On Observability and Detectability of Continuous-time Linear Switching Systems [1]

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Abstract

The notion of observability and detectability for a particular class of hybrid systems, linear continuous-time switching systems, is investigated. We compare some of the definitions of observability previously offered and we analyze their drawbacks. A novel definition of observability is proposed corresponding to the possibility of reconstructing the state of the system from the knowledge of the discrete and continuous outputs and inputs. The notion of detectability is also introduced. Sufficient and necessary conditions for these properties to hold for switching systems are presented.

1 Introduction

An important application of hybrid system technology is Air Traffic Management Systems where different types of dynamics coexist and interact in non obvious ways. For example, continuous dynamics are used to model the aircraft motion, while the change in cruising altitude and the transition from one sector to another are modeled using discrete dynamics. To complicate matters further, uncertainty affects these systems in substantial ways. Disturbances and human behavior including errors are examples of uncertainty that is important to deal with to ensure safe operation. Error detection and control must rely upon robust state estimation techniques, thus providing a strong motivation for a rigorous approach to observability and detectability based on tests of affordable computational complexity.

Observability, a fundamental property of systems, has been extensively studied both in the continuous ([9], [10]) and in the discrete domains (see e.g. [13], [14]). In particular, Sontag in [15] defined different observability concepts and analyzed their relations for polynomial systems. More recently, various researchers have approached the study of observability for hybrid systems, but the definitions of and the testing criteria for it varied depending on the class of systems under consideration and on the knowledge that is assumed at the output. Vidal et al. in [17] considered autonomous switching systems and proposed a definition of observability based on the concept of indistinguishability of continuous initial states and discrete state evolutions from the outputs in free evolution. Incremental observability was introduced in [3] for the class of piecewise affine (PWA) systems. Incremental observability implies that different initial states always give different outputs independently of the applied input. In [1], a methodology was presented for the design of dynamic observers of hybrid systems that reconstructs the discrete state and the continuous state from the knowledge of the continuous and discrete outputs. In [8] the definitions of observability of [16] and the results of [1] on the design of an observer for deterministic hybrid systems are extended to discrete–time stochastic linear autonomous hybrid systems. In [2], the notion of generic final-state determinability proposed by Sontag [15] is extended to hybrid systems and sufficient conditions are given for linear hybrid systems.

In this paper, we address the issues of observability and detectability, a property that to the best of our knowledge has not been addressed so far, for a class of hybrid systems, switching linear systems with minimum and maximum dwell time. The reason for this choice of a subclass of hybrid systems is our interest in observability and detectability testable conditions. The paper is organized as follows: in Section 2, we review a set of formal definitions for switching systems. In Section 3, we show, on the basis of some examples, that the observability notions based on state indistinguishability do not imply state reconstructability. We therefore propose a new definition of observability, and a weaker notion of detectability, based on the possibility of reconstructing the system state. We then give some necessary and sufficient testable conditions for observability. We also derive a Kalman-like decomposition of the switching system and propose conditions for detectability based on this decomposition. Conclusions are offered in Section 4. The results are given without proof for lack of space. A full version of this paper is in [7].

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2 Switching systems

The class of switching systems we consider in this paper are defined as in [4], following the general model of hybrid automata given in [11]:

Definition 1 *A continuous-time linear switching system $\mathcal{S}$ is a tuple $(\mathbf{Q}, \mathbf{P}, \mathbf{U}_D, \gamma, \mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p, \mathbf{S}_c, S, E, R)$ where $\mathbf{Q} = \{q_i, i \in J\}$ is the finite set of discrete states, being $J = \{1, 2, ..., N\}, N \in \mathbb{N}$; $\mathbf{P} = \{p_h, h \in J\}$ is the set of discrete outputs; $\gamma : \mathbf{Q} \to \mathbf{P}$ is a mapping that associates a discrete output to each discrete state; $\mathbf{U}_D$ is the finite set of the discrete disturbances; $\mathbb{R}^n, \mathbb{R}^m$ and $\mathbb{R}^p$ are respectively the continuous state, input and output spaces; we denote by $\mathcal{U}_c$ the class of piecewise continuous functions $u : \mathbb{R} \to \mathbb{R}^m$; $\mathbf{S}_c$ is a subclass of linear, continuous-time dynamical systems, and $S_i \in \mathbf{S}_c$ is defined by the equations $\dot{x}(t) = A_i x(t) + B_i u(t)$, $y(t) = C_i x(t)$, $i \in J$; $S : \mathbf{Q} \to \mathbf{S}_c$ is a mapping that associates a continuous-time dynamical system to every discrete state[1]; $E \subset \mathbf{Q} \times \mathbf{U}_D \times \mathbf{Q}$ is a collection of discrete transitions; $R : E \times \mathbb{R}^n \to \mathbb{R}^n$ is the reset function.*

Given a switching system $\mathcal{S} = (\mathbf{Q}, \mathbf{P}, \mathbf{U}_D, \gamma, \mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p, \mathbf{S}_c, S, E, R)$, the tuple $\mathcal{D}_{\mathcal{S}} = (\mathbf{Q}, \mathbf{P}, \mathbf{U}_D, \gamma, E)$ can be viewed as a Finite State Machine (FSM), having state set $\mathbf{Q}$, input set $\mathbf{U}_D$, output set $\mathbf{P}$, transition relation E and output function γ. This FSM characterizes the structure of the discrete transitions and without loss of generality (w.l.o.g.) is supposed to be connected. A strongly connected component of the FSM is a maximal set of mutually reachable discrete states. A strongly connected component of a FSM is said to be *proper* if it contains more than one discrete state. In what follows, $y_i(t, 0, u)$ denotes the continuous output of system $S(q_i)$ at time t, with continuous initial state $x_0 = 0$ and control law $u|_{[t_0, t)}$.

Following [11], we recall that a hybrid time basis τ is an infinite or finite sequence of sets I_j satisfying the following conditions: $I_j = \left\{t \in \mathbb{R} : t_j \le t \le t'_j\right\}$; if $card(\tau) = L + 1 < \infty$, then I_L may be of the form $I_L = \{t \in \mathbb{R} : t_L \le t < \infty\}$ and $t'_L = \infty$ or of the form $I_L = [t_L, t'_L)$ with $t'_L < \infty$; for all j, $t_j \le t'_j$ and for $j > 0$, $t_j = t'_{j-1}$. Denote by $\mathcal{T}$ the set of all hybrid time bases. The switching system temporal evolution is then defined as follows:

Definition 2 *(Switching System Execution) An execution χ of a switching system $\mathcal{S}$ is a collection $\chi = (\widehat{q}, \xi_0, \tau, \sigma, q, p, u, \xi, \eta)$ with $\widehat{q} \in \mathbf{Q}$, $\xi_0 \in \mathbb{R}^n$, $\tau \in \mathcal{T}$, $\sigma : \tau \to \mathbf{U}_D$, $q : \tau \to \mathbf{Q}$, $p : \tau \to \mathbf{P}$, $u \in \mathcal{U}_c$, $\xi : \mathbb{R} \times \mathbb{N} \to \mathbb{R}^n$, $\eta \in \mathbb{R} \times \mathbb{N} \to \mathbb{R}^p$ satisfying:*

- *Discrete evolution: $q(I_0) = \widehat{q}$; $q(I_{j+1})$ is such that $e_j \in E$ and $e_j = (q(I_j), \sigma(I_{j+1}), q(I_{j+1}))$; $p(I_j) = \gamma(q(I_j))$;*
- *Continuous evolution: $\forall t \in I_j$ the functions ξ and η are such that $\xi(t_0, 0) = \xi_0$, $\xi(t_{j+1}, j+1) = R\left(e_j, \xi\left(t'_j, j\right)\right)$, $\xi(t, j) = x(t)$ and $\eta(t, j) = y(t)$, where $x(t)$ (resp. $y(t)$) is the unique solution (resp. output) at time t of the dynamical system $S(q(I_j))$, with initial condition $x(t_j) = \xi(t_j, j)$ and input function u.*

[1] For the sake of notational simplicity, we assume $S_i = S(q_i)$.

Given an execution $\chi = (\widehat{q}, \xi_0, \tau, \sigma, q, p, u, \xi, \eta)$, we will say that χ is an execution of $\mathcal{S}$ with initial state $\begin{pmatrix} \xi_0 \\ \widehat{q} \end{pmatrix} \in \mathbb{R}^n \times \mathbf{Q}$ and that the sequence of discrete disturbances σ is *admissible* with respect to the discrete initial state $\widehat{q}$. Following [11], an execution is *infinite* if $card(\tau) = \infty$ or $t'_L = \infty$; *Zeno* if $card(\tau) = \infty$ and $\sum_{j=0}^{card(\tau)} t'_j - t_j < \infty$. A switching system is Zeno if at least one execution is Zeno. We assume the existence of a *minimum dwell time* [12] before which no discrete disturbance causes a discrete transition, and of a *maximum dwell time* within which a discrete disturbance is guaranteed to cause a transition according to the following

Assumption 1: (Minimum and maximum dwell time) Given the switching system $\mathcal{S}$, $0 < \delta_m \le t'_j - t_j \le \delta_M$, $\forall j = 0, 1, ..., card(\tau) - 1$, for any switching system execution. The value δ_M can be finite or infinite.

Assumption 1 implies that all executions of $\mathcal{S}$ are non-Zeno. Then, if there is no maximum dwell time, i.e. $\delta_M = \infty$, all executions may be assumed w.l.o.g. to be infinite. If there is a maximum dwell time, i.e. $\delta_M < \infty$, we assume that any execution is non-blocking as follows:

Assumption 2: Given the switching system $\mathcal{S}$, the FSM $\mathcal{D}_{\mathcal{S}}$ is alive [13], i.e. for any $q \in \mathbf{Q}$ there exist $\sigma \in \mathbf{U}_D$ and $q' \in \mathbf{Q}$ such that $(q, \sigma, q') \in E$.

The switching system $\mathcal{S}$ is said to be alive if $\mathcal{D}_{\mathcal{S}}$ is alive. Under Assumptions 1 and 2, any execution of $\mathcal{S}$ may be assumed w.l.o.g. to be infinite.

Assumption 3: The reset function is linear, i.e. $R(e, x) = M_e x$, $e \in E$.

Remark: Suppose that linear state space transformations for the systems S_i and S_j are applied, represented by the nonsingular matrices T_i and T_j, respectively. Then, for any transition $e = (q_i, \sigma, q_j)$, the matrix M_e representing the reset function is transformed into the matrix $T_j M_e T_i^{-1}$.

Assumptions 1, 2 and 3 hold throughout the paper. By abuse of notation, the symbol $\mathcal{T}$ will denote the set of all hybrid time bases that satisfy Assumption 1. The symbol $\mathcal{R}(.)$ will denote the range space and $f^{-1}(.)$ the inverse image operator of $f(.)$. Given a switching system $\mathcal{S}$ and an execution χ, consider the functions $y_c : \mathbb{R} \to \mathbb{R}^p$and $y_d : \mathbb{R} \to \mathbf{P}$, where $y_c(t) = \eta(t, j)$, $y_d(t) = p(I_j)$, $t \in [t_j, t'_j)$, $j = 0, 1, ..., card(\tau) - 1$. Let $\mathcal{Y}_c$ and $\mathcal{Y}_d$ be the classes of piecewise continuous functions y_c and y_d, respectively. The pair $\left(y_c|_{[t_0,t]}, y_d|_{[t_0,t]} \right)$ is said to be the *observed output* at time t of the switching system $\mathcal{S}$. Time t'_j is said to be a *switching time.*

3 Observability and detectability of switching systems

In this section, we show that, for the class of switching systems, some existing definitions of observability that have been proposed for hybrid systems and reviewed in [6] do not allow the reconstruction of the state of the system.

Vidal et al. considered in [17] "jump linear systems", that are autonomous switching systems (i.e. $B_i = 0$, $\forall i \in J$) having a minimum dwell time $\delta_m > 0$, and proposed a notion of observability based on the concept of indistinguishability of continuous initial states and discrete state evolutions. This notion of observability is rather strong since it considers only the free response to reconstruct the state. In fact, consider a switching system $\mathcal{S} = (\mathbf{Q}, \mathbf{P}, \mathbf{U}_D, \gamma, \mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p, \mathbf{S}_c, S, E, R)$ and let X_0 be a set of initial states such that, for any $x_0 \in X_0$, the systems in $\mathbf{S}_c$ have the same free continuous output. However, assume that all the systems in $\mathbf{S}_c$ are observable and that there exists an input $u \in \mathcal{U}_c$ and $\Delta \in (0, \delta_m)$ such that $\int_0^\Delta \|y_i(s) - y_j(s)\| \, ds > 0, \forall i, j \in J$, where $y_i(t)$ and $y_j(t)$ are the outputs at time t of systems S_i and S_j, respectively, starting from initial states in X_0, under the same input function u. Then, even though $\mathcal{S}$ is not observable in the sense of [17], at time $t_j + \Delta$ the discrete state $q(I_j)$ can be determined, $\forall j = 0, 1, ..., card(\tau) - 1$, and the continuous state $\xi(t, j)$ can be reconstructed $\forall t \in (t_j + \Delta, t'_j]$, $\forall j = 0, 1, ..., card(\tau) - 1$, for a suitable input function.

The forced response of the system is used in [3] where, for the class of piecewise affine (PWA) systems, a notion of observability, called incremental observability, is introduced. Informally, a PWA system is said to be incrementally observable if for any pair of continuous initial states in a given state set and for any input sequence in a given input set, the output trajectories are "sufficiently" different. In other words, incremental observability implies that different initial states always give different outputs independently of the applied input. The definition of incremental observability of [3] can be trivially extended to the class of continuous-time switching systems that we are considering. To better analyze the consequences of such a definition, consider a switching system $\mathcal{S} = (\mathbf{Q}, \mathbf{P}, \mathbf{U}_D, \gamma, \mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p, \mathbf{S}_c, S, E, R)$ with minimum dwell time $\delta_m > 0$. Assume that all dynamic systems in $\mathbf{S}_c$ are controllable, $\mathbb{R}^p = \mathbb{R}^n$, and suppose that the matrices C_i are nonsingular and are such that $\rho(C_i - C_j) = n$, $\forall i, j \in J$, $i \neq j$. In that case, $\forall x \in \mathbb{R}^n - \{0\}$, $C_i x \neq C_j x$. Therefore, for any pair of initial states $\begin{pmatrix} \xi \\ q_i \end{pmatrix}$ and $\begin{pmatrix} \xi \\ q_j \end{pmatrix} \in X_0 \subset \mathbb{R}^n - \{0\} \times \mathbf{Q}$, $i \neq j$, and for any input function, the output functions of the switching system $\mathcal{S}$ do not coincide, for any execution of $\mathcal{S}$. Hence, $\mathcal{S}$ is incrementally observable for any set of initial states $X_0 \subset \mathbb{R}^n - \{0\} \times \mathbf{Q}$. However, there exist input functions such that the discrete state evolution of $\mathcal{S}$ cannot be reconstructed. In fact, since the systems in $\mathbf{S}_c$ are controllable, for all x belonging to any subset of $\mathbb{R}^n - \{0\}$ and for any $\widehat{t} \in (t_0, t_0 + \delta_m)$ there exists an input function such that $\xi(t, 0) = 0$, $\forall t \geq \widehat{t}$. As a consequence, it is not always possible to reconstruct the discrete state evolution, even if the state $q(I_0)$ were known. This shows that, for switching systems, incremental observability, based on a distinguishability property that holds for any input, does not guarantee state reconstruction.

Consider now a definition of observability based on distinguishability of initial states from the observed output, for a *suitable* input function. The following example shows that this notion has problems too for state reconstruction. Suppose that the matrices describing the dynamic systems S_i in $\mathbf{S}_c$ are in the observability canonical form, i.e.

$$A_i = \begin{pmatrix} A_{i11} & 0 \\ A_{i21} & A_{i22} \end{pmatrix}, B_i = \begin{pmatrix} B_{i1} \\ B_{i2} \end{pmatrix}, C_i = \begin{pmatrix} C_{i1} & 0 \end{pmatrix} \tag{1}$$

where $A_{i22} \in \mathbb{R}^{d_i \times d_i}$ and d_i is the dimension of the unobservable subspace $\mathcal{O}_i$ of system S_i, $i \in J$. Consider the switching system $\mathcal{S} = (\mathbf{Q}, \mathbf{P}, \mathbf{U}_D, \gamma, \mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p, \mathbf{S}_c, S, E, R)$ with $\delta_m > 0$ and $\delta_M < \infty$, where $\mathbf{Q} = \{q_1, q_2, q_3, q_4\}$, $\mathbf{P} = \{p_1, p_2, p_3\}$, $\mathbf{U}_D = \{\sigma\}$, $\gamma(q_1) = p_1$, $\gamma(q_2) = p_2$, $\gamma(q_3) = \gamma(q_4) = p_3$, $E = \{(q_3, \sigma, q_1), (q_4, \sigma, q_1), (q_1, \sigma, q_2), (q_2, \sigma, q_1)\}$, the systems S_1 and S_2 are observable, the dynamical matrices describing systems S_3 and S_4 are such that $A_{3,22} = A_{4,22} \in \mathbb{R}^{d_3 \times d_3}$, $B_3 = B_4 = 0$ in the observability canonical form, and the reset function is the identity. Any pair of initial states $\begin{pmatrix} \xi_0 \\ q_3 \end{pmatrix}$ and $\begin{pmatrix} \xi_0 \\ q_4 \end{pmatrix}$, with $\xi_0 = \begin{pmatrix} 0 \\ \xi'_0 \end{pmatrix}$, $\xi'_0 \in \mathbb{R}^{d_3}$, is indistinguishable, since for any input function $u \in \mathcal{U}_c$, the same output functions are observed. However, after the first switching, the discrete state evolution can be uniquely determined from the discrete output and the continuous state evolution can be reconstructed for any continuous input function, since S_1 and S_2 are observable.

Consequently, observability notions based on state indistinguishability do not imply state reconstructability. One of the concepts introduced in [15] based on state reconstruction is the so-called generic finite-state determinability. Generic finite-state determinability implies that *any* input/output experiment allows the determination of the state. In [2], this property was extended to hybrid systems and testable sufficient conditions were given. In this paper, we modify the notion of observability given in [15] for switching systems by focusing on state reconstruction. We therefore propose the following new definition.

Definition 3 *A switching system* $\mathcal{S} = (\mathbf{Q}, \mathbf{P}, \mathbf{U}_D, \gamma, \mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p, \mathbf{S}_c, S, E, R)$ *is **observable** if there exist a function* $\varphi : \mathcal{Y}_c \times \mathcal{Y}_d \times \mathcal{U}_c \rightarrow \mathbb{R}^n \times \mathbf{Q}$*, an integer* $\mathbf{j} \geq 0$ *and a real* $\Delta \in (0, \delta_m)$ *such that* $\forall \begin{pmatrix} \xi_0 \\ \widehat{q} \end{pmatrix} \in \mathbb{R}^n \times \mathbf{Q}$*,* $\forall \tau \in \mathcal{T}$*,* $\forall \sigma$ *admissible w.r.t.* $\widehat{q}$ *there exists an execution* $\chi = (\widehat{q}, \xi_0, \tau, \sigma, q, p, u, \xi, \eta)$ *such that* $\varphi\left(y_c|_{[t_0,t]}, y_d|_{[t_0,t]}, u|_{[t_0,t)} \right) = \begin{pmatrix} \xi(t,j) \\ q(I_j) \end{pmatrix}$*,* $\forall t \in \left(t_j + \Delta, t'_j\right], \forall j = \mathbf{j}, ..., card(\tau) - 1$*.*

Definition 4 *A switching system* $\mathcal{S} = (\mathbf{Q}, \mathbf{P}, \mathbf{U}_D, \gamma, \mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p, \mathbf{S}_c, S, E, R)$ *is **detectable** if there exist a function* $\varphi : \mathcal{Y}_c \times \mathcal{Y}_d \times \mathcal{U}_c \rightarrow \mathbb{R}^n \times \mathbf{Q}$*, an integer* $\mathbf{j} \geq 0$ *and a real* $\Delta \in (0, \delta_m)$ *such that, by setting* $\varphi(.) = \begin{pmatrix} \varphi_{\mathbb{R}^n}(.) \in \mathbb{R}^n \\ \varphi_{\mathbf{Q}}(.) \in \mathbf{Q} \end{pmatrix}$*,* $\forall \begin{pmatrix} \xi_0 \\ \widehat{q} \end{pmatrix} \in \mathbb{R}^n \times \mathbf{Q}, \forall \tau \in \mathcal{T}$*,* $\forall \sigma$ *admissible w.r.t.* $\widehat{q}$*,* $\forall \varepsilon > 0$*, there exist an execution* $\chi = (\widehat{q}, \xi_0, \tau, \sigma, q, p, u, \xi, \eta)$ *and* $\mathbf{t} > t_0$ *such that*

$$(i) \quad \varphi_{\mathbf{Q}}\left(y_c|_{[t_0,t]}, y_d|_{[t_0,t]}, u|_{[t_0,t)} \right) = q(I_j), \quad \forall j = \mathbf{j}, ..., card(\tau) - 1$$

$$(ii) \quad \left\| \varphi_{\mathbb{R}^n}\left(y_c|_{[t_0,t]}, y_d|_{[t_0,t]}, u|_{[t_0,t)} \right) - \xi(t,j) \right\| \leq \varepsilon; \quad \forall t \in [\mathbf{t}, \infty) \cap \left(t_j + \Delta, t'_j\right], \forall j = \mathbf{j}, ..., card(\tau) - 1.$$

The above definitions reduce to the standard concepts of observability and detectability for classical dynamical linear systems.

3.1 Switching systems with $\delta_m > 0$ and $\delta_M \leq \infty$

In this subsection, we give conditions that are sufficient and in some cases necessary for observability and detectability of switching systems. In the case $\delta_M = \infty$, no switching might occur. Hence, the switching system is observable in the sense of Definition 3 if and only if it is possible to reconstruct any hybrid initial state and the first switching time from the input function and the observed output. Such condition becomes sufficient if $\delta_M < \infty$.

We need the following definition:

Definition 5 *Given a switching system, the hybrid initial state is reconstructable if there exist a function* $\widetilde{\varphi} : \mathcal{Y}_c \times \mathcal{Y}_d \times \mathcal{U}_c \rightarrow \mathbb{R}^n \times \mathbf{Q}$ *and a real* $\Delta > 0$ *such that* $\forall \begin{pmatrix} \xi_0 \\ \widehat{q} \end{pmatrix} \in \mathbb{R}^n \times \mathbf{Q}$*,* $\forall \tau \in \mathcal{T}$*,* $\forall \sigma$ *admissible w.r.t.* $\widehat{q}$ *there exists an execution* $\chi = (\widehat{q}, \xi_0, \tau, \sigma, q, p, u, \xi, \eta)$ *such that* $\widetilde{\varphi}\left(y_c|_{[t_0,t_0+\Delta]}, y_d|_{[t_0,t_0+\Delta]}, u|_{[t_0,t_0+\Delta)} \right) = \begin{pmatrix} \xi_0 \\ \widehat{q} \end{pmatrix}$*. The first switching time is reconstructable if there exist a function* $\widehat{\varphi} : \mathcal{Y}_c \times \mathcal{Y}_d \times \mathcal{U}_c \rightarrow \mathbb{R}$ *and a nonnegative real* $\Delta < \delta_m$ *such that* $\forall \begin{pmatrix} \xi_0 \\ \widehat{q} \end{pmatrix} \in \mathbb{R}^n \times \mathbf{Q}$*,* $\forall \tau \in \mathcal{T}$*,* $\forall \sigma$ *admissible w.r.t.* $\widehat{q}$ *there exists an execution* $\chi = (\widehat{q}, \xi_0, \tau, \sigma, q, p, u, \xi, \eta)$ *such that* $\widehat{\varphi}\left(y_c|_{[t_0,t_0+\Delta]}, y_d|_{[t_0,t_0+\Delta]}, u|_{[t_0,t_0+\Delta)} \right) = t'_0$*.*

Our first result completely characterizes the hybrid initial state reconstruction.

Proposition 6 *Given a switching system* $\mathcal{S} = (\mathbf{Q}, \mathbf{P}, \mathbf{U}_D, \gamma, \mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p, \mathbf{S}_c, S, E, R)$*, the hybrid initial state is reconstructable if and only if* $S(q_i)$ *is observable for any* $q_i \in Q$ *and*

$$\begin{array}{c} \forall p \in \mathcal{R}(\gamma), \exists u^* : \forall q_i, q_j \in \gamma^{-1}(p) \\ y_i(t, 0, u^*) \neq y_j(t, 0, u^*), a.e.\ t \geq 0 \end{array} \tag{2}$$

or, equivalently,

$$\begin{array}{c} \forall p \in \mathcal{R}(\gamma), \forall q_i, q_j \in \gamma^{-1}(p), \exists k \in \mathbb{N} \cup \{0\} : \\ C_i A_i^k B_i \neq C_j A_j^k B_j \end{array} \tag{3}$$

The following result characterizes the first switching time reconstruction.

Proposition 7 *Given a switching system* $\mathcal{S} = (\mathbf{Q}, \mathbf{P}, \mathbf{U}_D, \gamma, \mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p, \mathbf{S}_c, S, E, R)$*, the first switching time is reconstructable if*

$$\begin{array}{c} \forall q_i \in \mathbf{Q}, \forall q_j \in J_i, \exists k \in \mathbb{N} \cup \{0\} : \\ C_i A_i^k B_i \neq C_j A_j^k B_j \end{array} \tag{4}$$

where $J_i = \{q \in \mathbf{Q} : (q_i, \sigma, q) \in E$ *for some* $\sigma \in U_D$ *and* $\gamma(q_i) = \gamma(q)\}$*.*

By combining Propositions 6 and 7, we obtain the following result.

Theorem 8 *A switching system $\mathcal{S} = (\mathbf{Q}, \mathbf{P}, \mathbf{U}_D, \gamma, \mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p, \mathbf{S}_c, S, E, R)$ is observable if the following conditions are satisfied:*

(i) $S(q_i)$ is observable for any $q_i \in \mathbf{Q}$;

(ii) $\forall p \in \mathcal{R}(\gamma)$, $\forall q_i, q_j \in \gamma^{-1}(p)$, $\exists k \in \mathbb{N} \cup \{0\}$: $C_i A_i^k B_i \neq C_j A_j^k B_j$;

Moreover, if $\delta_M = \infty$, then conditions (i) and (ii) are necessary.

We now state conditions for detectability of a switching system.

As in the case of a linear dynamic system, the switching system can be decomposed into two subsystems. Given $\mathcal{S} = (\mathbf{Q}, \mathbf{P}, \mathbf{U}_D, \gamma, \mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p, \mathbf{S}_c, S, E, R)$, where it is assumed w.l.o.g. that the dynamical systems are in observability canonical form (1), define the switching system $\widetilde{\mathcal{S}} = (\mathbf{Q}, \mathbf{P}, \mathbf{U}_D, \gamma, \mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p, \widetilde{\mathbf{S}}_c, \widetilde{S}, E, \widetilde{R})$ with $\widetilde{S}(q_i) \in \widetilde{\mathbf{S}}_c$ described by

$$\begin{array}{ll} \dot{x}(t) = \begin{pmatrix} 0 & 0 \\ 0 & A_{i22} \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ B_{i2} \end{pmatrix} u(t), & \\ y(t) = \begin{pmatrix} 0 & 0 \end{pmatrix} x(t), & i \in J, \end{array}$$

and $\widetilde{R}(e, x) = \widetilde{M}_e x$, $e = (q_i, \sigma, q_j)$, $\widetilde{M}_e = \begin{pmatrix} 0 & 0 \\ 0 & I_{d_j \times d_j} \end{pmatrix} M_e \in \mathbb{R}^{n \times n}$ and $I_{d_j \times d_j} \in \mathbb{R}^{d_j \times d_j}$ is the identity matrix. Define now the switching system $\mathcal{S}_o = \left(\mathbf{Q}, \mathbf{P}, \mathbf{U}_D, \gamma, \mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p, \widehat{\mathbf{S}}_c, \widehat{S}, E, \widehat{R}\right)$ with $\widehat{S}(q_i) \in \widehat{\mathbf{S}}_c$ described by

$$\begin{array}{ll} \dot{x}(t) = \begin{pmatrix} A_{i11} & 0 \\ 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} B_{i1} \\ 0 \end{pmatrix} u(t), & \\ y(t) = \begin{pmatrix} C_{i1} & 0 \end{pmatrix} x(t), & i \in J, \end{array}$$

and $\widehat{R}(e, x) = \widehat{M}_e x$, $e = (q_i, \sigma, q_j)$, $\widehat{M}_e = \begin{pmatrix} I_{(n-d_j) \times (n-d_j)} & 0 \\ 0 & 0 \end{pmatrix} M_e \in \mathbb{R}^{n \times n}$. The decomposition above can be seen as an extension of the classical Kalman decomposition for linear dynamical system to switching systems. The following result, based on this decomposition, characterizes detectability of a switching system in terms of properties related to the observability of $\mathcal{S}_o$ and to the asymptotic stability of $\widetilde{\mathcal{S}}$.

Theorem 9 *A switching system $\mathcal{S} = \{\mathbf{Q}, \mathbf{P}, \mathbf{U}_D, \gamma, \mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p, \mathbf{S}_c, S, E, R\}$ is detectable if the following conditions hold:*

(i) $\forall p \in \mathcal{R}(\gamma)$, $\forall q_i, q_j \in \gamma^{-1}(p)$, $\exists k \in \mathbb{N} \cup \{0\}$: $C_{i1} A_{i11}^k B_{i1} \neq C_{j1} A_{j11}^k B_{j1}$;

(ii) for any initial state $\begin{pmatrix} \xi_0 \\ q_i \end{pmatrix} \in \mathcal{O}_i \times \mathbf{Q}$, $i \in J$, and for any $\varepsilon > 0$ there exists $\mathbf{t} > t_0$ such that $\|\xi(t, j)\| \leq \varepsilon$, for any $t \geq \mathbf{t}$, for any execution of $\widetilde{\mathcal{S}}$ with $u(t) = 0$, for any $t \geq t_0$.

As it is for stabilizability and safety properties (see [5]), observability and detectability of a switching system may be assessed on its strongly connected components. More precisely,

Proposition 10 *A switching system $\mathcal{S}$ with $\delta_m > 0$ and $\delta_M = \infty$ is observable (resp. detectable) if and only if each strongly connected component of $\mathcal{S}$ is observable (resp. detectable) .*

3.2 Switching systems with $\delta_m > 0$ and $\delta_M < \infty$
In this subsection we characterize observability of switching systems with minimum dwell time and finite maximum dwell time. In this case, under the liveness assumption, each infinite execution is such that $card\,(\tau) = \infty$. Moreover, the observability (resp. detectability) of the switching system $\mathcal{S}$ with $\delta_m > 0$ and $\delta_M < \infty$ does not imply the observability (resp. detectability) of the systems S_i, $\forall i \in J$. Observability and detectability of a switching system with $\delta_m > 0$ and $\delta_M < \infty$ may be assessed on its proper strongly connected components. More precisely, Proposition 10 becomes :

Proposition 11 *A switching system $\mathcal{S}$ with $\delta_m > 0$ and $\delta_M < \infty$ is observable (resp. detectable) if and only if each proper strongly connected component is observable (resp. detectable).*

We first recall from [1] that an alive finite state machine is **current-state observable** if there exists a positive integer K such that, for every $h \geq K$ and for any unknown initial state $q(I_0)$, the state $q(I_h)$ can be determined from the output sequence $p(I_i)$, $i = 0 \ldots K$, for every possible sequence $\sigma(I_i)$, $i = 0 \ldots K-1$. It is important to highlight that current state observability of $\mathcal{D}_{\mathcal{S}}$ does not imply in general discrete state observability of the switching system $\mathcal{S}$, since the switching times cannot always be determined from the observed discrete output y_d. This is why, in the following theorem, we need to assume that the switching times t_j can be determined from y_d.

Theorem 12 *Given a switching system $\mathcal{S}$ with $\delta_m > 0$ and $\delta_M < \infty$, assume $S(q_i)$ observable, $\forall i \in J'$ where J' denote the set of all indices associated with discrete states belonging to proper strongly connected components of the FSM $\mathcal{D}_{\mathcal{S}}$. $\mathcal{S}$ is observable if $\mathcal{D}_{\mathcal{S}}$ is current-state observable. Moreover $\mathcal{S}$ is detectable if $\mathcal{D}_{\mathcal{S}}$ is current-state observable and $\forall \begin{pmatrix} \xi_0 \\ q_i \end{pmatrix} \in \mathcal{O}_i \times \mathbf{Q}$, $i \in J$,*

$\forall \varepsilon > 0$ *there exists* $\mathbf{t} > t_0$ *such that* $\|\xi(t,j)\| \leq \varepsilon$, *for any* $t \geq \mathbf{t}$, *for any free execution of* $\widetilde{S}$.

4 Conclusions

We addressed observability and detectability for linear continuous-time switching systems. We compared existing definitions, presented some of their drawbacks and proposed a new definition of observability, and a weaker notion of detectability, related to the possibility of reconstructing the system state. To the best of our knowledge, detectability has not been addressed as yet in the literature on hybrid systems. We gave some sufficient and necessary testable conditions for observability. We also derived a Kalman-like decomposition of the switching system and we proposed conditions for detectability based on this decomposition.

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