ)^T V_t^{-1} (\vartheta - \vartheta^\circ) \leq \Delta\}$, for a suitably chosen $\Delta > 0$. Indeed, a $\Delta > 0$ such that $\{\vartheta \in \mathbb{R}^{n_p+m_p} : (\vartheta - \vartheta^\circ)^T V_0^{-1} (\vartheta - \vartheta^\circ) \leq \Delta\} \subseteq \Theta$ can be found since ϑ° is an interior point of Θ (Assumption 1). The fact that $\{\vartheta \in \mathbb{R}^{n_p+m_p} : (\vartheta - \vartheta^\circ)^T V_t^{-1} (\vartheta - \vartheta^\circ) \leq \Delta\} \subseteq \Theta$, $\forall t \geq 0$, then follows from the fact that V_t^{-1} is increasing since it is given by

$$V_t^{-1} = \frac{1}{\sigma^2} \sum_{s=1}^{t} \varphi_{s-1} \varphi_{s-1}^T + V^{-1}. \tag{17}$$

The right-hand-side of (16) can be further bounded as follows:

$$\begin{aligned} \int_{\mathcal{D}_t} f_g(\vartheta; M_t, V_t) d\vartheta &\geq e^{-(\vartheta^\circ - M_t)^T V_t^{-1} (\vartheta^\circ - M_t)} \int_{\mathcal{D}_t} e^{-\frac{1}{2}(\vartheta - \vartheta^\circ)^T V_t^{-1} (\vartheta - \vartheta^\circ)} f_g(\vartheta; \vartheta^\circ, V_t) d\vartheta \\ &\geq e^{-(\vartheta^\circ - M_t)^T V_t^{-1} (\vartheta^\circ - M_t)} e^{-\frac{\Delta}{2}} \int_{\mathcal{D}_t} f_g(\vartheta; \vartheta^\circ, V_t) d\vartheta \\ &\geq e^{-(\vartheta^\circ - M_t)^T V_t^{-1} (\vartheta^\circ - M_t)} e^{-\frac{\Delta}{2}} (1 - \eta(\Delta)), \end{aligned} \tag{18}$$

where the first inequality follows from $(\vartheta-M_t)^T V_t^{-1}(\vartheta-M_t) \leq 2(\vartheta-\vartheta^\circ)^T V_t^{-1}(\vartheta-\vartheta^\circ)+2(\vartheta^\circ-M_t)^T V_t^{-1}(\vartheta^\circ-M_t)$, the second from the definition of $\mathcal{D}_t$, and the third from the definition (13) of η. We now appeal to [24, Theorem 4.1] and [25] to claim that

$$(\vartheta^\circ - M_t)^T V_t^{-1}(\vartheta^\circ - M_t) = O(\log(\sum_{s=1}^{t} \|\varphi_{s-1}\|^2)), \quad \text{a.s.} \tag{19}$$

This jointly with (10) and (12) implies that there exists a.s. a time instant $\tilde{t} \geq 0$ such that $e^{-(\vartheta^\circ-M_t)^T V_t^{-1}(\vartheta^\circ-M_t)} \geq e^{-\frac{1}{8}v_t}$, $\forall t \geq \tilde{t}$. Using this bound in (18) and recalling equation (16), we get

$$\int_\Theta f_g(\vartheta; M_t, V_t)d\vartheta \geq \int_{\mathcal{D}_t} f_g(\vartheta; M_t, V_t)d\vartheta \geq e^{-\frac{1}{8}v_t} e^{-\frac{\Delta}{2}}(1-\eta(\Delta)), \quad \forall t \geq \tilde{t}. \tag{20}$$

¿From (15) and (20): the left-hand-side of (11) is a $o\big(e^{-\frac{1}{4}v_t}/e^{-\frac{1}{8}v_t}\big) = o\big(e^{-\frac{1}{8}v_t}\big)$, which, by (12), leads to the fact that the left-hand-side of (11) is a $o\big(e^{-\frac{1}{8}\log^{1+\beta}(t)}\big)$. By noticing that $e^{-\frac{1}{8}\log^{1+\beta}(t)} = t^{-\frac{1}{8}\log^{\beta}(t)} = o(t^{-2})$, (11) is proven to hold.

The next step in the proof consists of showing that, based on (11), there exists with probability 1 a $\bar{t} \geq 0$ such that, $\forall t \geq \bar{t}$, the parameters $\vartheta_{1,t}, \ldots, \vartheta_{M(\epsilon,\delta),t}$ are outside S_t, which in turn implies that

$$(\vartheta_{i,t} - M_t)^T V_t^{-1}(\vartheta_{i,t} - M_t) = o\Big(\sum_{s=1}^{t} \|\varphi_{s-1}\|^2\Big), \text{ a.s.,} \quad i = 1, \ldots, M(\epsilon, \delta). \tag{21}$$

Consider the event where at least one of the parameters extracted at time t belongs to S_t, i.e. $A_t := \{\vartheta_{1,t} \in S_t \text{ or } \ldots \text{ or } \vartheta_{M(\epsilon,\delta),t} \in S_t\}$, and let $\mathcal{F}_t$ be the σ-algebra generated by all variables at any time instant up to t except the last extractions of ϑ parameters $\vartheta_{1,t}, \ldots, \vartheta_{M(\epsilon,\delta),t}$. Since the extractions of $\vartheta_{1,t}, \ldots, \vartheta_{M(\epsilon,\delta),t}$ are independent, conditionally to $\mathcal{F}_t$, we have:

$$\Pr\{A_t/\mathcal{F}_t\} = M(\epsilon,\delta)\Pr\{\vartheta_{1,t} \in S_t/\mathcal{F}_t\} = \frac{\displaystyle\int_{S_t\cap\Theta} f_g(\vartheta; M_t, V_t)d\vartheta}{\displaystyle\int_\Theta f_g(\vartheta; M_t, V_t)d\vartheta}$$
$$\leq M(\epsilon,\delta)\frac{\displaystyle\int_{S_t} f_g(\vartheta; M_t, V_t)d\vartheta}{\displaystyle\int_\Theta f_g(\vartheta; M_t, V_t)d\vartheta} = o\left(\frac{1}{t^2}\right), \quad \text{a.s.,}$$

where we have used (11) in the last equality. So,

$$\sum_{t=0}^{\infty} \Pr\{A_t/\mathcal{F}_t\} < \infty, \quad \text{a.s..}$$

An application of the conditional Borel-Cantelli Lemma (see e.g. Exercise 7 of Section 7.4 in [26]) permits one to conclude that $\Pr\{A_t \text{ i.o.}\} = 0$, where i.o. means infinitely often. Thus, there exists with probability 1 a $\bar{t} \geq 0$ such that, $\forall t \geq \bar{t}$, all model parameters $\vartheta_{1,t}, \ldots, \vartheta_{M(\epsilon,\delta),t}$ are outside S_t.

Finally, observe now that by equation (17), for any $i \in \{1, \dots, M(\epsilon,\delta)\}$, $(\vartheta_{i,t} - \vartheta^\circ)^T \sum_{s=1}^{t} \varphi_{s-1}\varphi_{s-1}^T(\vartheta_{i,t} - \vartheta^\circ) \le \sigma^2(\vartheta_{i,t} - \vartheta^\circ)^T V_t^{-1}(\vartheta_{i,t} - \vartheta^\circ) \le 2\sigma^2(\vartheta_{i,t} - M_t)^T V_t^{-1}(\vartheta_{i,t} - M_t) + 2\sigma^2(\vartheta^\circ - M_t)^T V_t^{-1}(\vartheta^\circ - M_t)$. In view of equations (19) and (21), this concludes the proof. ■

We can now state and prove the main result of the paper.

Theorem 1. *Under Assumptions 1 and 2, the cautious switching control scheme (8) is L^2-stable.*

Proof. We start by showing that for each switching time t_i there exists a parameter value in the set $\xi_{t_i} = (\vartheta_{1,t_i}, \dots, \vartheta_{M(\epsilon,\delta),t_i})$ such that the closed-loop system formed by the model with this parameter and the controller with parameter γ_{t_i} is λ-stable. This is proved by contradiction.

Suppose that the claim is false. Then, by (3), $\hat{\mathrm{E}}_{\mathcal{P}_{t_i}}[J(\vartheta, \gamma_{t_i})] = 1$. Pick any $k \in \{1, \dots, M(\epsilon,\delta)\}$ and consider ϑ_{k,t_i}. Since $\vartheta_{k,t_i} \in \Theta$, by Assumption 2 there exists a $\bar{\gamma} \in \Gamma$ such that $J(\vartheta_{k,t_i}, \bar{\gamma}) < 1$. Then, $\hat{\mathrm{E}}_{\mathcal{P}_{t_i}}[J(\vartheta, \bar{\gamma})] \le \frac{1}{M(\epsilon,\delta)}(M(\epsilon,\delta) - 1 + J(\vartheta_{k,t_i}, \bar{\gamma})) < 1 = \hat{\mathrm{E}}_{\mathcal{P}_{t_i}}[J(\vartheta, \gamma_{t_i})]$, which is a contradiction with the fact that γ_{t_i} is optimal.

Now, let us call ϑ_{j,t_i} the parameter in ξ_{t_i} (the existence of which has just been proved) such that the closed-loop system formed by the model with parameter ϑ_{j,t_i} and the controller with parameter γ_{t_i} is λ-stable. Then, $F(\vartheta_{j,t_i}, \gamma_{t_i})$ in equation (7) is stable and we let k be the integer such that equation (7) holds.

Define

$$\vartheta_t = \begin{cases} \vartheta_{j,t_i} & \text{if } t = t_i \\ \vartheta_{t-1} & \text{otherwise.} \end{cases} \tag{22}$$

We represent the closed-loop system (8) as a perturbation system with respect to the time varying closed-loop system formed by the model with parameter ϑ_t and the controller with parameter γ_t as follows

$$\begin{cases} y_{t+1} = [1 - \mathcal{A}(\vartheta_t, z^{-1})]\, y_{t+1} + \mathcal{B}(\vartheta_t, z^{-1})\, u_t + e_t + w_{t+1} \\ u_t = \mathcal{D}(\gamma_t, z^{-1})\, y_t + [1 - \mathcal{C}(\gamma_t, z^{-1})]\, u_t, \end{cases} \tag{23}$$

where $e_t := \varphi_t^T(\vartheta^\circ - \vartheta_t)$ (the perturbation term) is regarded as an exogenous input.

This representation has two nice properties: (a) over each dwell-time interval $[t_i, t_{i+1})$, the closed-loop system (23) is time invariant and has a λ-stable dynamic matrix $F(\vartheta_{t_i}, \gamma_{t_i})$; and (b) $\vartheta^\circ - \vartheta_t$ appearing in the exogenous input e_t is bounded at each switching time by $(\vartheta_{t_i} - \vartheta^\circ)^T \sum_{s=1}^{t_i} \varphi_{s-1}\varphi_{s-1}^T(\vartheta_{t_i} - \vartheta^\circ) = o(\sum_{s=1}^{t_i} \|\varphi_{s-1}\|^2)$ (Proposition 2).

The rest of the proof is structured as follows. We first show that

i) the dwell-time interval sequence $\{t_{i+1} - t_i\}_{i \ge 0}$ is bounded;
ii) the time-varying system (23) where e_t is seen as an exogenous input is exponentially stable, uniformly in time;
iii) the perturbation term feeding the system (23) is bounded as indicated in (25).

Equation (9) then follows from i)–iii).

Proof of i) To each $\gamma \in \Gamma$, we can associate the set of parameters $\Theta_\gamma \subseteq \Theta$ such that the closed-loop system formed by the model with parameter ϑ and the controller with parameter γ is λ-stable: $\Theta_\gamma := \{\vartheta \in \Theta : |\lambda_{\max}(F(\vartheta, \gamma))| \le \lambda\}$, where $\lambda_{\max}(F(\vartheta, \gamma))$ is the maximum eigenvalue of the closed-loop matrix $F(\vartheta,\gamma)$ defined in Section 2.2. Note that since $\lambda_{\max}(F(\cdot,\gamma))$, $\gamma\in\Gamma$, is a continuous function of ϑ and Θ is compact, Θ_γ is a compact set.

Consider now a parameter $\gamma\in\Gamma$ and fix $\nu\in(\lambda,1)$. The matrix $\frac{1}{\nu}F(\vartheta,\gamma)$ is exponentially stable $\forall\,\vartheta\in\Theta_\gamma$. Hence, the solution $P_\gamma(\vartheta)$ to the Lyapunov equation

$$\frac{1}{\nu}F(\vartheta,\gamma)^T P\frac{1}{\nu}F(\vartheta,\gamma)-P=-I$$

is positive definite. Thus,

$$x^T\frac{1}{\nu}F(\vartheta,\gamma)^T P_\gamma(\vartheta)\,\frac{1}{\nu}F(\vartheta,\gamma)x\le x^T P_\gamma(\vartheta)x,\quad \forall x\in\mathbb{R}^{n+m-1},$$

and, by applying k times this equation, we get

$$x^T(\frac{1}{\nu^k}F(\vartheta,\gamma)^k)^T P_\gamma(\vartheta)\,\frac{1}{\nu^k}F(\vartheta,\gamma)^k x\le x^T P_\gamma(\vartheta)x,\quad \forall x\in\mathbb{R}^{n+m-1},$$

which leads to

$$\|F(\vartheta,\gamma)^k x\|\le c_\gamma\nu^k\|x\|,\quad \forall x\in\mathbb{R}^{n+m-1},\quad \forall\vartheta\in\Theta_\gamma, \tag{24}$$

where $c_\gamma:=\max_{\vartheta\in\Theta_\gamma}\sqrt{\lambda_{max}(P_\gamma(\vartheta))/\lambda_{min}(P_\gamma(\vartheta))}$. Note that $c_\gamma<\infty$ because $P_\gamma(\vartheta)$ is continuous on the compact set Θ_γ (see e.g. [27]).

Define $T_\gamma:=\inf\{k\in\mathbb{N}:c_\gamma\,\nu^k\le\mu\}$. Then, $\|F(\vartheta,\gamma)^{T_\gamma}\|=\sup_{\|x\|\neq0}\frac{\|F(\vartheta,\gamma)^{T_\gamma}x\|}{\|x\|}\le\mu$, $\forall\,\vartheta\in\Theta_\gamma$, $\forall\gamma\in\Gamma$. Therefore, $\{t_{i+1}-t_i\}_{i\ge0}$ is uniformly bounded by $\bar{T}:=\max_{\gamma\in\Gamma}T_\gamma$, as one concludes from observing that:

$$t_{i+1}-t_i=\min\{k\ge1:\|F(\vartheta_{t_i},\gamma_{t_i})^k\|\le\mu\}\le\min\{k\ge1:\sup_{\vartheta\in\Theta_{\gamma_{t_i}}}\|F(\vartheta,\gamma_{t_i})^k\|\le\mu\}\le\bar{T}.$$

Proof of ii) Consider the state space representation $x_{t+1}=F(\vartheta_t,\gamma_t)x_t+C(e_t+w_{t+1})$ of the time-varying system (23). In each interval $[t_i,t_{i+1})$ this system is time invariant and by (24) its dynamic matrix $F(\vartheta_{t_i},\gamma_{t_i})$ satisfies $\|F(\vartheta_{t_i},\gamma_{t_i})^k\|=\sup_{\|x\|\neq0}\frac{\|F(\vartheta_{t_i},\gamma_{t_i})^k x\|}{\|x\|}\le c\nu^k$ with $c:=\max_{\gamma\in\Gamma}c_\gamma$. Also, $\|F(\vartheta_{t_i},\gamma_{t_i})^{t_{i+1}-t_i}\|\le\mu$.

Consider two time instants $0\le t'<t$. If t',t do not belong to the same dwell-time interval, say $0\le t'<t_i<\cdots<t_{j+1}<t$, then,

$$\begin{aligned}\|x_t\|&=\|F(\vartheta_{t_{j+1}},\gamma_{t_{j+1}})^{t-t_{j+1}}F(\vartheta_{t_j},\gamma_{t_j})^{t_{j+1}-t_j}\dots F(\vartheta_{t_i},\gamma_{t_i})^{t_{i+1}-t_i}F(\vartheta_{t_{i-1}},\gamma_{t_{i-1}})^{t_i-t'}x_{t'}\|\\&\le\|F(\vartheta_{t_{j+1}},\gamma_{t_{j+1}})^{t-t_{j+1}}\|\|F(\vartheta_{t_j},\gamma_{t_j})^{t_{j+1}-t_j}\|\dots\|F(\vartheta_{t_i},\gamma_{t_i})^{t_{i+1}-t_i}\|\|F(\vartheta_{t_{i-1}},\gamma_{t_{i-1}})^{t_i-t'}\|\|x_{t'}\|\\&\le c\nu^{t-t_{j+1}}\mu^{j+1-i}c\nu^{t_i-t'}\|x_{t'}\|\le c^2\bar{\nu}^{t-t'}\|x_{t'}\|,\end{aligned}$$

where $\bar{\nu}:=\max\{\nu,\mu^{1/\bar{T}}\}$.
If instead t',t belong to the same dwell-time interval, we still have $\|x_t\|\le c\nu^{t-t'}\|x_{t'}\|\le c^2\bar{\nu}^{t-t'}\|x_{t'}\|$. This proves the uniform exponential stability of (23).

Proof of iii) Since $\{t_{i+1}-t_i\}_{i\ge0}$ is bounded and $(\vartheta_{t_i}-\vartheta^\circ)^T\sum_{s=1}^{t_i}\varphi_{s-1}\varphi_{s-1}^T(\vartheta_{t_i}-\vartheta^\circ)=o(\sum_{s=1}^{t_i}\|\varphi_{s-1}\|^2)$, a.s. (see (b) at the beginning of the proof), by Proposition 3.3 in [16] we have that e_t is bounded as follows

$$\sum_{t=0,\ t\notin\mathcal{Q}_N}^{N-1}e_t^2=\ o(\sum_{t=0}^{N-1}\|\varphi_t\|^2+N),\quad\text{a.s.}, \tag{25}$$

where $\mathcal{Q}_N$ is a set of times which may depend on N but whose cardinality is uniformly bounded: $|\mathcal{Q}_N| \leq K, \forall N$.

To conclude the proof of the theorem, it is convenient to adopt the following representation for (23). Fix $N > 0$. For every $t \in [0, N)$ the system (23) can be represented as

$$x_{t+1} = \begin{cases} F(\vartheta^\circ, \gamma_t)\, x_t + C w_{t+1} & t \in \mathcal{Q}_N \\ F(\vartheta_t, \gamma_t)\, x_t + C[e_t + w_{t+1}] & t \notin \mathcal{Q}_N. \end{cases} \quad (26)$$

From the uniform exponential stability of $x_{t+1} = F(\vartheta_t, \gamma_t) x_t$, it is straightforward to show that the x_t generated by (26) can be bounded as follows:

$$\|x_t\| \leq k_1 \{ \sum_{i=1}^{t} \nu^{t-i} |w_i| + \sum_{i=0, i \notin \mathcal{Q}_N}^{t-1} \nu^{t-i} |e_i| \}, \quad t \leq N,$$

where k_1 and $\nu \in (0, 1)$ are suitable constants, from which we get

$$\frac{1}{N} \sum_{t=1}^{N} \|x_t\|^2 \leq k_2 \{ \frac{1}{N} \sum_{t=1}^{N} w_t^2 + \frac{1}{N} \sum_{t=0, t \notin \mathcal{Q}_N}^{N-1} e_t^2 \},$$

where k_2 is a suitable constant, independent of N. Because of $\sum_{t=1}^{N} w_t^2 = O(1)$ and equation (25), we then have $\frac{1}{N} \sum_{t=1}^{N} \|x_t\|^2 = O(1) + o(\frac{1}{N} \sum_{t=0}^{N-1} \|\varphi_t\|^2)$, a.s.. Since $\frac{1}{N} \sum_{t=0}^{N-1} \|\varphi_t\|^2 \leq \frac{1}{N} \sum_{t=0}^{N} \|x_t\|^2$, this implies that $\frac{1}{N} \sum_{t=0}^{N-1} \|\varphi_t\|^2$ remains bounded, thus concluding the proof. ■

4. Conclusions

In this paper, we combined cautious randomized control and switching control to overcome existing difficulties of both methods, while preserving their positive features. For the class of systems described by a linear input-output model affected by Gaussian noise, we introduced a randomized cautious switching control scheme that is robust in finite time and asymptotically stable.

Based on this stability result, it is also possible to obtain self-tuning properties for the proposed scheme. For example, if a dither noise is added to the control input as suggested in [13], then the results in [13] can be applied to the present context so as to prove that the switching controller is optimal in the long run.

Still, important issues remain open. These include taking into consideration the presence of unmodeled dynamics when updating $\mathcal{P}_t$, and studying an easy-to-implement procedure for the construction of a hierarchical controller cover structure. These problems represent a stimulus for future research.

ACKNOWLEDGEMENTS

Research supported by the European Commission under the project HYBRIDGE IST-2001-32460, by MIUR under the project "New techniques for the identification and adaptive control of industrial systems" and by the National Science Foundation under the Grant No. ECS-0242798.

REFERENCES

1. G.C. Goodwin and K.S. Sin. *Adaptive filtering prediction and control*. Prentice-Hall, 1984.
2. R.H. Middleton, G.C. Goodwin, D.J. Hill, and D.Q. Mayne. Design issues in adaptive control. *IEEE Trans. on Automatic Control*, AC-33:50–58, 1988.
3. A.S. Morse. Control using logic-based switching. In A. Isidori, editor, *Trends in Control*, pages 69–113. Springer-Verlag, 1995.
4. A.S. Morse. Supervisory control of families of linear set-point controllers—Part 1: Exact matching. *IEEE Trans. on Automatic Control*, AC-41:1413–1431, 1996.
5. J.P. Hespanha. *Logic-based switching algorithms in control*. PhD thesis, Dept. of Electrical Engineering, Yale University, 1998.
6. D. Borrelli, A.S. Morse, and E. Mosca. Discrete-time supervisory control of families of 2-DOF linear set-point controllers. *IEEE Trans. on Automatic Control*, AC-44:178–181, 1999.
7. M. Prandini and M.C. Campi. Logic-based switching for the stabilization of stochastic systems in presence of unmodeled dynamics. In *Proc. 40th CDC Conf.*, Orlando, USA, Dec. 2001.
8. B.D. O. Anderson, T.S. Brinsmead, F. de Bruyne, J. Hespanha, D. Liberzon, and A. S. Morse. Multiple model adaptive control. I: Finite controller coverings. *Int. J. of Robust and Nonlinear Control*, 10:909–929, 2000. George Zames Special Issue.
9. F.M. Pait and F. Kassab. On a class of switched, robustly stable, adaptive systems. *Int. J. of Adaptive Control and Signal Processing*, 15(3):213–238, May 2001.
10. Y. Bar-Shalom. Stochastic dynamic programming: caution and probing. *IEEE Trans. on Automatic Control*, AC-26:1184–1195, 1981.
11. M. Vidyasagar. Statistical learning theory and its applications to randomized algorithms for robust controller synthesis. In *Plenary Lectures and Mini-Courses at the Europ. Control Conf.*, Brussels, Belgium, July, 1997.
12. M. Vidyasagar. Randomized algorithms for robust controller synthesis using statistical learning theory. *Automatica*, 37(10):1515–28, 2001.
13. M.C. Campi and M. Prandini. Randomized algorithms for the synthesis of cautious adaptive controllers. *Syst. & Contr. Lett.*, 49(1):21–36, 2003.
14. A.S. Morse. Supervisory control of families of linear set-point controllers—Part 2: Robustness. *IEEE Trans. on Automatic Control*, AC-42:1500–1515, 1997.
15. M. Prandini, S. Bittanti, and M.C. Campi. A penalized identification criterion for securing controllability in adaptive control. *J. of Math. Syst., Estim. and Control*, 8:491–494, 1998. Full paper electronic access code: 29460.
16. M. Prandini and M.C. Campi. Adaptive LQG control of input-output systems - A cost-biased approach. *SIAM J. Control and Optim.*, 39(5):1499–1519, 2001.
17. J. Sternby. On consistency for the method of least squares using martingale theory. *IEEE Trans. on Automatic Control*, AC-22:346–352, 1977.
18. H. Rootzen and J. Sternby. Consistency in least-squares estimation: a Bayesian approach. *Automatica*, 20:471–475, 1984.
19. P.R. Kumar. Convergence of adaptive control schemes using least-squares parameter estimates. *IEEE Trans. on Automatic Control*, AC-35:416–424, 1990.
20. M.C. Campi. The problem of pole-zero cancellation in transfer function identification and application to adaptive stabilization. *Automatica*, 32:849–857, 1996.
21. M.C. Campi and P.R. Kumar. Adaptive linear quadratic Gaussian control: the cost-biased approach revisited. *SIAM J. Control and Optim.*, 36(6):1890–1907, 1998.
22. H.F. Chen, P.R. Kumar, and J.H. van Schuppen. On Kalman filtering for conditionally Gaussian systems with random matrices. *Syst. & Contr. Lett.*, 13:397–404, 1989.
23. M. Vidyasagar. *A theory of learning and generalization: with applications to neural networks and control systems*. Springer-Verlag, 1997.
24. H.F. Chen and L. Guo. *Identification and Stochastic Adaptive Control*. Birkhäuser, 1991.
25. T.L. Lai and C.Z. Wei. Least squares estimates in stochastic regression models with applications to identification and control of dynamic systems. *Ann. Statist.*, 10:154–166, 1982.
26. Y.S. Chow and H. Teicher. *Probability theory*. Springer, 1997.
27. D.F. Delchamps. Analytic feedback control and the algebraic Riccati equation. *IEEE Trans. on Automatic Control*, AC-29:1031–1033, 1984.