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Tracking Multiple Maneuvering Targets by Joint Combinations of IMM and PDA

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***Abstract*— For the problem of tracking multiple manoeuvering targets in false and missing measurements the paper develops a characterization of the exact Bayesian equations of the conditional density. Since in these exact equations both IMM and PDA are Jointly performed over all targets, we also develop two Joint IMMPDA type of filters and compare them with other combinations of IMM and JPDA through Monte Carlo simulation for a simple example.**

I. Introduction

Particle filtering [1] forms an elegant approach towards the numerical evaluation of the exact conditional density in nonlinear filtering and maneuvering target tracking. As such the aim of this paper is to develop such a particle filter and to compare its performance with those of Gaussian approximation type of filters.

A prerequisite for developing a Particle filter is to first characterize the exact conditional density. In order to prepare for this, the multitarget tracking problem is shown to be one of filtering for a descriptor system with both i.i.d. and Markovian coefficients [2], [3]. For this descriptor system we develop a Bayesian characterization of the evolution of the exact conditional density function. The specialty of this exact equation is that both the IMM step and the PDA step are performed jointly for all targets. Next, from these exact equations we develop a Joint IMMPDA Particle filter which evaluates the exact density through using the bootstrap approach of [4]. In addition, a Joint IMMPDA filter is obtained by adopting a Gaussian approximation of the conditional density for the joint target state given the joint target mode.

Through Monte Carlo simulations for a simple example, the Joint IMMPDA Particle filter is compared with the JIMMPDA filter and also with the IMMJPDA of [5] and the IMMJPDA* (track coalescence avoiding IMMJPDA) of [2], [3]. The last two approaches perform the PDA step jointly for all targets, but the IMM step per single target. Table 1 provides an overview of the characteristics of these different filtering algorithms, including the single target IMMPDA of [6], [7].

The paper is organized as follows. Section 2 formulates the filtering problem considered. Section 3 develops an exact Bayesian characterization of the evolution of the conditional density for the state of the multiple targets. Section 4 develops the Joint IMMPDA Particle filter. Section 5 considers the single Gaussian assumption for the joint targets per joint mode value, and presents the Joint IMMPDA algorithm. Section 6 shows Monte Carlo simulation results.

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TABLE I

Characteristics of different filtering algorithms

	Joint measure-ments	Joint manoeuvre modes	Hypotheses merging	Hypotheses pruning	Particle filter
IMMPDA [6],[7]	-	-	yes	-	-
IMMJPDA [5]	yes	-	yes	-	-
IMMJPDA* [2],[3]	yes	-	yes	yes	-
JIMMPDAP [9]	yes	yes	-	-	yes
JIMMPDA	yes	yes	yes	-	-

II. Problem formulation

Following [2], [3] the problem of tracking multiple linear Markovian mode switching targets in false and missed detections is formulated in terms of filtering for a jump linear descriptor system with both Markovian switching and i.i.d. coefficients:

$$x_{t+1} = A(\theta_{t+1})x_t + B(\theta_{t+1})w_t \tag{1.a}$$

$$z_t = H(\theta_t)x_t + G(\theta_t)v_t \tag{1.b}$$

$$\underline{\Phi}(\psi_t^*)\mathrm{y}_t = v_t^* \quad \text{if } L_t > D_t, \tag{2}$$

$$\underline{\Phi}(\psi_t)\mathrm{y}_t = \underline{\chi}_t\underline{\Phi}(\phi_t)z_t \quad \text{if } D_t > 0 \tag{3}$$

<u>Target evolution eq. (1.a) and potential measurements (1.b)</u>

The underlying model components of (1.a) are as follows:

$x_t \triangleq \text{Col}\{x_t^1, ..., x_t^M\}$,

$\theta_t \triangleq \text{Col}\{\theta_t^1, ..., \theta_t^M\}$,

$A(\theta) \triangleq \text{Diag}\{a^1(\theta^1), ..., a^M(\theta^M)\}$,

$B(\theta) \triangleq \text{Diag}\{b^1(\theta^1), ..., b^M(\theta^M)\}$,

$w_t \triangleq \text{Col}\{w_t^1, ..., w_t^M\}$,

where x_t^i is the n-vectorial state of the i-th target at moment t, θ_t^i is the mode of the i-th target at moment t and assumes values from $\mathbb{M} = \{1, .., N\}$ according to a transition probability matrix Π^i, $a^i(\theta^i)$ and $b^i(\theta^i)$ are $(n \times n)$- and $(n \times n')$-matrices, w_t^i is a sequence of i.i.d. standard Gaussian variables of dimension n' with w_t^i , w_t^j independent for all $i \neq j$ and w_t^i ,x_0^i, x_0^j independent for all $i \neq j$. With this, x_t is a vector of size Mn, $A(\theta)$ and $B(\theta)$ are of size $Mn \times Mn$ and $Mn \times Mn'$ respectively, and $\{\theta_t\}$ assumes values from

$\mathbb{M}^M$ according to transition probability matrix $\Pi = [\Pi_{\eta,\theta}]$. If the M targets switch mode independently of each other, then:

$$\Pi_{\eta,\theta} = \prod_{i=1}^{M} \Pi^i_{\eta^i,\theta^i}, \text{ for every } (\eta,\theta) \in \mathbb{M}^M$$

The coefficients in eq. (1.b) are:
$H(\theta) \triangleq \text{Diag}\{h^1(\theta^1), ..., h^M(\theta^M)\}$,
$G(\theta) \triangleq \text{Diag}\{g^1(\theta^1), ..., g^M(\theta^M)\}$,
$h^i(\theta^i)$ is an $(m \times n)$-matrix,
$g^i(\theta^i)$ is an $(m \times m')$-matrix,
$v_t \triangleq \text{Col}\{v^1_t, ..., v^M_t\}$,
where v^i_t is a sequence of i.i.d. standard Gaussian variables of dimension m' with v^i_t and v^j_t independent for all $i \neq j$. Moreover v^i_t is independent of x^j_0 and w^j_t for all i,j.

Measurements
We next describe the relation between the potential measurement vector z_t and the measurement vector y_t.
$\text{y}_t \triangleq \text{Col}\{\text{y}_{1,t}, ..., \text{y}_{L_t,t}\}$ is the measurement vector that contains a random mixture of target- and false measurements within a given volume V. Here $\text{y}_{i,t}$ denotes the i-th m-vectorial measurement at moment t, L_t denotes the number of measurements at moment t within volume V, and D_t denotes the number of detected targets at moment t.

False measurements eq. (2)
The number of false measurements at moment t, F_t, is assumed to be Poisson distributed

$$\begin{aligned} p_{F_t}(F) &= \tfrac{(\lambda V)^F}{F!}\exp\big(-\lambda V\big), \qquad F = 0,\ 1,\ 2,\ . .. \\ &= 0, \qquad\qquad\qquad\qquad \text{else} \end{aligned}$$

where λ is the spatial density of false measurements (i.e. the average number per unit volume). Thus λV is the expected number of false measurements in volume V.
v^*_t is a column vector of F_t i.i.d. false measurements within volume V. The prior density of these false measurements is assumed to be uniform on V.
$\psi^*_t \triangleq \text{Col}\{\psi^*_{1,t}, ..., \psi^*_{L_t,t}\}$ is a false indicator vector of size L_t $(= F_t + D_t)$ with $\psi^*_{i,t} \in [0,1]$ the false indicator at moment t for measurement i. It assumes the value one if measurement i is a false measurement and zero if measurement i belongs to a target.

In order to select the false measurements by simple matrix multiplication, a matrix operator Φ is defined, producing $\Phi(\psi')$ as a $(0,1)$-valued matrix of size $D(\psi') \times M'$ of which the ith row equals the ith non-zero row of $\text{Diag}\{\psi'\}$, where $D(\psi') \triangleq \sum_{i=1}^{M'} \psi'_i$ for an arbitrary (0,1)-valued M'-vector ψ'. To take into account the measurement vector size m, $\Phi(\psi^*_t)$ needs to be "inflated" to the proper size of $D_t m$ by means of the tensor product with I_m. To this end, $\underline{\Phi}(\psi') \triangleq \Phi(\psi') \otimes I_m$ with I_m a unit-matrix of size m, and $\otimes$ the tensor product. Hence $\underline{\Phi}(\psi^*_t)\text{y}_t$ is a column vector that contains only false measurements from y_t.

Target measurement eq. (3)
Equation (3) is a descriptor system [8], with stochastic i.i.d. coefficients $\underline{\Phi}(\psi_t)$ and $\underline{\chi}_t\underline{\Phi}(\phi_t)$. $\psi_t \triangleq \text{Col}\{\psi_{1,t}, ..., \psi_{L_t,t}\}$ is the target indicator vector, where $\psi_{i,t} \in$\{0,1\} is a target indicator at moment t for measurement i, which assumes the value one if measurement i belongs to a detected target and zero if measurement i is false.
To select the target measurements, which are indicated by the target indicator vector, by simple matrix multiplication, the matrix operator Φ is used again. Hence $\underline{\Phi}(\psi_t)\text{y}_t$ is a column vector that contains target measurements from y_t only, in a random order.

$\phi_t \triangleq \text{Col}\{\phi_{1,t}, ..., \phi_{M,t}\}$ is the detection indicator vector, where $\phi_{i,t} \in$\{0,1\} is the detection indicator for target i, which assumes the value one with probability $P^i_{\text{d}} > 0$, independently of $\phi_{j,t}$, $j \neq i$, where P^i_{d} denotes the detection probability of target i. $\{\phi_t\}$ is a sequence of i.i.d. vectors, and $D_t \triangleq \sum_{i=1}^{M} \phi_{i,t}$ denotes the number of detected targets. Hence $L_t - D_t$ is the number of false measurements. As before, by using the matrix operator Φ, $\underline{\Phi}(\phi_t)H(\theta_t)x_t$ is a column vector of potential detected measurements of targets in a fixed order.

Finally the detected target measurements in the observation vector y_t are in random order. Hence the potential detected measurements of targets need to be randomly mixed. To perform this by a simple matrix multiplication, a sequence of independent stochastic permutation matrices $\{\chi_t\}$ of size $D_t \times D_t$ is defined and assumed to be independent of $\{\phi_t\}$. To take into account the measurement vector size m, χ_t needs to be "inflated" to the proper size of $D_t m$ by means of the tensor product with I_m. To this end, $\underline{\chi}_t \triangleq \chi_t \otimes I_m$ with I_m a unit-matrix of size m, and $\otimes$ the tensor product. Hence $\underline{\chi}_t\underline{\Phi}(\phi_t)H(\theta_t)x_t$ is a column vector of potential detected measurements of targets in random order.

III. Exact filter equations

In this section a Bayesian characterization of the conditional density $p_{x_t,\theta_t|\ Y_t}(x,\theta)$ is given where Y_t denotes the σ-algebra generated by measurements y_t up to and including moment t. In preparation to this eq. (3) is first transformed following [9].
Because χ_t has an inverse, (3) can be transformed into

$$\underline{\chi}^T_t\underline{\Phi}(\psi_t)\text{y}_t = \underline{\Phi}(\phi_t)z_t, \quad \text{if } D_t > 0 \tag{4}$$

We introduce an auxiliary indicator matrix process $\tilde{\chi}_t$ of size $D_t \times L_t$, as follows:

$$\tilde{\chi}_t \triangleq \chi^T_t\Phi(\psi_t) \ \ \text{if } D_t > 0. \tag{5.a}$$

and an auxiliary measurement process

$$\tilde{y}_t \triangleq \underline{\tilde{\chi}}_t y_t \tag{5.b}$$

With this we get a simplified version of (4):

$$\tilde{y}_t = \underline{\tilde{\chi}}_t y_t = \underline{\Phi}(\phi_t) z_t, \quad \text{if } D_t > 0, \tag{6}$$

where the size of $\underline{\tilde{\chi}}_t$ is $D_t m \times L_t m$ and the size of $\underline{\Phi}(\phi_t)$ is $D_t m \times M m$.

The right-hand side of (6) shows that all relevant combinations of detected potential target measurements can be covered by ϕ_t hypotheses. The left-hand side of (6) shows that all relevant selections of sets of target originating measurements out of the set of all measurements, can be covered by $\tilde{\chi}_t$ hypotheses. Thus from (6), it follows that for $D_t > 0$ all relevant measurement-to-target associations can be covered by $(\phi_t, \tilde{\chi}_t)$-hypotheses. We extend this to $D_t = 0$ by adding the combination $\phi_t = \{0\}^M$ and $\tilde{\chi}_t = \{\}^{L_t}$. Next, by defining the weights

$$\beta_t(\phi, \tilde{\chi}, \theta) \triangleq \text{Prob}\{\phi_t = \phi, \tilde{\chi}_t = \tilde{\chi}, \theta_t = \theta \mid Y_t\},$$

the law of total probability yields:

$$p_{x_t \theta_t | Y_t}(x, \theta) = \sum_{\tilde{\chi}, \phi} \beta_t(\phi, \tilde{\chi}, \theta) p_{x_t | \theta_t, \phi_t, \tilde{\chi}_t, Y_t}(x \mid \theta, \phi, \tilde{\chi}) \tag{7}$$

The terms in the last summation are characterized in the following Proposition.

Proposition 1. *For any $\phi \in \{0,1\}^M$, such that $D(\phi) \triangleq \sum_{i=1}^{M} \phi_i \leq L_t$, and any $\tilde{\chi}_t$ matrix realization $\tilde{\chi}$ of size $D(\phi) \times L_t$, the following holds true:*

$$p_{x_t | \theta_t, \phi_t, \tilde{\chi}_t, Y_t}(x \mid \theta, \phi, \tilde{\chi}) = \\ = \frac{p_{\tilde{y}_t | x_t, \theta_t, \phi_t}(\underline{\tilde{\chi}} y_t \mid x, \theta, \phi) \cdot p_{x_t | \theta_t, Y_{t-1}}(x \mid \theta)}{F_t(\phi, \tilde{\chi}, \theta)} \tag{8}$$

$$\beta_t(\phi, \tilde{\chi}, \theta) = F_t(\phi, \tilde{\chi}, \theta) \lambda^{(L_t - D(\phi))} \cdot \\ \cdot [\prod_{i=1}^{M} (1 - P_d^i)^{(1-\phi_i)} (P_d^i)^{\phi_i}] \cdot p_{\theta_t | Y_{t-1}}(\theta) / c_t \tag{9}$$

where $\underline{\tilde{\chi}} \triangleq \tilde{\chi} \otimes I_m$, and $F_t(\phi, \tilde{\chi}, \theta)$ and c_t are such that they normalize $p_{x_t | \theta_t, \phi_t, \tilde{\chi}_t, Y_t}(x \mid \theta, \phi, \tilde{\chi})$ and $\beta_t(\phi, \tilde{\chi}, \theta)$ respectively.

Proof: Here omitted.

IV. JOINT IMMPDA PARTICLE FILTER

In this section a JIMMPDA Particle filter of the exact filter characterization of Proposition 1 is developed following [10]. In order to prepare for a particle filter approach, substituting (8) and (9) into (7) yields

$$p_{x_t, \theta_t | Y_t}(x, \theta) = \\ \sum_{\tilde{\chi}, \phi} \frac{p_{\tilde{y}_t | x_t, \theta_t, \phi_t}(\underline{\tilde{\chi}} y_t \mid x, \theta, \phi) \cdot p_{x_t | \theta_t, Y_{t-1}}(x \mid \theta)}{F_t(\phi, \tilde{\chi}, \theta)} \cdot \\ F_t(\phi, \tilde{\chi}, \theta) \lambda^{(L_t - D(\phi))} \cdot [\prod_{i=1}^{M} (1 - P_d^i)^{(1-\phi_i)} (P_d^i)^{\phi_i}] \cdot \\ \cdot p_{\theta_t | Y_{t-1}}(\theta) / c_t \tag{10}$$

Simplifying (10) and rearranging terms yields:

$$p_{x_t, \theta_t | Y_t}(x, \theta) = \\ \sum_{\tilde{\chi}, \phi} p_{\tilde{y}_t | x_t, \theta_t, \phi_t}(\underline{\tilde{\chi}} y_t \mid x, \theta, \phi) \cdot p_{x_t, \theta_t | Y_{t-1}}(x, \theta) \\ \lambda^{(L_t - D(\phi))} \cdot [\prod_{i=1}^{M} (1 - P_d^i)^{(1-\phi_i)} (P_d^i)^{\phi_i}] / c_t \tag{11}$$

with

$$p_{\tilde{y}_t | x_t, \theta_t, \phi_t}(\tilde{y} \mid x, \theta, \phi) = \\ N\{\tilde{y}; \underline{\Phi}(\phi) H(\theta) x, \underline{\Phi}(\phi) G(\theta) G(\theta)^T \underline{\Phi}(\phi)^T\} \tag{12}$$

Define

$$\tilde{F}_t(\phi, \tilde{\chi}, x, \theta) \triangleq p_{\tilde{y}_t | x_t, \theta_t, \phi_t}(\underline{\tilde{\chi}} y_t \mid x, \theta, \phi) \tag{13}$$

Hence from (12) we get:

$$\tilde{F}_t(\phi, \tilde{\chi}, x, \theta) = [(2\pi)^{mD(\phi)} \text{Det}\{\tilde{Q}_t(\phi, \theta)\}]^{-\frac{1}{2}} \cdot \\ \cdot \exp\{-\frac{1}{2} \tilde{\mu}_t^T(\phi, \tilde{\chi}, x, \theta) \tilde{Q}_t(\phi, \theta)^{-1} \tilde{\mu}_t(\phi, \tilde{\chi}, x, \theta)\} \tag{14}$$

where

$$\tilde{\mu}_t(\phi, \tilde{\chi}, x, \theta) \triangleq \underline{\tilde{\chi}} y_t - \underline{\Phi}(\phi) H(\theta) x \\ \tilde{Q}_t(\phi, \theta) \triangleq \underline{\Phi}(\phi) \big(G(\theta) G(\theta)^T\big) \underline{\Phi}(\phi)^T$$

Substituting (13) into (11) and rearranging terms yields

$$p_{x_t, \theta_t | Y_t}(x, \theta) = \\ \frac{1}{c_t} \sum_{\tilde{\chi}, \phi} \tilde{F}_t(\phi, \tilde{\chi}, x, \theta) \cdot \lambda^{(L_t - D(\phi))} \cdot \\ \cdot [\prod_{i=1}^{M} (1 - P_d^i)^{(1-\phi_i)} (P_d^i)^{\phi_i}] \cdot p_{x_t, \theta_t | Y_{t-1}}(x, \theta) \tag{15}$$

With this we are prepared to specify a McGinnity & Irwin [11] type of particle filter, but now for JIMMPDA. One cycle of this JIMMPDA Particle filter consists of the following seven steps, where a particle is defined as a triplet (w, x, θ), $w \in [0, 1]$, $x \in \mathbb{R}^{Mn}$, $\theta \in \mathbb{M}^M$.

JIMMPDA Particle filter Step 1: Start with the mode probabilities

$$\hat{\gamma}_{t-1}(\theta) \triangleq p_{\theta_{t-1} | Y_{t-1}}(\theta)$$

and for each $\theta \in \mathbb{M}^M$ a set of S^θ particles in $[0,1] \times \mathbb{R}^{Mn} \times \mathbb{M}^M$, i.e.:

$$\{(w_{t-1}^{\theta,j}, x_{t-1}^{\theta,j}, \theta_{t-1}^{\theta,j} = \theta);\ j \in [1, S^\theta], \theta \in \mathbb{M}^M\}$$

with

$$w_{t-1}^{\theta,j} = \hat{\gamma}_{t-1}(\theta)/S^\theta$$

Thus in total there are $S = \sum_\theta S^\theta$ particles.

JIMMPDA Particle filter Step 2: (Interaction) Determine the new set of particles (the weights $w_{t-1}^{\theta,j}$ are not changed)

$$\{(w_{t-1}^{\theta,j}, x_{t-1}^{\theta,j}, \bar{\theta}_t^{\theta,j});\ j \in [1, S^\theta], \theta \in \mathbb{M}^M\}$$

by generating for each particle a new value $\bar{\theta}_t^{\theta,j}$ according to the model

$$\text{Prob}\{\bar{\theta}_t^{\theta,j} = \bar{\theta} \mid \theta_{t-1}^{\theta,j} = \theta\} = \Pi_{\theta,\bar{\theta}}$$

JIMMPDA Particle filter Step 3: Determine the new set of particles (the weights $w_{t-1}^{\theta,j}$ are not changed)

$$\{(w_{t-1}^{\theta,j}, \bar{x}_t^{\theta,j}, \bar{\theta}_t^{\theta,j});\ j \in [1, S^\theta], \theta \in \mathbb{M}^M\}$$

by running for each particle a Monte Carlo simulation from $(t-1)$ to t according to the model

$$\bar{x}_t^{\theta,j} = A(\bar{\theta}_t^{\theta,j})x_{t-1}^{\theta,j} + B(\bar{\theta}_t^{\theta,j})w_{t-1}$$

JIMMPDA Particle filter Step 4: Determine new weights for the set of particles, i.e.

$$\{(\bar{w}_t^{\theta,j}, \bar{x}_t^{\theta,j}, \bar{\theta}_t^{\theta,j});\ j \in [1, S^\theta], \theta \in \mathbb{M}^M\}$$

with for the new weights

$$\bar{w}_t^{\theta,j} = w_{t-1}^{\theta,j} \cdot \frac{1}{c_t} \sum_{\tilde{\chi},\phi} \tilde{F}_t(\phi, \tilde{\chi}, \bar{x}_t^{\theta,j}, \bar{\theta}_t^{\theta,j}) \cdot \lambda^{(L_t - D(\phi))} \cdot \\ \cdot [\prod_{i=1}^{M} (1-P_d^i)^{(1-\phi_i)} (P_d^i)^{\phi_i}]$$

where

$$\tilde{F}_t(\phi, \tilde{\chi}, x, \theta) = [(2\pi)^{mD(\phi)} \text{Det}\{\tilde{Q}_t(\phi,\theta)\}]^{-\frac{1}{2}} \cdot \\ \cdot \exp\{-\frac{1}{2}\tilde{\mu}_t^T(\phi,\tilde{\chi},x,\theta)\tilde{Q}_t(\phi,\theta)^{-1}\tilde{\mu}_t(\phi,\tilde{\chi},x,\theta)\} \quad (16)$$

with

$$\begin{aligned} \tilde{\mu}_t(\phi, \tilde{\chi}, x, \theta) &\triangleq \underline{\tilde{\chi}}\mathrm{y}_t - \underline{\Phi}(\phi)H(\theta)x \\ \tilde{Q}_t(\phi,\theta) &\triangleq \underline{\Phi}(\phi)\big(G(\theta)G(\theta)^T\big)\underline{\Phi}(\phi)^T \end{aligned}$$

and c_t such that

$$\sum_{\theta \in \mathbb{M}^M} \sum_{j=1}^{S^\theta} \bar{w}_t^{\theta,j} = 1$$

JIMMPDA Particle filter Step 5: θ_t conditional filter estimates:

$$\hat{\gamma}_t(\theta) = \sum_{\eta \in \mathbb{M}^M} \sum_{j=1}^{S^\eta} \bar{w}_t^{\eta,j} 1_{\bar{\theta}_t^{\eta,j}}(\theta)$$

$$\hat{x}_t(\theta) \cong \sum_{\eta \in \mathbb{M}^M} \sum_{j=1}^{S^\eta} \bar{w}_t^{\eta,j} \bar{x}_t^{\eta,j} 1_{\bar{\theta}_t^{\eta,j}}(\theta)$$

$$\hat{P}_t(\theta) \cong \sum_{\eta \in \mathbb{M}^M} \sum_{j=1}^{S^\eta} \bar{w}_t^{\eta,j} [\bar{x}_t^{\eta,j} - \hat{x}_t(\theta)][\bar{x}_t^{\eta,j} - \hat{x}_t(\theta)]^T 1_{\bar{\theta}_t^{\eta,j}}(\theta)$$

JIMMPDA Particle filter Step 6: θ dependent resampling: Generate the new set of particles

$$\{(w_t^{\theta,j}, x_t^{\theta,j}, \theta_t^{\theta,j});\ j \in [1, S^\theta], \theta \in \mathbb{M}^M\}$$

by applying the following equations per θ value:

$$\theta_t^{\theta,j} = \theta$$

$$w_t^{\theta,j} = \hat{\gamma}_t(\theta)/S^\theta$$

$x_t^{\theta,j}$ is the j-th of the S^θ samples drawn from the particle spanned joint conditional density for (x_t, θ_t) given Y_t:

$$\sum_{\eta \in \mathbb{M}^M} \sum_{l=1}^{S^\eta} \bar{w}_t^{\eta,l} 1_{\bar{\theta}_t^{\eta,l}}(\theta) \delta_{\bar{x}_t^{\eta,l}}(x)$$

JIMMPDA Particle filter Step 7: MMSE output equations:

$$\hat{x}_t = \sum_{\theta \in \mathbb{M}^M} \hat{\gamma}(\theta)\hat{x}_t(\theta)$$

$$\hat{P}_t = \sum_{\theta \in \mathbb{M}^M} \hat{\gamma}(\theta) \Big(\hat{P}_t + (\theta)[\hat{x}_t(\theta) - \hat{x}_t][\hat{x}_t(\theta) - \hat{x}_t]^T\Big)$$

V. Joint IMMPDA

Although in theory not as optimal as a good particle filter, for practical applications Gaussian approximations have been proven to be of use. In this section we develop a new one, which adopts the assumption of a single joint Gaussian per joint target mode. To accomplish this, we use the following Theorem from [3]:

Theorem 1. *For each $\theta \in \{1, ..., N\}^M$, let $p_{x_t|\underline{\theta}_t, Y_{t-1}}(x \mid \theta)$ be Gaussian with mean $\bar{x}_t(\theta)$ and covariance $\bar{P}_t(\theta)$ and let $\beta_t(\phi,\tilde{\chi},\theta)$ and $F_t(\phi,\tilde{\chi},\theta)$ be defined by Proposition 1. Then $F_t(\{0\}^M, \{\}^{L_t}, \theta) = 1$, whereas for $\phi \neq \{0\}^M$:*

$$F_t(\phi,\tilde{\chi},\theta) = [(2\pi)^{mD(\phi)} Det\{Q_t(\phi,\theta)\}]^{-\frac{1}{2}} \cdot \\ \cdot \exp\{-\frac{1}{2}\mu_t^T(\phi,\tilde{\chi},\theta)Q_t(\phi,\theta)^{-1}\mu_t(\phi,\tilde{\chi},\theta)\} \quad (17)$$

where

$$\begin{aligned} \mu_t(\phi,\tilde{\chi},\theta) &\triangleq \underline{\tilde{\chi}}y_t - \underline{\Phi}(\phi)H(\theta)\bar{x}_t(\theta) \\ Q_t(\phi,\theta) &\triangleq \underline{\Phi}(\phi)\big(H(\theta)\bar{P}_t(\theta)H(\theta)^T + \\ &\quad + G(\theta)G(\theta)^T\big)\underline{\Phi}(\phi)^T \end{aligned}$$

Moreover, $p_{x_t|\theta_t,Y_t}(x \mid \theta)$ is a Gaussian mixture, with overall weight $p_{\theta_t|Y_t}(\theta)$, mean $\hat{x}_t(\theta)$ and its covariance $\hat{P}_t(\theta)$ satisfying:

$$p_{\theta_t|Y_t}(\theta) = \sum_{\phi,\tilde{\chi}} \beta_t(\phi,\tilde{\chi},\theta) \quad (18)$$

$$\hat{x}_t(\theta) = \bar{x}_t(\theta) + \sum_{\substack{\phi \\ \phi\neq 0}} K_t(\phi,\theta)\Big(\sum_{\tilde{\chi}} \beta_{t|\theta}(\phi,\tilde{\chi})\mu_t(\phi,\tilde{\chi},\theta)\Big) \tag{19}$$

$$\begin{aligned}\hat{P}_t(\theta) = \bar{P}_t(\theta) + \\ &- \sum_{\substack{\phi \\ \phi\neq 0}} K_t(\phi,\theta)\underline{\Phi}(\phi)H(\theta)\bar{P}_t(\theta)\Big(\sum_{\tilde{\chi}} \beta_{t|\theta}(\phi,\tilde{\chi})\Big) + \\ &+ \sum_{\substack{\phi \\ \phi\neq 0}} K_t(\phi,\theta)\Big(\sum_{\tilde{\chi}} \beta_{t|\theta}(\phi,\tilde{\chi})\mu_t(\phi,\tilde{\chi},\theta)\mu_t^T(\phi,\tilde{\chi},\theta)\Big)\cdot \\ &\cdot K_t^T(\phi,\theta) + \\ &- \Bigg(\sum_{\substack{\phi \\ \phi\neq 0}} K_t(\phi,\theta)\Big(\sum_{\tilde{\chi}} \beta_{t|\theta}(\phi,\tilde{\chi})\mu_t(\phi,\tilde{\chi},\theta)\Big)\Bigg)\cdot \\ &\cdot \Bigg(\sum_{\substack{\phi' \\ \phi'\neq 0}} K_t(\phi',\theta)\Big(\sum_{\tilde{\chi}'} \beta_{t|\theta}(\phi',\tilde{\chi}')\mu_t(\phi',\tilde{\chi}',\theta)\Big)\Bigg)^T\end{aligned} \tag{20}$$

with:

$$\begin{aligned} K_t(\phi,\theta) &\triangleq \bar{P}_t(\theta)H(\theta)^T\underline{\Phi}(\phi)^T Q_t(\phi,\theta)^{-1} \quad \textit{if } \phi\neq 0, \\ &\triangleq 0 \quad \textit{else} \end{aligned} \tag{21.a}$$

$$\beta_{t|\theta}(\phi,\tilde{\chi}) \triangleq \beta_t(\phi,\tilde{\chi},\theta)/p_{\theta_t|Y_t}(\theta) \tag{21.b}$$

Theorem 1 provides a conditional characterization for the joint targets modes and states given that for each $\theta \in \mathbb{M}^M$, the conditional density $p_{x_t|\theta_t,Y_{t-1}}(x \mid \theta)$ is Gaussian. Although this condition is usually not satisfied, the resulting characterization can be used as an approximation in a recursive algorithm. We refer to this recursive algorithm as the Joint IMMPDA (JIMMPDA) filter, which consists of the following six subsequent steps.

JIMMPDA Step 1: Interaction:
For all $\theta \in \mathbb{M}^M$, starting with the weights

$$\hat{\gamma}_{t-1}(\theta) \triangleq p_{\theta_{t-1}|Y_{t-1}}(\theta),$$

the means $\hat{x}_{t-1}(\theta)$ and the associated covariances $\hat{P}_{t-1}(\theta)$ one evaluates the mixed initial condition for the filter matched to $\theta_t = \theta$ as in IMM [12]:

$$\bar{\gamma}_t(\theta) = \sum_{\eta\in\mathbb{M}^M} \Pi_{\eta,\theta}\cdot\hat{\gamma}_{t-1}(\eta)$$

$$\hat{x}_{t-1|\theta_t}(\theta) = \sum_{\eta\in\mathbb{M}^M} \Pi_{\eta,\theta}\cdot\hat{\gamma}_{t-1}(\eta)\cdot\hat{x}_{t-1}(\eta)/\bar{\gamma}_t(\theta)$$

$$\begin{aligned}\hat{P}_{t-1|\theta_t}(\theta) = \sum_{\eta\in\mathbb{M}^M} \Pi_{\eta,\theta}\cdot\hat{\gamma}_{t-1}(\eta)\cdot \\ \cdot\Big(\hat{P}_{t-1}(\eta) + [\hat{x}_{t-1}(\eta) - \hat{x}_{t-1|\theta_t}(\theta)]\cdot \\ \cdot[\hat{x}_{t-1}(\eta) - \hat{x}_{t-1|\theta_t}(\theta)]^T\Big)/\bar{\gamma}_t(\theta)\end{aligned}$$

JIMMPDA Step 2: Prediction for all $\theta \in \{1,...,N\}^M$:

$$\bar{x}_t(\theta) = A(\theta)\hat{x}_{t-1|\theta_t}(\theta) \tag{22.a}$$

$$\bar{P}_t(\theta) = A(\theta)\hat{P}_{t-1|\theta_t}(\theta)A(\theta)^T + B(\theta)B(\theta)^T \tag{22.b}$$

JIMMPDA Step 3: Gating, based on [13].
Evaluate for each i and θ the following covariance:

$$\bar{Q}_t(\theta) = H(\theta)\bar{P}_t(\theta)H(\theta)^T + G(\theta)G(\theta)^T$$

Let $\bar{Q}_t^i(\theta)$ be the i-th $m\times m$ diagonal block matrix of $\bar{Q}_t(\theta)$. Subsequently identify for each target the mode for which Det $\bar{Q}_t^i(\theta)$ is largest:

$$\theta_t^{*i} = \text{Arg}\max_\theta\{\text{Det } \bar{Q}_t^i(\theta)\}$$

and use this to define for each target i a gate $G_t^i \in I\!R^m$ as follows:

$$\begin{aligned} G_t^i \triangleq \{z^i \in I\!R^m; [z^i - h^i(\theta_t^{*i})\bar{x}_t^i(\theta_t^{*i})]^T\cdot \\ \cdot\bar{Q}_t^i(\theta_t^{*i})^{-1}[z^i - h^i(\theta_t^{*i})\bar{x}_t^i(\theta_t^{*i})] \leq \gamma\}\end{aligned}$$

with γ the gate size. If the j-th measurement y_t^j falls outside gate G_t^i; i.e. $\mathrm{y}_t^j \notin G_t^i$, then the j-th component of the i-th row of $[\Phi(\phi)^T\tilde{\chi}]$ is assumed to equal zero at moment t. This reduces the set of possible detection/permutation hypotheses to be evaluated at moment t for various ϕ to $\tilde{\mathcal{X}}_t(\phi)$.

JIMMPDA Step 4: Evaluation of the hypotheses by using (9) and (17) as approximation:

$$\begin{aligned}\beta_t(\phi,\tilde{\chi},\theta) &= F_t(\phi,\tilde{\chi},\theta)\lambda^{(L_t - D(\phi))}\cdot \\ &\cdot[\textstyle\prod_{i=1}^M(1-P_{\mathrm{d}}^i)^{(1-\phi_i)}(P_{\mathrm{d}}^i)^{\phi_i}]\cdot \\ &\cdot\bar{\gamma}_t(\theta)/c_t \quad \text{for } \tilde{\chi}\in\tilde{\mathcal{X}}_t(\phi), \\ &= 0 \quad \text{else}\end{aligned} \tag{23.a}$$

$$\begin{aligned}F_t(\phi,\tilde{\chi},\theta) \cong [(2\pi)^{mD(\phi)}\text{Det}\{Q_t(\phi,\theta)\}]^{-\frac{1}{2}}\cdot \\ \cdot\exp\{-\frac{1}{2}\mu_t^T(\phi,\tilde{\chi},\theta)Q_t(\phi,\theta)^{-1}\mu_t(\phi,\tilde{\chi},\theta)\}\end{aligned} \tag{23.b}$$

with c_t normalizing $\beta_t(\phi,\tilde{\chi},\theta)$, and

$$\mu_t(\phi,\tilde{\chi},\theta) \triangleq \underline{\tilde{\chi}}\mathrm{y}_t - \underline{\Phi}(\phi)H(\theta)\bar{x}_t(\theta) \tag{23.c}$$

$$Q_t(\phi,\theta) \triangleq \underline{\Phi}(\phi)\big(H(\theta)\bar{P}_t(\theta)H(\theta)^T + G(\theta)G(\theta)^T\big)\underline{\Phi}(\phi)^T \tag{23.d}$$

JIMMPDA Step 5: Measurement-based update equations, using (18), (19) and (20) as approximations:

$$\hat{\gamma}_t(\theta) = \sum_{\phi,\tilde{\chi}} \beta_t(\phi,\tilde{\chi},\theta) \tag{24}$$

$$\hat{x}_t(\theta) \cong \bar{x}_t(\theta) + \sum_{\substack{\phi \\ \phi\neq 0}} K_t(\phi,\theta)\Big(\sum_{\tilde{\chi}} \beta_{t|\theta}(\phi,\tilde{\chi})\mu_t(\phi,\tilde{\chi},\theta)\Big) \tag{25}$$

$$\hat{P}_t(\theta) \cong \bar{P}_t(\theta) + \text{ other four right hand terms of (20)} \quad (26)$$

with $K_t(\phi, \theta)$ and $\beta_{t|\theta}(\phi, \tilde{\chi})$ as in (21a,b).

JIMMPDA Step 6: Output equations:

$$\hat{x}_t = \sum_{\theta \in \mathbb{M}^M} \hat{\gamma}_t(\theta) \cdot \hat{x}_t(\theta) \quad (27)$$

$$\hat{P}_t = \sum_{\theta \in \mathbb{M}^M} \hat{\gamma}_t(\theta) \big(\hat{P}_t(\theta) + [\hat{x}_t(\theta) - \hat{x}_t] \cdot [\hat{x}_t(\theta) - \hat{x}_t]^T\big) \quad (28)$$

VI. Monte Carlo simulations

In this section some Monte Carlo simulation results are given for the JIMMPDA Particle filter, the JIMMPDA, IMMJPDA* and IMMJPDA filter algorithms, and for an IMMPDA which updates an individual track using PDA by assuming the measurements from the adjacent targets as false. The JIMMPDA Particle filter ran on a total of $S = 10000$ particles, with for each of the four modes $S^\theta = 2500$ particles.
The four scenarios and underlying model equations are the same as in [2], [3], [10]. The Monte Carlo simulation results for the four scenarios are presented in Table 2.

TABLE II
Monte Carlo simulation results.

	Average % Both Tracks O.K.				Average % Both Tracks O.K. or Swapped			
	1	2	3	4	1	2	3	4
IMMPDA	19	10	6	4	28.34	18.9	8.5	5.6
IMMJPDA	66	56	63	41	99.96	92.5	99.8	76.6
IMMJPDA*	73	68	69	50	100	96.8	100	80.96
JIMPDAP	75	70	72	57	96.2	94.6	95.8	82.3
JIMPDA	54	47	52	35	79.6	77.3	80.1	65.6
	Average number of Coalescing scans				**Average CPU time per scan (in seconds)**			
	1	2	3	4	1	2	3	4
IMMPDA	9.7	11.0	18.9	14.5	0.016	0.038	0.014	0.038
IMMJPDA	1.5	2.1	1.7	2.6	0.022	0.054	0.020	0.061
IMMJPDA*	0.4	0.3	0.5	0.5	0.023	0.048	0.020	0.056
JIMPDAP	1.3	1.4	1.3	1.5	0.439	7.959	0.438	7.810
JIMPDA	3.3	3.7	3.4	3.8	0.042	0.070	0.037	0.085

As expected, there is a significant CPU-time increase for JIMMPDA Particle filter relative to the others. The increase is one order of magnitude for the scenarios without false measurements and two orders of magnitude for the scenarios with false measurements. For the example considered, the averages in the tables show that JIMMPDA performs less well than all others except IMMPDA. In contrast with this, the JIMMPDA Particle filter (JIMMPDAP) outperforms the other filter algorithms when it comes to ”Both tracks O.K.”. Nevertheless, IMMJPDA* performs slightly better regarding the ”both tracks O.K. or swapped” criterion on scenarios 1-3 and on track coalescence avoidance for all scenarios.

VII. Acknowledgements

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VIII. References

[1] Doucet, A., N. J. Gordon and V. Krishnamurthy, “Particle Filters for State Estimation of Jump Markov Linear Systems,” *IEEE Tr. on Signal Processing*, Vol. 49, 2001, pp. 613-624.

[2] Blom, H. A. P., and E. A. Bloem, “Combining IMM and JPDA for tracking multiple maneuvering targets in clutter,” *Proc. 5th Int. Conf. on Inf. Fusion*, July 8-11, 2002, Annapolis, MD, USA, Vol. 1, pp. 705-712.

[3] Blom, H. A. P., and E. A. Bloem, “Interacting Multiple Model Joint Probabilistic Data Asscoiation avoiding track coalescence,” *Proc. IEEE CDC*, December 2002, pp. 3408-3415.

[4] Gordon, N. J., D. J. Salmond and A. F. M. Smith, “Novel approach to nonlinear/non-Gaussian Bayesian state estimation,” *IEE Proceedings-F*, Vol. 140, pp. 107-113, 1993.

[5] Chen, B., and J. K. Tugnait, “Tracking of multiple maneuvering targets in clutter using IMM/JPDA filtering and fixed-lag smoothing,” *Automatica*, vol. 37, 2001, pp. 239-249.

[6] Blom, H. A. P., “A sophisticated tracking algorithm for Air Traffic Control surveillance radar data,” *Proc. Int. Conf. on Radar*, Paris, May 1984, pp. 393-398.

[7] Houles, A., and Y. Bar-Shalom, “Multisensor tracking of a maneuvering target in clutter,” *IEEE Tr. AES*, Vol. 25, 1989, pp. 176-188.

[8] Dai, L., “Singular control systems,” *Lecture notes in Control and information sciences*, 118, Springer, 1989.

[9] Blom, H. A. P., and E. A. Bloem, “Probabilistic Data Association Avoiding Track Coalescence,” *IEEE Tr. AC*, Vol. 45 (2000), pp. 247-259.

[10] Blom, H. A. P., and E. A. Bloem, “Joint IMMPDA Particle filter,” *Proc. 6th Int. Conf. on Inf. Fusion*, July 8-11, 2003, Cairns, Australia, Vol. 1, pp. 785-792.

[11] McGinnity, S. and G. W. Irwin, “Multiple Model Bootstrap Filter for Maneuvering Target Tracking,” *IEEE Tr. AES*, Vol. 36, 2000, pp. 1006-1012.

[12] Blom, H. A. P., and Y. Bar-Shalom, “The Interacting Multiple Model algorithm for systems with Markovian switching coefficients,” *IEEE Tr. AC*, Vol. 33 (1988), pp. 780-783.

[13] Bar-Shalom, Y., and X. R. Li, “Multitarget-Multisensor Tracking: Principles and Techniques,” *YBS Publishing*, Storrs, CT, 1995.