Decentralized Stabilization and Collision Avoidance of Multiple Air Vehicles with Limited Sensing Capabilities

Dimos V. Dimarogonas and Kostas J. Kyriakopoulos

Control Systems Laboratory,Mechanical Eng. Dept.
National Technical University of Athens,Greece
ddimar,kkyria@mail.ntua.gr

***Abstract*— Motivated by the needs of distributed Air Traffic Management, we extend the decentralized navigation function methodology, established in previous work for navigation of multiple holonomic agents with global sensing capabilities to the case of local sensing capabilities. Each agent plans its actions without knowing (i) the destinations of the others and (ii) the positions of agents outside its sensing neighborhood. The overall system is modelled as a deterministic switched system and we use tools from nonsmooth analysis to check its stability properties. The collision avoidance and global convergence properties are verified through simulations.**

I. INTRODUCTION

Navigation of mobile agents has been an area of significant interest in robotics and control communities. Most efforts have focused on the case of a single agent navigating in an environment with obstacles [13]. Recently, decentralized navigation for multiple agents has gained increasing attention. The motivation comes from many application domains, among which decentralized conflict resolution in air traffic management(ATM) has gained increasing attention in the past few years.

Today's air traffic systems remain to a large extent widely centralized [18]. A central authority, namely the Air Traffic Controllers (ATC), is responsible for issuing instructions to conflict-bound aircraft . To resolve conflicts they ask aircraft to climb/descend or vector them away from the path in the flight plan and then back on to it. Flight plans are completely pre-defined and aircraft fly along fixed corridors and at specified altitude. Only minor deviations from the original flight plan are permitted on line. Autonomous decision-making by aircraft is allowed under the Traffic Alert Collision Avoidance System (TCAS)[14], which issues advisories in order to avoid potential collisions, yet is used only in extreme situations.

On the other hand, the demand for air transportation is constantly increasing and threatens to exceed the capacity of the current centralized ATM structure. The number of passengers using air traffic is predicted to increase up to 120 % in the next ten years [4], and studies in [18] indicate that, with the current ATM structure, a major accident could occur every 7 to 10 days by the year 2015. Moreover, recent technological advances in avionics such as satellite positioning systems (the Global Positioning System (GPS)), inter-communication systems (the Automatic Dependent Surveillance-Broadcast (ADSB)-although its current use in air traffic is rather limited), and powerful on-board computers, are used in terms of the current centralized ATM system and provide an improvement on it, but not a radical change in the air traffic community.

These facts have resulted in the growing will of the air traffic world for new architectures, which employ these new technological innovations towards a more user-centered system. The purpose is to supply pilots with more decisional freedom and to reduce the authority and influence of the ATCs. The ultimate purpose of these efforts is *free-flight*, a concept in which aircraft will be allowed to plan their en-route trajectories and resolve any conflicts with other aircraft in a distributed and cooperative manner. In this case, the ATC will play the role of a passive observer.

The need for development of decentralized conflict resolution algorithms is therefore evident. The level of decentralization depends on the perception an agent has on the other agents' actions and the knowledge of their objectives. In our previous work ([5],[16]) the decentralization factor lied in the fact that each agent had knowledge only of its own desired destination, but not of the desired destinations of the others. Clearly, this is a suitable model for a futuristic distributed ATM system, where each aircraft will have knowledge of the actions and positions of the other aircraft at each time instant, but not of their destinations.

Nevertheless, in practice, the sensing capabilities of each agent are limited. Consequently, each agent can not have knowledge of the positions and/or velocities of every agent in the workspace but only of the agents within its sensing zone at each time instant. The interpretation of the sensing zone of an agent that we use in this paper is a circle of constant radius around its center of mass. Taking those aspects into consideration, the multi agent navigation problem treated in this paper can be stated as follows: *derive a set of control laws (one for each agent) that drives a team of agents from any initial configuration to a desired goal* *configuration avoiding, at the same time, collisions. Each agent has no knowledge of the others desired destinations and has only local knowledge of their positions at each instant*. The same problem has been dealt in [1],[11] under a game theoretic perspective, while the concept of a sensing zone has also been used in [8] . In [20] a nonsmooth controller was designed to achieve flocking behavior in an environment with multiple agents with limited sensing capabilities. In this paper we use the navigation function method established in [12], [15],[5] and solve the problem in a closed loop fashion.

The rest of the paper is organized as follows: in section 2 the concept of decentralized navigation functions introduced in [5]. In section 3 we present the class of nonsmooth navigation functions we use to cope with the limited sensing capabilities of the agents. Section 4 is a summary of the tools from nonsmooth analysis needed to check the stability properties of the system in section 5. Simulation results are presented in section 6 while section 7 summarizes the conclusions and indicates our current research.

II. Decentralized Navigation Functions

In this section, we review the decentralized navigation function method used in [5] for the case of multiple holonomic agents. Consider a system of *n* agents operating in the same workspace $W \subset R^2$. Each agent *i* occupies a disk: $R = \{q \in R^2 : \|q - q_i\| \leq r_i\}$ in the workspace where $q_i \in R^2$ is the center of the disk and r_i is the radius of the agent. The dynamics of each agent are given by $\dot{q}_i = u_i$ and the configuration space is spanned by $q = [q_1, \ldots, q_n]^T$. The proposed control law for each agent is given by

$$u_i = -K_i \cdot \frac{\partial \varphi_i}{\partial q_i} \tag{1}$$

where K_i is a positive gain and the *decentralized navigation function* φ_i is defined as

$$\varphi_i = \frac{\gamma_{di} + f_i}{((\gamma_{di} + f_i)^k + G_i)^{1/k}} \tag{2}$$

The term $\gamma_{di} = \|q_i - q_{di}\|^2$ in the potential function is the squared metric of the agent's i configuration from its desired destination q_{di}. The function G_i expresses all possible collisions of agent i with the others, while f_i guarantees that the φ_i attains positive values in proximity situations even when i has already reached its destination.

A. Construction of the G_i function

We review now the construction of the "collision" function G_i for each agent i. The "Proximity Function"between agents i and j is given by

$$\beta_{ij} = \|q_i - q_j\|^2 - (r_i + r_j)^2$$

We will use the term *relation* to describe the possible collision schemes that can occur in a multiple agents scene with respect to agent i. A *binary relation* is a relation between agent i and another. We will call the number of binary relations in a relation, the *relation level*. With this terminology in hand, the relation of figure (1a) is a level-1 relation (one binary relation) and that of figure (1b) is a level-3 relation (three binary relations), always with respect to the specific agent R.

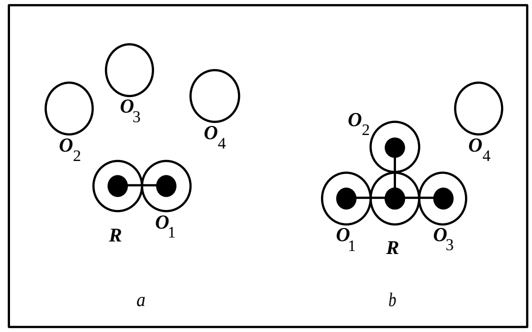

Fig. 1. Part *a* represents a level-1 relation and part *b* a level-3 relation wrt agent R.

A "Relation Proximity Function" (RPF) provides a measure of the distance between agent i and the other agents involved in the relation. Each relation has its own RPF. Let R_k denote the k^{th} relation of level l. The RPF of this relation is given by: $(b_{R_k})_l = \sum_{j \in (R_k)_l} \beta_{ij}$ or $b_k = \sum_{j \in P_r} \beta_{ij}$ for simplicity, where P_r denotes the set of agents participating in the specific relation.

A "Relation Verification Function" (RVF) is defined by:

$$(g_{R_k})_l = (b_{R_k})_l + \frac{\lambda (b_{R_k})_l}{(b_{R_k})_l + (B_{R_k^C})_l^{1/h}}$$

where λ, h are positive scalars and $(B_{R_k^C})_l = \prod_{m \in (R_k^C)_l} (b_m)_l$ where $(R_k^C)_l$ is the complementary set of relations of level-l, i.e. all the other relations with respect to agent i that have the same number of binary relations with the relation R_k. For simplicity we denote $(B_{R_k^C})_l \equiv \tilde{b}_i = \prod_{m \in (R_k^C)_l} b_m$. The RVF can be written as $g_i = b_i + \frac{\lambda b_i}{b_i + \tilde{b}_i^{1/h}}$ It is obvious that for the highest level $l = n-1$ only one relation is possible so that $(R_k^C)_{n-1} = \emptyset$ and $(g_{R_k})_l = (b_{R_k})_l$ for $l = n-1$. The function G_i is now defined as $G_i = \prod_{l=1}^{n_L^i} \prod_{j=1}^{n_{R_l}^i} (g_{R_j})_l$ where n_L^i the number of levels and $n_{R_l}^i$ the number of relations in level-l with respect to agent i.

B. Construction of the f_i function

The key difference of the decentralized method with respect to the centralized case is that the control law of each agent ignores the destinations of the others. By using $\varphi_i = \frac{\gamma_{di}}{\left((\gamma_{di})^k + G_i\right)^{1/k}}$ as a navigation function for agent i, there is no potential for i to cooperate in a possible collision scheme when its initial condition coincides with its final destination. In order to overcome this limitation,we add a function f_i to γ_i so that the cost function φ_i attains positive values in proximity situations even when i has already reached its destination. A preliminary definition for this function was given in [5], [21]. Here, we modify the previous definitions to ensure that the destination point is a non-degenerate local minimum of φ_i with minimum requirements on assumptions. We define the function f_i by:

$$f_i(G_i) = \begin{cases} a_0 + \sum_{j=1}^{3} a_j G_i^j, \ G_i \leq X \\ 0, \ G_i > X \end{cases}$$

where $X, Y = f_i(0) > 0$ are positive parameters the role of which will be made clear in the following. The parameters a_j are evaluated so that f_i is maximized when $G_i \to 0$ and minimized when $G_i = X$. We also require that f_i is continuously differentiable at X. Therefore we have:

$$a_0 = Y, a_1 = 0, a_2 = \frac{-3Y}{X^2}, a_3 = \frac{2Y}{X^3}$$

The parameter X serves as a sensing parameter that activates the f_i function whenever possible collisions are bound to occur. The only requirement we have for X is that it must be small enough whenever the system has reached its equilibrium, i.e. when everyone has reached its destination. In mathematical terms:

$$X < G_i\left(q_{d1}, \ldots, q_{dN}\right) \forall i$$

That's the minimum requirement we have regarding knowledge of the destinations of the team. Intuitively, the destinations should be far enough from one another.

It has been proven that this class of potential fields are navigation functions. For further information regarding terminology the reader is referred to [15],[6].

III. The Case of Limited Sensing Capabilities

It has been shown in [6] that with a suitable choice of the exponent k the proposed control law satisfies the collision avoidance and destination convergence properties in a bounded workspace. The decentralization feature of the whole scheme lied in the fact that each agent didn't have knowledge of the desired destinations of the rest of the team. On the other hand, each one had global knowledge of the positions of the others at each time instant. This is far from realistic in real world applications.

In this work we take the limited sensing capabilities of each agent into account. We consider a bounded workspace with n agents. Each agent has only local knowledge of the positions of the others at each time instant. Specifically, it only knows the position of agents which are in a cyclic neighborhood of specific radius d_C around its center. Therefore the Proximity Function between two agents has to be redefined in this case. We propose the following nonsmooth function:

$$\beta_{ij} = \begin{cases} \|q_i - q_j\|^2 - (r_i + r_j)^2, \text{for } \|q_i - q_j\| \leq d_C \\ d_C^2 - (r_i + r_j)^2, \text{for } \|q_i - q_j\| > d_C \end{cases}$$

The whole scheme is now modelled as a (deterministic) switched system in which switches occur whenever a agent enters or leaves the neighborhood of another. In [6], we have used $\varphi = \sum_{i=1}^{n} \varphi_i$ as a Lyapunov function for the whole system. In this case this function is continuous everywhere, but nonsmooth whenever a switching occurs, i.e. whenever $\|q_i - q_j\| = d_c$ for some i, j. We define the *switching surface* as:

$$S = \{q : \exists i, j, i \neq j \| \|q_i - q_j\| = d_c\} \quad (3)$$

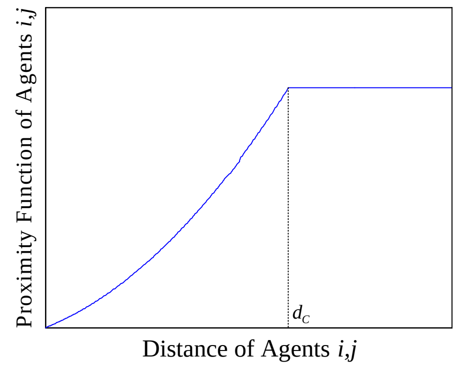

Fig. 2. The function β_{ij} for $r_i + r_j = 1, d_C = 4$.

We have proved that the system converges whenever $q \notin S$. On the switching surface the Lyapunov function is no longer smooth so we must use stability theorems for nonsmooth systems. A summary of the theory we use is given in the following section.

IV. Elements from Nonsmooth Analysis

We consider the vector differential equation with discontinuous right-hand side:

$$\dot{x} = f(x) \quad (4)$$

where $f : R^n \to R^n$ is measurable and essentially locally bounded.

Definition 4.1 [9] *In the case when n is finite, the vector function $x(.)$ is called a solution of (4) in $[t_0, t_1]$ if it is absolutely continuous on $[t_0, t_1]$ and there exists $N_f \subset R^n, \mu(N_f) = 0$ such that for all $N \subset R^n, \mu(N) = 0$ and for almost all $t \in [t_0, t_1]$*

$$\dot{x} \in K[f](x) \equiv \overline{co}\{\lim_{x_i \to x} f(x_i) | x_i \notin N_f \cup N\}$$

The above definition along with the assumption that f is measurable guarantees the uniqueness of solutions of (4)[9].

Lyapunov stability theorems have been extended for nonsmooth systems in [19],[2]. The authors use the concept of *generalized gradient* which for the case of finite-dimensional spaces is given by the following definition:

Definition 4.2 [3]*Let $V : R^n \to R$ be a locally Lipschitz function. The generalized gradient of V at x is given by*

$$\partial V(x) = \overline{co}\{\lim_{x_i \to x} \nabla V(x_i) | x_i \notin \Omega_V\}$$

where Ω_V is the set of points in R^n where V fails to be differentiable.

Lyapunov theorems for nonsmooth systems require the energy function to be *regular*. Regularity is based on the concept of *generalized derivative* which was defined by Clarke as follows:

Definition 4.3 [3] *Let f be Lipschitz near x and v be a vector in R^n. The generalized directional derivative of f at x in the direction v is defined*

$$f^0(x;v) = \lim_{y\to x}\sup_{t\downarrow 0}\frac{f(y+tv)-f(y)}{t}$$

Regularity of a function is defined:

Definition 4.4 [3] *The function $f : R^n \to R$ is called regular if*

1) $\forall v$, the usual one-sided directional derivative $f'(x;v)$exists and

2) $\forall v, f'(x;v) = f^0(x;v)$

The following chain rule provides a calculus for the time derivative of the energy function in the nonsmooth case:

Theorem 4.5 [19] *Let x be a Filippov solution to $\dot{x} = f(x)$ on an interval containing t and $V : R^n \to R$ be a Lipschitz and regular function. Then $V(x(t))$ is absolutely continuous, $(d/dt)V(x(t))$ exists almost everywhere and*

$$\frac{d}{dt}V(x(t)) \in^{a.e.} \dot{\tilde{V}}(x) := \bigcap_{\xi\in\partial V(x(t))} \xi^T K[f](x(t))$$

We shall use the following nonsmooth version of LaSalle's invariance principle to prove the convergence of the prescribed system:

Theorem 4.6 [19] *Let Ω be a compact set such that every Filippov solution to the autonomous system $\dot{x} = f(x), x(0) = x(t_0)$ starting in Ω is unique and remains in Ω for all $t \geq t_0$. Let $V : \Omega \to R$ be a time independent regular function such that $v \leq 0 \forall v \in \dot{\tilde{V}}$(if $\dot{\tilde{V}}$ is the empty set then this is trivially satisfied). Define $S = \{x \in \Omega | 0 \in \dot{\tilde{V}}\}$. Then every trajectory in Ω converges to the largest invariant set,M, in the closure of S.*

V. Stability Analysis

In [5], we used the sum of the separate decentralized navigation functions $\varphi = \sum\varphi_i$ as a candidate Lyapunov function for the whole system and showed that the derivative of the energy function assumes negative values up to a set of measure zero of initial conditions.We show by the following theorem that this is also the case in the limited sensing capabilities case:

Theorem 5.1 *The system is asymptotically stabilized to $q_d = [q_{d1},\ldots,q_{dN}]^T$ up to a set of initial conditions of measure zero if the exponent k assumes values bigger than a finite lower bound.*

It has been proven in [6] that the statement is true for $q \notin S$. In the case when $q \in S$ we shall make use of theorem 4.6. First we must use the following lemma to ensure that φ is regular.

Lemma 5.2 *The function φ is regular $\forall q \in S$.*

Proof of Lemma 5.2: We show first that β_{ij} is regular whenever $\|q_i - q_j\| = d_C$. The directional derivative at d_C is

$$\beta'_{ij}(d_C;v) = \lim_{t\to 0}\frac{\beta_{ij}(d_C+tv)-\beta_{ij}(d_C)}{t} = \left\{\begin{array}{l} 0, v\geq 0 \\ c<0, v<0\end{array}\right.$$

The generalized directional derivative is

$$\beta^0_{ij}(d_C;v) = \lim_{\substack{t\to 0\\ y\to d_C}}\sup\frac{\beta_{ij}(y+tv)-\beta_{ij}(y)}{t} = \left\{\begin{array}{l} 0, v\geq 0 \\ c<0, v<0\end{array}\right.$$

so that $\beta^0_{ij}(d_C;v) = \beta'_{ij}(d_C;v)\ \forall v$. It is easy to check that the terms $\frac{\partial b_i}{\partial\beta_{ij}}, \frac{\partial G_i}{\partial b_i}$ are nonnegative so by virtue of Theorem 2.3.9 (i),[3], the function G_i is regular at $q \in S$.

Function φ_i is continuously differentiable wrt G_i. In this case the term $\frac{\partial\varphi_i}{\partial G_i}$ is nonpositive but we are fortunate that G_i is 1-dimensional. Following the proof of theorem 2.3.9 (ii),[3] we can see that the generalized derivative of φ_i satisfies the following inequality: $\varphi^0_i(q;v) \leq \frac{\partial\varphi_i}{\partial G_i}G^0_i(q;v) = \frac{\partial\varphi_i}{\partial G_i}G'_i(q;v) = \varphi'_i(q;v)$. But we always have $\varphi'_i(q;v) \leq \varphi^0_i(q;v)$, so that $\varphi'_i(q;v) = \varphi^0_i(q;v)$, ensuring the regularity of φ_i. The function φ is regular as the finite linear combination of regular functions.$\diamondsuit$

We now proceed with a sketch of the proof of theorem 5.1. The complete proof can be found in [7]. We make use of the following matrix theorems in our analysis:

Theorem 5.3 (Gersgorin) [10]: *Given a matrix $A \in \Re^{n\times n}$ then all its eigenvalues lie in the union of n discs:*

$$\bigcup_{i=1}^{n}\left\{z : |z-a_{ii}| \leq \sum_{\substack{j=1\\ j\neq i}}^{n}|a_{ij}|\right\} \triangleq \bigcup_{i=1}^{n} R_i(A) \triangleq R(A)$$

Each of these discs is called a Gersgorin disc of A.

Corollary 5.4 [10]: *Given a matrix $A \in \Re^{n\times n}$ and n positive real numbers $p_1,\ldots,p_n$ then all its eigenvalues of A lie in the union of n discs:*

$$\bigcup_{i=1}^{n}\left\{z : |z-a_{ii}| \leq \frac{1}{p_i}\sum_{\substack{j=1\\ j\neq i}}^{n}p_j|a_{ij}|\right\}$$

A key point of Corollary 5.4 is that if we bound the first $n/2$ Gersgorin discs of a matrix A sufficiently away from zero, then an appropriate choice of the numbers $p_1,\ldots,p_n$ renders the remaining $n/2$ discs sufficiently close to the corresponding diagonal elements. Hence, by ensuring the positive definiteness of the eigenvalues of the matrix M corresponding to the first $n/2$ rows, then we can render the remaining ones sufficiently close to the corresponding diagonal elements. This fact will be made clearer in the analysis that follows.

Proof Sketch of Theorem 5.1: We immediately note that the following proof is existential rather than computational.

We show that a finite k that renders the system almost everywhere asymptotically stable *exists*, but we do not provide an analytical expression for this lower bound. However, practical values of k will be provided in the simulation section.

Consider the global sensing case at first. In this case, the Proximity function between agents i and j is given by:

$$\beta_{ij}(q) = \|q_i - q_j\|^2 - (r_i + r_j)^2 = q^T D_{ij} q - (r_i + r_j)^2$$

where the $2N \times 2N$ matrix D_{ij} is defined in [15]:

$$D_{ij} = \begin{bmatrix} & & O_{2(i-1)\times 2N} & & \\ O_{2\times 2(i-1)} & I_{2\times 2} & O_{2\times 2(j-i-1)} & -I_{2\times 2} & O_{2\times 2(N-j)} \\ & & O_{2(j-i-1)\times 2N} & & \\ O_{2\times 2(i-1)} & -I_{2\times 2} & O_{2\times 2(j-i-1)} & I_{2\times 2} & O_{2\times 2(N-j)} \\ & & O_{2(N-j)\times 2N} & & \end{bmatrix}$$

We can also write $b_r^i = q^T P_r^i q - \sum\limits_{j\in P_r} (r_i + r_j)^2$,where $P_r^i = \sum\limits_{j\in P_r} D_{ij}$, and P_r denotes the set of binary relations in relation r. It can easily be seen that $\nabla b_r^i = 2P_r^i q, \nabla^2 b_r^i = 2P_r^i$. We also use the following notation for the r-th relation wrt agent i:

$$g_r^i = b_r^i + \frac{\lambda b_r^i}{b_r^i + \left(\tilde{b}_r^i\right)^{1/h}}, \tilde{b}_r^i = \prod_{\substack{s\in S_r \\ s\neq r}} b_s^i,$$

$$\nabla \tilde{b}_r^i = \sum_{\substack{s\in S_r \\ s\neq r}} \underbrace{\prod_{\substack{t\in S_r \\ t\neq s,r}} b_i^t}_{\tilde{b}_{s,r}^i} \cdot 2P_s^i q$$

where S_r denotes the set of relations in the same level with relation r. An easy calculation shows that

$$\nabla g_r^i = \ldots = 2\left[d_r^i P_r^i - w_r^i \tilde{P}_r^i\right] q \triangleq Q_r^i q, \tilde{P}_r^i \triangleq \sum_{\substack{s\in S_r \\ s\neq r}} \tilde{b}_{s,r}^i P_s^i$$

where $d_r^i = 1 + (1 - \frac{b_r^i}{b_r^i + (\tilde{b}_r^i)^{1/h}})\frac{\lambda}{b_r^i + (\tilde{b}_r^i)^{1/h}}$, $w_r^i = \frac{\lambda b_r^i (\tilde{b}_r^i)^{\frac{1}{h}-1}}{h(b_r^i + (\tilde{b}_r^i)^{1/h})^2}$. The gradient of the G_i function is given by:

$$G_i = \prod_{r=1}^{N_i} g_r^i \Rightarrow \nabla G_i = \sum_{r=1}^{N_i} \underbrace{\prod_{\substack{l=1 \\ l\neq r}}^{N_i} g_l^i}_{\tilde{g}_r^i} \nabla g_r^i = \sum_{r=1}^{N_i} \tilde{g}_r^i Q_r^i q \triangleq Q_i q$$

where N_i all the relations with respect to agent i. We define

$$\nabla G \triangleq \begin{bmatrix} \nabla G_1 \\ \vdots \\ \nabla G_N \end{bmatrix} = \begin{bmatrix} Q_1 \\ \vdots \\ Q_N \end{bmatrix} q \triangleq Qq$$

Remembering that $u_i = -K_i \frac{\partial \varphi_i}{\partial q_i}$ and that $\varphi_i = \frac{\gamma_{di} + f_i}{\left((\gamma_{di}+f_i)^k + G_i\right)^{1/k}}, f_i = \sum\limits_{j=0}^{3} a_i G_i^j$ the closed loop dynamics of the system are given by:

$$\dot{q} = \begin{bmatrix} -K_1 A_1^{-(1+1/k)} \left\{ G_1 \frac{\partial \gamma_{d1}}{\partial q_1} + \sigma_1 \frac{\partial G_1}{\partial q_1} \right\} \\ \vdots \\ -K_N A_N^{-(1+1/k)} \left\{ G_N \frac{\partial \gamma_{dN}}{\partial q_N} + \sigma_N \frac{\partial G_N}{\partial q_N} \right\} \end{bmatrix} = \ldots$$
$$= -A_K G\left(\partial \gamma_d\right) - A_K \Sigma Q q$$

where $\sigma_i = G_i \sigma(G_i) - \frac{\gamma_{di}+f_i}{k}, \sigma(G_i) = \sum\limits_{j=1}^{3} j a_j G_i^{j-1}$,$A_i = (\gamma_{di} + f_i)^k + G_i$ and the matrices

$$A_K \triangleq \underbrace{diag\begin{pmatrix} K_1 A_1^{-(1+1/k)}, K_1 A_1^{-(1+1/k)}, \ldots \\ , K_N A_N^{-(1+1/k)}, K_N A_N^{-(1+1/k)} \end{pmatrix}}_{2N\times 2N}$$

$$G \triangleq \underbrace{diag\left(G_1, G_1, \ldots, G_N, G_N\right)}_{2N\times 2N}, \left(\partial \gamma_d\right) = \left[\frac{\partial \gamma_{d1}}{\partial q_1} \ldots \frac{\partial \gamma_{dN}}{\partial q_N}\right]$$

$$\Sigma \triangleq \underbrace{\left[\underbrace{\Sigma_1}_{2N\times 2N}, \ldots, \underbrace{\Sigma_N}_{2N\times 2N}\right]}_{2N\times 2N^2}, \Sigma_i = diag\begin{pmatrix} 0, 0, \ldots, \underbrace{\sigma_i, \sigma_i}_{2i-1,2i} \\ , \ldots, 0, 0 \end{pmatrix}$$

By using $\varphi = \sum\limits_i \varphi_i$ as a candidate Lyapunov function we have $\varphi = \sum\limits_i \varphi_i \Rightarrow \dot{\varphi} = \left(\sum\limits_i \left(\nabla \varphi_i\right)^T\right) \dot{q}$,$\nabla \varphi_i = A_i^{-(1+1/k)} \left\{G_i \nabla \gamma_{di} + \sigma_i \nabla G_i\right\}$ and after some trivial calculation

$$\sum_i \left(\nabla \varphi_i\right)^T = \ldots = \left(\partial \gamma_d\right)^T A_G + q^T Q^T A_\Sigma$$

where $A_G = \underbrace{diag\begin{pmatrix} G_1 A_1^{-(1+1/k)}, G_1 A_1^{-(1+1/k)}, \ldots, \\ G_N A_N^{-(1+1/k)}, G_N A_N^{-(1+1/k)} \end{pmatrix}}_{2N\times 2N}$

and

$$A_\Sigma = \underbrace{\begin{bmatrix} \underbrace{A_{\Sigma_1}}_{2N\times 2N} \\ \vdots \\ \underbrace{A_{\Sigma_N}}_{2N\times 2N} \end{bmatrix}}_{2N^2\times 2N}, A_{\Sigma_i} = \underbrace{diag\left(A_i^{-(1+1/k)}\sigma_i, \ldots, A_i^{-(1+1/k)}\sigma_i\right)}_{2N\times 2N}$$

The derivative of the candidate Lyapunov function is calculated as

$$\dot{\varphi} = \left(\sum_i \left(\nabla \varphi_i\right)^T\right) \cdot \dot{q} = \ldots$$
$$= -\begin{bmatrix} \left(\partial \gamma_d\right)^T & q^T \end{bmatrix} \underbrace{\begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix}}_{M} \begin{bmatrix} \partial \gamma_d \\ q \end{bmatrix}$$

where $M_1 = A_G A_K G, M_2 = A_G A_K \Sigma Q, M_3 = Q^T A_\Sigma A_K G, M_4 = Q^T A_\Sigma A_K \Sigma Q$.

Let's return to the local sensing case.Let $S_1 = \{q : \exists i, j, i \neq j | (\|q_i - q_j\| = d_c) \bigwedge (\|q_k - q_l\| \neq d_c \forall k, l : k \neq i, j, l \neq i, j)\}$ denote the subset of S which corresponds to the simplest case of switching that involves only two agents. System dynamics are given by:

$$\dot{q} = f(q) = \left[-K_1 \frac{\partial \varphi_1}{\partial q_1}, ..., -K_n \frac{\partial \varphi_n}{\partial q_n}\right]^T$$

The vector function $f(q)$ is nonsmooth at S_1 so that $\dot{q} \in K[f](q), q \in S_1$. We have $K[f](q \in S_1) = \overline{co}\{f_{S_1}^-, f_{S_1}^+\}$ where $S_1^{-(+)} = \{q : \|q_i - q_j\| < (>) d_C\}$ and

$$f_{S_1}^{-(+)}(q \in S_1) = \lim_{\substack{q^* \to q, \\ q^* \in S_1^{-(+)}}} f(q^*)$$

Likewise, the generalized gradient of the candidate Lyapunov function at the discontinuity surface is given by $\partial\varphi(q \in S_1) = \overline{co}\{\nabla\varphi_{S_1}^-, \nabla\varphi_{S_1}^+\}$ where

$$\nabla\varphi_{S_1}^{-(+)}(q \in S_1) = \lim_{\substack{q^* \to q, \\ q^* \in S_1^{-(+)}}} \nabla\varphi(q^*)$$

Each $\rho \in \partial\varphi(q \in S_1)$ is the convex combination of the limit points of the convex hull: $\rho = \mu\left(\nabla\varphi_{S_1}^-\right) + (1 - \mu)\left(\nabla\varphi_{S_1}^+\right), \mu \in [0, 1]$. Similarly, each $\eta \in K[f](q \in S_1)$ as $\eta = \lambda f_{S_1}^- + (1 - \lambda) f_{S_1}^+, \lambda \in [0, 1]$, so that $\rho^T \eta = \lambda\mu\left(\nabla\varphi_{S_1}^-\right)^T f_{s_1}^- + (1-\lambda)\mu\left(\nabla\varphi_{S_1}^-\right)^T f_{s_1}^+ + \lambda(1 - \mu)\left(\nabla\varphi_{S_1}^+\right)^T f_{s_1}^- + (1-\lambda)(1-\mu)\left(\nabla\varphi_{S_1}^+\right)^T f_{s_1}^+$. By virtue of theorem 4.5 one has

$$\dot{\varphi}(q \in S_1) \in \bigcap_{\rho \in \partial\varphi(q \in S_1)} \rho^T \eta, \; \eta \in K[f](q \in S_1)$$

Going back to the previous analysis, it is easy to see that the matrices $A_G, A_K, G, \Sigma, A_\Sigma$ are continuous in the discontinuity surface. The matrix Q is discontinuous at S_1 and that's due to the nonsmoothness of the functions G_i, G_j. By using the notation $Q^{-(+)}(q \in S_1) = \lim_{\substack{q^* \to q, \\ q^* \in S_1^{-(+)}}} Q(q^*)$ and noting that $\bigcap_{\rho \in \partial\varphi(q \in S_1)} \rho^T \eta = \bigcap_{\mu \in [0,1]} \{\rho^T \eta | \lambda \in [0, 1]\}$ we conclude after some trivial calculation that

$$\dot{\varphi}(q \in S_1) \in \bigcap_{\mu \in [0,1]} \left\{ \begin{array}{l} -\left[\begin{array}{cc} (\partial\gamma_d)^T & q^T \end{array}\right] M \left[\begin{array}{c} \partial\gamma_d \\ q \end{array}\right] \\ |\lambda \in [0, 1] \end{array} \right\}$$

with $M = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix}$ where $M_1 = A_G A_K G$, $M_2 = A_G A_K \Sigma\left(\lambda Q^- + (1-\lambda) Q^+\right)$, $M_3 = \left(\mu\left(Q^-\right)^T + (1-\mu)\left(Q^+\right)^T\right) A_\Sigma A_K G$ and $M_4 = \lambda\mu\left(Q^-\right)^T A_\Sigma A_K \Sigma Q^- + (1-\lambda)\mu\left(Q^-\right)^T A_\Sigma A_K \Sigma Q^+ - \lambda(1-\mu)\left(Q^+\right)^T A_\Sigma A_K \Sigma Q^- - (1-\lambda)(1-\mu)\left(Q^+\right)^T A_\Sigma A_K \Sigma Q^+$

In [7], we make use of corollary 5.4 to prove that the matrix M is positive definite up to a measure zero set of initial conditions that lead to saddle points. We first proceed by examining the Gersgorin discs of the first half rows of the matrix M. We denote this procedure as $M_1 - M_2$, as the main diagonal elements of M_1 are "compared" with the corresponding raw elements of M_2. Note that the submatrices M_1, M_2 are both diagonal, therefore the only nonzero elements of raw i of the $4N \times 4N$ matrix M are the elements $M_{ii}, M_{i,2N+i}$ where of course $1 \leq i \leq 2N$ as we calculate the Gersgorin discs of the first half rows of the matrix M. With respect to corollary 5.4,we have:

$$|z - M_{ii}| \leq \tfrac{1}{p_i} \sum_{j \neq i} p_j |M_{ij}|, 1 \leq i \leq 2N \Rightarrow$$
$$\Rightarrow |z - (M_1)_{ii}| \leq \tfrac{p_{2N+i}}{p_i} |(M_2)_{ii}|$$

where $(M_1)_{ii} = A_i^{-2(1+1/k)} K_i G_i^2$,$|(M_2)_{ii}| = \left|A_i^{-2(1+1/k)} \sigma_i K_i G_i \cdot \left\{\lambda\left(Q_{ii}^i\right)^+ + (1-\lambda)\left(Q_{ii}^i\right)^-\right\}\right|$,$\lambda \in [0, 1]$. Denote $\left|\lambda\left(Q_{ii}^i\right)^+ + (1-\lambda)\left(Q_{ii}^i\right)^- | \lambda \in [0, 1]\right| \triangleq \left|\left(Q_{ii}^i\right)^\pm\right|$. It is then obvious that $\left|\left(Q_{ii}^i\right)^\pm\right|_{\max} = \max\left\{\left|\left(Q_{ii}^i\right)^-\right|_{\max}, \left|\left(Q_{ii}^i\right)^+\right|_{\max}\right\}$, which is always bounded in a bounded workspace. Therefore we have:

$$\left|z - A_i^{-2(.)} K_i G_i^2\right| \leq \tfrac{p_{2N+i}}{p_i} \left|A_i^{-2(.)} \sigma_i K_i G_i \left(Q_{ii}^i\right)^\pm\right|$$
$$\Rightarrow z \geq A_i^{-2(.)} K_i G_i^2 - \tfrac{p_{2N+i}}{p_i} \left|A_i^{-2(.)} \sigma_i K_i G_i \left(Q_{ii}^i\right)^\pm\right|$$

We examine the following three cases:

- $G_i \to 0$ In [5] we have proven that this situation occurs whenever the potential function reaches a saddle point. However, the third property of the definition of a navigation function indicates that φ_i is a Morse function, hence its critical points are isolated[12]. Thus the set of initial conditions that lead to saddle points are sets of measure zero[17].
- $G_i > X$ The corresponding eigenvalue is guaranteed to be positive as long as:

$$z > 0 \Leftarrow A_i^{-2(.)} K_i \left(G_i - \tfrac{p_{2N+i}}{p_i} |\sigma_i| \left|\left(Q_{ii}^i\right)^\pm\right|\right) > 0$$
$$\Leftarrow G_i \geq X > \tfrac{p_{2N+i}}{p_i} |\sigma_i| \left|\left(Q_{ii}^i\right)^\pm\right| = \tfrac{\gamma_{di}}{k} \tfrac{p_{2N+i}}{p_i} \left|\left(Q_{ii}^i\right)^\pm\right|$$
$$\Leftarrow k > \tfrac{(\gamma_{di})_{\max}}{X} \tfrac{p_{2N+i}}{p_i} \left|\left(Q_{ii}^i\right)^\pm\right|_{\max}$$

- $0 < \varepsilon \leq G_i \leq X$ In [7], we prove that $|\sigma_i(\varepsilon)| \leq Y\left|\frac{1}{k} + \frac{8}{9}\right| + \left|\frac{\gamma_{di}}{k}\right|$ The corresponding eigenvalue is guaranteed to be positive as long as:

$$z > 0 \Leftarrow \varepsilon > \left\{Y\left|\tfrac{1}{k} + \tfrac{8}{9}\right| + \left|\tfrac{\gamma_{di}}{k}\right|\right\} \tfrac{p_{2N+i}}{p_i} \left|\left(Q_{ii}^i\right)^\pm\right|_{\max}$$
$$\overset{Y \leq \frac{\Theta_1}{k}}{\Longleftarrow}$$

$$k > 2\max\left\{2\sqrt{\frac{\Theta_1}{\varepsilon}}, \frac{16\Theta_1}{9\varepsilon}, \frac{(\gamma_{di})_{\max}}{\varepsilon}\right\} \frac{p_{2N+i}}{p_i} \left|\left(Q_{ii}^i\right)^\pm\right|_{\max}$$

A key point is that there is no restriction on how to select the terms $\frac{p_{2N+i}}{p_i}$. This will help us in deriving bounds that guarantee the positive definiteness of the matrix M.

We are now left to examine the Gersgorin discs of the second half rows of the matrix M. Likewise, we denote this procedure as $M_3 - M_4$. The details of these calculations are omitted here due to lack of space. The discs of Corollary 5.4 are evaluated:

$$|z - M_{ii}| \leq \sum_{j \neq i} \frac{p_j}{p_i} |M_{ij}|, 2N+1 \leq i \leq 4N, 1 \leq j \leq 4N$$
$$\Rightarrow |z - (M_4)_{ii}| \leq R_i(M_3) + R_i(M_4)$$

where $R_i(M_3) = \sum_{j=1}^{2N} \frac{p_j}{p_i} \left|(M_3)_{ij}\right|$, $R_i(M_4) = \sum_{\substack{j=2N+1 \\ j \neq i}}^{4N} \frac{p_j}{p_i} \left|(M_4)_{ij}\right|$. We proceed by proving that $R_i(M_3) \geq R_i(M_4) \forall i$. This is proven in [7] and apparently, it is a consequence of the bounds obtained for k in procedure $M_1 - M_2$.

The corresponding eigenvalue is guaranteed to be positive as long as:

$$z > 0 \Leftarrow (M_4)_{ii} > R_i(M_3) + R_i(M_4)$$
$$\Leftarrow (M_4)_{ii} > \max\{2R_i(M_3), 2R_i(M_4)\} = 2R_i(M_4)$$

It can be shown that this is guaranteed to be the case provided that the parameters $p_i, 2N+1 \leq i \leq 4N$ corresponding to the second half rows of the matrix M are chosen bigger than a finite lower bound.

The preceding analysis guarantees the positive definiteness of the matrix M up to a measure zero set of initial conditions that lead to saddle points.

We can now directly apply theorem 4.6 to our case. We have proved that $v \leq 0 \ \forall v \in \dot{\tilde{\varphi}}$ and that the only invariant subset of the set $S = \{q | 0 \in \dot{\tilde{\varphi}}(q)\}$ is $\left\{q_d = [q_{d1}, ..., q_{dn}]^T\right\}$. Hence the nonsmooth version of LaSalle's invariance principle guarantees convergence to the destination points.

VI. Simulations

To demonstrate the navigation properties of our decentralized approach, we present a simulation of four holonomic agents that have to navigate from an initial to a final configuration, avoiding collision with each other. Each agent has no knowledge of the positions of agents outside its sensing zone, which is the big circle around its center of mass in Fig.3, Pic.1. In this picture A-i,T-i denote the initial condition and desired destination of agent i respectively. The chosen configurations constitute non-trivial setups since the straight-line paths connecting initial and final positions of each agent are obstructed by other agents. The following have been chosen for the simulation of figure 3:

Initial Conditions:

$$q_1(0) = \begin{bmatrix} -.1732 & -.1 \end{bmatrix}^T, q_2(0) = \begin{bmatrix} .1732 & -.1 \end{bmatrix}^T,$$
$$q_3(0) = \begin{bmatrix} 0 & .2 \end{bmatrix}^T, q_4(0) = \begin{bmatrix} 0 & -.2 \end{bmatrix}^T$$

Final Conditions:

$$q_{d1} = \begin{bmatrix} .1732 & .1 \end{bmatrix}^T, q_{d2} = \begin{bmatrix} - .1732 & .1 \end{bmatrix}^T,$$
$$q_{d3} = \begin{bmatrix} 0 & -.1 \end{bmatrix}^T, q_{d4} = \begin{bmatrix} 0 & .25 \end{bmatrix}^T$$

Parameters:

$$k = 110, r_1 = r_2 = r_3 = r_4 = .05, d_C = .11$$

Pictures 1-6 of Figure 3 show the evolution of the team configuration within a horizon of 6000 time units. One can observe that the collision avoidance as well as destination convergence properties are fulfilled.

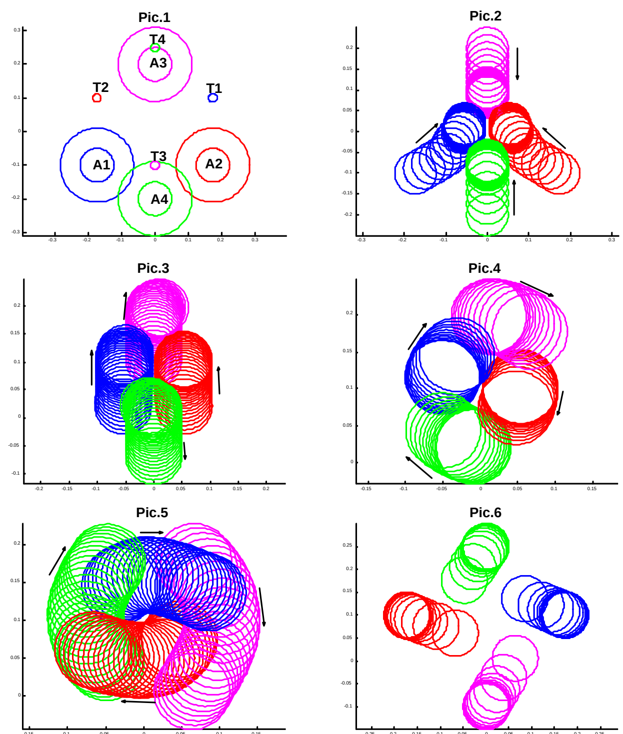

Fig. 3. Simulation A

In the next simulation (Fig.4) the sensing zone of the red agent A2 is shown in all the screenshots. The following have been chosen for the simulation:

Initial Conditions:

$$q_1(0) = \begin{bmatrix} -.1732 & -.1 \end{bmatrix}^T, q_2(0) = \begin{bmatrix} .1732 & -.1 \end{bmatrix}^T,$$
$$q_3(0) = \begin{bmatrix} 0 & .2 \end{bmatrix}^T, q_4(0) = \begin{bmatrix} 0 & -.2 \end{bmatrix}^T$$

Final Conditions:

$$q_{d1} = \begin{bmatrix} .15 & .05 \end{bmatrix}^T, q_{d2} = \begin{bmatrix} - .1732 & .2 \end{bmatrix}^T,$$
$$q_{d3} = \begin{bmatrix} 0 & -.1 \end{bmatrix}^T, q_{d4} = \begin{bmatrix} 0 & .25 \end{bmatrix}^T$$

Parameters:

$$k = 100, r_1 = r_2 = r_3 = r_4 = .03, d_C = .08$$

The collision avoidance and destination requirements are met in this case as well. We point out that since the sensing zone of the red agent is always empty,i.e. it does not participate in a conflict situation, its trajectory is the straight line between its initial and final destination. This is due to

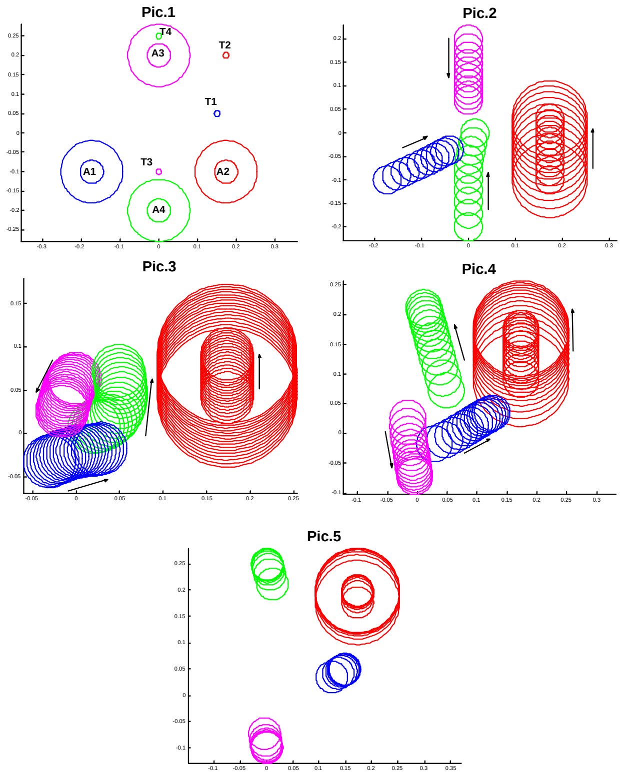

Fig. 4. Simulation B

the fact that the sensing parameter d_C is small in this case. A bigger choice of d_C would force the red agent to alter its trajectory, because in this case it would have to take the positions of the other agents into account at each time instant.

VII. Conclusions

In this paper we extended the decentralized navigation method to the case of multiple holonomic agents with limited sensing capabilities. We proposed a nonsmooth extension of the navigation function of [5] and proved system convergence using tools from nonsmooth stability analysis. The effectiveness of the methodology is verified through computer simulations.

Current research includes applying this method to the case of distributed nonholonomic agents [16] as well as introducing new definitions of the sensing zone of an agent. Extensions of this method to 3-dimensional dynamics are also under investigation.

VIII. Acknowledgements

The authors want to acknowledge the contribution of the European Commission through contract HYBRIDGE:*Distributed Control and Stochastic Analysis of Hybrid Systems Supporting Safety Critical Real-Time Systems Design*(IST-2001-32460).

References

[1] A. Bicchi and L. Pallottino, On Optimal Cooperative Conflict Resolution for Air Traffic Management Systems, *IEEE Transactions on Intelligent Transportation Systems*,1(4): 221-232, 2000.
[2] F. Ceragioli, Discontinuous Ordinary Differential Equations and Stabilization, PhD Thesis, Dept. of Mathematics, Universita di Firenze, 1999.
[3] F.H. Clarke,*Optimization and Nonsmooth Analysis*, Wiley & Sons, 1983.
[4] Department of Transport UK Government, Air Traffic Forecasts for the United Kingdom 1997, Technical Report, 1997.
[5] D. V. Dimarogonas, M. M. Zavlanos, S. G. Loizou and K. J. Kyriakopoulos, Decentralized Motion Control of Multiple Holonomic Agents under Input Constraints ,in *42th IEEE Conf. Decision and Control*, Maui, Hawaii, 2003.
[6] D. V. Dimarogonas, M. M. Zavlanos, S. G. Loizou and K. J. Kyriakopoulos, Decentralized Motion Control of Multiple Holonomic Agents under Input Constraints , Technical Report, http://users.ntua.gr/ddimar/TechRep0301.pdf, NTUA, 2003.
[7] D. V. Dimarogonas and K. J. Kyriakopoulos, Decentralized Motion Control of Multiple Holonomic Agents with Limited Sensing Capabilities , Technical Report, NTUA, 2004.
[8] M.Egerstedt,X.Hu, A Hybrid Control Approach to Action Coordination for Mobile Robots, *Automatica*,vol.38,pp.125-130,2002.
[9] A.F. Filippov,*Differential Equations with Discontinuous Righthand Sides*, Mathematics and its Applications (Soviet Series),Kluwer Academic Publishers, 1988.
[10] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, 1996.
[11] G. Inalhan, D.M. Stipanovic and C.J. Tomlin, Decentralized Optimization, with Application to Multiple Aircraft Coordination ,in *41th IEEE Conf. Decision and Control*, Las Vegas, 2002.
[12] D.E. Koditscheck and E. Rimon, Robot Navigation Functions on Manifolds with Boundary, *Advances Appl. Math.* , 11:412-442, 1990.
[13] J.C. Latombe, *Robot Motion Planning*, Kluwer Academic Publishers, 1991.
[14] C. Livadas, J. Lygeros and N.A. Lynch, High-Level Modeling and Analysis of the Traffic Alert and Collision Avoidance System (TCAS), *Proceedings of the IEEE*, 88(7): 949-970, 2000.
[15] S.G. Loizou and K.J. Kyriakopoulos, Closed Loop Navigation for Multiple Holonomic Vehicles,in *2002 IEEE/RSJ International Conference on Intelligent Robots and Systems*, pp 2861 - 2866, 2002.
[16] S.G. Loizou, D.V. Dimarogonas and K.J. Kyriakopoulos, Decentralized Feedback Stabilization of Multiple Nonholonomic Agents, in *2004 IEEE International Conference on Robotics and Automation* .
[17] J. Milnor, *Morse theory*, Annals of Mathematics Studies, Princeton University Press, Princeton, NJ, 1963.
[18] T.S. Perry, In Search of the Future of Air Traffic Control, *IEEE Spectrum*, 34(8):18-34, 1997.
[19] D. Shevitz and B. Paden, Lyapunov Stability Theory of Nonsmooth Systems, *IEEE Transactions on Automatic Control*, 39(9):1910-1914, 1994.
[20] H.G. Tanner, A. Jadbabaie and G.J. Pappas, Stable Flocking of Mobile Agents Part II: Dynamic Topology , in *42th IEEE Conf. Decision and Control*, Maui, Hawaii, 2003.
[21] M.M. Zavlanos and K.J. Kyriakopoulos, Decentralized Motion Control of Multiple Mobile Agents, in *11th Mediterranean Conference on Control and Automation*, Rhodes, Greece, 2003.