APPROXIMATION OF FIRST PASSAGE TIMES OF SWITCHING DIFFUSION

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Abstract. This paper studies the approximation of first passage time of a multi-dimensional switching diffusion process to a given target domain. We propose a discrete-time strong approximation scheme for switching diffusion processes with state-dependent switching rates. If τ and τ^h are the first passage times of the continuous and discretized processes respectively, then under some conditions we show that τ^h converges in distribution to τ as the discretization step tends to zero.

1. Introduction

In this paper we study the approximation of first passage time of a multi-dimensional switching diffusion process with state-dependent switching rates to a given target set. The problem of finding the probability of hitting the target set before some fixed time is of great importance in many applications, e.g. air traffic management [2], reliability analysis [1], finance applications etc. An analytical solution is available only in some special simple cases. Thus, approximate method is needed. Weak approximations of killed (or stopped) diffusions were studied in [5], [6], [7], [8] and [10]. They develop and prove the convergence of numerical schemes that approximate the expected value $E[g(x(\tau),\tau)]$ of a given function g depending on the solution x of an Itô stochastic differential equation and on the first exit time τ from a given domain.

We will consider one hybrid model, also called a switching diffusion, where switching rates of the discrete component may depend on continuous component [3]. Let $\{x_t, \theta_t\}$ be the switching diffusion taking its values in $\mathbb{R}^n \times \mathbb{M}$ defined by

$$dx_t = a(\theta_t, x_t)dt + b(\theta_t, x_t)dW_t, \tag{1.1}$$

$$P_{\theta_{t+\delta}|\theta_t, x_t}(\theta|\eta, x) = \lambda_{\eta\theta}(x)\delta + o(\delta),\ \ \eta \neq \theta, \tag{1.2}$$

where $\mathbb{M}$ is a finite set of modes and $(W_t)_{t\geq 0}$ is a Brownian motion in $\mathbb{R}^n$. We set $\tau \stackrel{\triangle}{=} \inf\{t > 0 : x_t \in D\}$ for the first passage time to the cylinder set $D \times \mathbb{M}$, where $D \subset \mathbb{R}^n$ is a closed connected set. We want to estimate

$$P(\tau \leq T) \tag{1.3}$$

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where T is a fixed time. We propose to approximate the switching diffusion (1.1)-(1.2) by a discrete-time strong approximation $(X^h_{t_i}, \theta^h_{t_i})_{t_i \in I}$ (where I is a time discretization), and approximate probability (1.3) by

$$P(\tau^h \leq T), \tag{1.4}$$

where $\tau^h \triangleq \inf\{t > 0 : X^h_t \in D\}$.

Time discrete approximations of an Ito diffusion are well explained in [9]. A discretization scheme for jump-diffusion process with state-dependent intensities was considered in [4]. We propose an Euler-type discretization scheme for hybrid model (1.1)-(1.2) and prove its convergence. Following the approach of Gobet (1999) [5], we then show that

$$P(\tau \leq T) - P(\tau^h \leq T) \longrightarrow 0$$

as the discretization step tends to zero.

Note that probability (1.4) can be estimated by Monte Carlo methods whatever the dimension n is.

The organization of the paper is the following. Section 2 describes the hybrid model in more detail. The strong approximation scheme and its convergence proof are presented in section 3. The approximation of first passage time is discussed in section 4.

2. Stochastic hybrid model

Let $\mathbb{M} = \{e_1, e_2, \dots, e_N\}$ be a finite set of unit vectors, i.e. $e_i \in \mathbb{M}$ is a i-th unit vector in $\mathbb{R}^N$. Let

$$\begin{aligned} a &: \mathbb{R}^n \times \mathbb{M} \to \mathbb{R}^n \\ b &: \mathbb{R}^n \times \mathbb{M} \to \mathbb{R}^{n\times n} \\ \lambda_{ij} &: \mathbb{R}^n \to \mathbb{R},\ i,j = 1,2,\dots,N \end{aligned}$$

For each $\theta \in \mathbb{M}$, $a(\cdot,\theta)$ and $b(\cdot,\theta)$ are assumed to be bounded, continuous and Lipschitz. For all $i,j \in \{1,\dots,N\}$ $\lambda_{ij}(\cdot)$ is assumed to be bounded, continuous and Lipschitz, $\lambda_{ij}(\cdot) \geq 0$ for $i \neq j$ and $\sum_{j=1}^N \lambda_{ij}(\cdot) = 0$ for any $i \in \{1,\dots,N\}$.

For $i,j \in \{1,\dots,N\}$, $i \neq j$, $x \in \mathbb{R}^n$ we construct the intervals $\Delta_{ij}(x)$ of the real line in the following manner [3]:

$$\begin{aligned} \Delta_{12}(x) &= [0, \lambda_{12}(x)) \\ \Delta_{13}(x) &= [\lambda_{12}(x), \lambda_{12}(x) + \lambda_{13}(x)) \\ &\vdots \\ \Delta_{1N}(x) &= \Big[\textstyle\sum_{j=2}^{N-1} \lambda_{1j}(x), \sum_{j=2}^{N} \lambda_{1j}(x)\Big) \\ \Delta_{21}(x) &= \Big[\textstyle\sum_{j=2}^{N} \lambda_{1j}(x), \sum_{j=2}^{N} \lambda_{1j}(x) + \lambda_{21}(x)\Big) \end{aligned}$$

and so on. Thus, in general,

$$\Delta_{ij}(x) = \Big[\sum_{i'=1}^{i-1} \sum_{\substack{j'=1 \\ j' \neq i'}}^{N} \lambda_{i'j'}(x) + \sum_{\substack{j'=1 \\ j' \neq i}}^{j-1} \lambda_{ij'}(x), \sum_{i'=1}^{i-1} \sum_{\substack{j'=1 \\ j' \neq i'}}^{N} \lambda_{i'j'}(x) + \sum_{\substack{j'=1 \\ j' \neq i}}^{j} \lambda_{ij'}(x)\Big).$$

For fixed x these are disjoint intervals, and the length of $\Delta_{ij}(x)$ is $\lambda_{ij}(x)$. Now we define a function $c : \mathbb{R}^n \times \mathbb{M} \times \mathbb{R} \to \mathbb{R}^N$:

$$c(x, e_i, z) = \begin{cases} e_j - e_i & \text{if } z \in \Delta_{ij}(x) \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Then the $(\mathbb{R}^n \times \mathbb{M})$-valued switching diffusion process (1.1)-(1.2) can be represented as a solution of the following SDE [3]:

$$dx_t = a(x_t, \theta_t)dt + b(x_t, \theta_t)dW_t \tag{2.2}$$

$$d\theta_t = \int_{\mathbb{R}} c(x_t, \theta_{t-}, z)p(dt, dz) \tag{2.3}$$

for $t \geq 0$, with (x_0, θ_0) a prescribed $(\mathbb{R}^n \times \mathbb{M})$-valued random variable; $p(dt, dz)$ is a Poisson random measure with intensity $dt \cdot dz$; (W_t) is an n-dimensional Wiener process independent of (x_0, θ_0) and $p(dt, dz)$. Under assumption on W_t, $p(dt, dz)$, (x_0, θ_0), and on functions a, b and λ, equation (2.2)-(2.3) admits an a.s. pathwise unique solution.

Define the following interval:

$$U(x) \triangleq \bigcup_{i=1}^{N} \Big(\bigcup_{\substack{j=1 \\ j \neq i}}^{N} \Delta_{ij}(x) \Big),$$

it includes all intervals $\Delta_{ij}(x), i, j = 1, \ldots, N, i \neq j$. Since the length of each interval $\Delta_{ij}(x)$ is $\lambda_{ij}(x)$, and this is continuous and bounded function for $i, j = 1, \ldots, N$, $i \neq j$, it follows that the length of interval $U(x)$ (denote $l(U(x))$) is bounded and is a continuous function of x. Therefore, it has a maximum at some point $x^\star$:

$$l(U(y)) \leq l(U(x^\star)) \text{ for all } y \in \mathbb{R}^n.$$

Then we denote the interval of maximum length as follows

$$U_{\max} \triangleq U(x^\star)$$

and the length of $U_{\max}$ is denoted as $\lambda_{\max} \triangleq l(U_{\max})$. We can rewrite equation (2.3) as follows

$$d\theta_t = \int_{U_{\max}} c(X_t, \theta_{t-}, z)p(dt, dz).$$

We can think of a Poisson random measure $p(dt, dz)$ as assigning unit mass to (τ_n, z_n) if there is a jump at time τ_n of size z_n.

Let $N(t)$ be a standard Poisson process with intensity $\lambda_{\max}$. We denote by τ_n, $n = 1, 2, \ldots$ the jump times of $N(t)$. Let $U_{\max}$ be the "mark" space, and $(Z_n)_{n\geq 1}$ be a sequence of i.i.d. random variables with uniform distribution on $U_{\max}$, independent of $N(t)$. In this special case we can represent the random Poisson measure $p(dt, dz)$ with intensity $dt \cdot dz$ as a random counting measure associated to the marked point process $(\tau_n, Z_n)_{n\geq 0}$, i.e. for each Lebesgue measurable $A \subset U_{\max}$

$$p((0, t], A) = \sum_{n\geq 1} \mathbf{1}_{\{\tau_n \leq t\}} \cdot \mathbf{1}_{\{Z_n \in A\}}. \tag{2.4}$$

We check that

$$\mathbb{E}[p((0,t],A)] = \lambda_{\max}\cdot t\cdot \mathbb{P}(Z_n\in A) = \lambda_{\max}\cdot t\cdot \frac{l(A)}{\lambda_{\max}} = t\cdot l(A).$$

The representation (2.4) is very convenient for practical problems. We see that $p(dt,dz)$ can be generated just by sampling independent random variables τ_n and Z_n, $n=1,2,\dots$.

3. Strong approximation of switching diffusion

3.1. **Discretization Scheme.** Now we turn our attention to numerical solution of SDE (2.2)-(2.3). We will develop an Euler type discretization scheme which allows to obtain a strong approximation process to the solution of switching diffusion precess. To start with, we should define the appropriate discretization of time interval $[0,T]$. Let us denote by $I_d = \{t_n^d : n=0,1,\dots,L\}$ the usual equidistant time discretization of a bounded interval $[0,T]$ with discretization step $h=T/L$. Suppose $\tau_1,\tau_2,\dots$ are the jump times of the component θ_t driven by equations (2.2-2.3). Then we take a new time discretization $I=\{t_n : n=0,1,\dots\}$ which is the superposition of the random jump times τ_n of a component θ_t on interval $[0,T]$ and a deterministic grid I_d.

For a given time discretization I an Euler type approximation is a continuous time stochastic process $\{X_t^h,\theta_t^h\}$ satisfying the following equation with "delayed" coefficients[1]:

$$X_t^h = X_0 + \int_0^t a^h(s,X^h,\theta^h)ds + \int_0^t b^h(s,X^h,\theta^h)dW_s \tag{3.1}$$

$$\theta_t^h = \theta_0 + \int_0^t\int_{U_{\max}} c^h(s-,X^h,\theta^h,z)p(ds,dz) \tag{3.2}$$

here

$$\begin{aligned} a^h(s,X^h,\theta^h) &\triangleq a(X_{t_k},\theta_{t_k}), && s\in[t_k,t_{k+1}),\\ b^h(s,X^h,\theta^h) &\triangleq b(X_{t_k},\theta_{t_k}), && s\in[t_k,t_{k+1}),\\ c^h(s-,X^h,\theta^h,z) &\triangleq c(X_{t_k},\theta_{t_k-},z), && s\in[t_k,t_{k+1}). \end{aligned}$$

The corresponding recursive discretization scheme

$$X_{t_i}^h = X_{t_{i-1}}^h + a(X_{t_{i-1}}^h,\theta_{t_{i-1}}^h)(t_i-t_{i-1}) + b(X_{t_{i-1}}^h,\theta_{t_{i-1}}^h)(W_{t_i}-W_{t_{i-1}}), \tag{3.3}$$

$$\theta_{t_i}^h = \theta_{t_{i-1}}^h + \int_{U_{\max}} c(X_{t_i}^h,\theta_{t_{i-1}}^h,z)p(\{t_i\},dz), \tag{3.4}$$

determines values of the approximating process (3.1)-(3.2) at discretization times only. Thus, approximation $(X_{t_i}^h,\theta_{t_i}^h)$ is iteratively computed from the initial condition (X_0,θ_0) using the scheme (3.3)-(3.4). At the grid point, (3.4) computes the jump of θ^h exactly, conditional on $(X_{t_i}^h,\theta_{t_i^-}^h)=(X_{t_i}^h,\theta_{t_{i-1}}^h)$, if t_i is indeed a point of the Poisson random measure. Otherwise, the jump term is zero. (The integral in (3.4) entails at most a single evaluation of the function c because $p(\{t_i\},dz)$ is a point mass at the mark z that arrives at t_i if it is a jump time.)

[1]Here h denotes the dependence on the time discretization step h.

3.2. **Convergence.**

Proposition 3.1. *Suppose, functions a, b, c and λ_{ij} are defined as in section 2 and the Euler type approximating process $\{X_t^h, \theta_t^h\}$ is defined as in section 3.1. We assume, that*

(1) *functions $\lambda_{ij}(\cdot)$ $(i, j = 1, \dots N)$ are Lipschitz, i.e.*

$$|\lambda_{ij}(x) - \lambda_{ij}(y)| \leq C_\lambda |x - y|, \text{ for all } x, y \in \mathbb{R}^n; \tag{3.5}$$

(2) *for all $x, y \in \mathbb{R}^n$ and $\theta, \eta \in \mathbb{M}$*

$$|a(x,\theta) - a(y,\eta)|^2 + |b(x,\theta) - b(y,\eta)|^2 \leq C_{ab}(|x-y|^2 + 1) \tag{3.6}$$

for $\theta \neq \eta$ and

$$|a(x,\theta) - a(y,\eta)|^2 + |b(x,\theta) - b(y,\eta)|^2 \leq C_{ab}(|x-y|^2) \tag{3.7}$$

for $\theta = \eta$.

Then

$$\sup_{s \leq T} \mathbb{E}(|X_s^h - X_s|^2 + |\theta_s^h - \theta_s|^2) \leq e^{-\lambda_{\max} T} K^2 \sum_{k=0}^{\infty} h^{2^{-k}} \cdot \frac{(\lambda_{\max} T K)^k}{k!}, \tag{3.8}$$

$$\sup_{s \leq T} \mathbb{E}(|X_s^h - X_s|^2 + |\theta_s^h - \theta_s|^2) \longrightarrow 0, \text{ as } h \longrightarrow 0, \tag{3.9}$$

and

$$\mathbb{E}[\sup_{s \leq T} |X_s^h - X_s|] \leq \Big(2T(T+4)C_{ab} \cdot e^{-\lambda_{\max} T} K^2 \sum_{k=0}^{\infty} h^{2^{-k}} \cdot \frac{(\lambda_{\max} T K)^k}{k!}\Big)^{1/2}, \tag{3.10}$$

$$\mathbb{E}[\sup_{s \leq T} |X_s^h - X_s|] \longrightarrow 0, \quad \text{as } h \longrightarrow 0. \tag{3.11}$$

Here the constant K does not depend on h.

To prove the Proposition 3.1 one needs the following lemmas.

Let $\mathcal{F}_t$ denote the σ algebra generated by all random variables W_s, $p((0,s], U_{\max})$, $s \leq t$ (see section 2). Let $\mathcal{L}_T^2$ denote the space of all $\mathcal{F}_t$-adapted stochastic processes that are square integrable:

$$||f||_{\mathcal{L}_T^2} = \int_0^T \int_\Omega f^2(t,\omega) \mathbb{P}(d\omega) dt < \infty.$$

To shorten expressions we introduce the following notation:

$$\mathbb{E}^i[\cdot] \triangleq \mathbb{E}[\cdot | N(T) = i].$$

Lemma 3.2. *Suppose, W_t is independent of $p(dt, dz)$. Then for every $f \in \mathcal{L}_T^2$,*

$$\mathbb{E}^i[(\int_0^T f(t,\omega) dW_t(\omega))^2] = \mathbb{E}^i[\int_0^T f^2(t,\omega) dt]. \tag{3.12}$$

Proof. First we consider the step processes, and then extend the result to arbitrary processes.

Let ϕ be a bounded step process in $\mathcal{L}_T^2$:

$$\phi(t,\omega) = \sum_{j=0}^{n-1} c_j(\omega) 1_{[t_j, t_{j+1})}(t). \tag{3.13}$$

By adaptedness, c_i in (3.13) is independent of $\Delta W_j \triangleq W_{t_{j+1}} - W_{t_j}$ for $i \leq j$. Therefore

$$\begin{aligned}
\mathbb{E}^i[(\int_0^T \phi(t)dW_t)^2] &= \mathbb{E}^i[(\sum_{j=0}^{n-1} c_j \Delta W_j)^2] \\
&= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \mathbb{E}^i[c_i c_j \Delta W_i \Delta W_j] \\
&= \sum_{j=0}^{n-1} \mathbb{E}^i[c_j^2 \Delta W_j^2] + 2\sum_{i<j} \mathbb{E}^i[c_i c_j \Delta W_i]\mathbb{E}[\Delta W_j] \\
&= \sum_{j=0}^{n-1} \mathbb{E}^i[c_j^2]\mathbb{E}[\Delta W_j^2] = \sum_{j=0}^{n-1} \mathbb{E}^i[c_j^2]\Delta t_j \\
&= \mathbb{E}^i[\sum_{j=0}^{n-1} c_j^2 \Delta t_j] = \mathbb{E}^i[\int_0^T \phi^2(t)dt].
\end{aligned}$$

To go from step processes to arbitrary processes we use the known fact that every process $\phi \in \mathcal{L}_T^2$ can be approximated arbitrarily well by step processes in $\mathcal{L}_T^2$. Now, suppose $\phi \in \mathcal{L}_T^2$ is an arbitrary process. We can approximate it by step processes $\phi^n \in \mathcal{L}_T^2$, i.e. $\phi^n \longrightarrow \phi$ in $\mathcal{L}_T^2$. To get the claim (3.12) we pass to the limit in the following equality, $n \longrightarrow \infty$

$$\mathbb{E}^i[\int_0^T \phi_t^n dW_t]^2 = \mathbb{E}^i[\int_0^T (\phi_t^n)^2 dt],\ n \in \mathbb{N}.$$

Indeed, since $Y^n \triangleq \int_0^T \phi_t^n dW_t \longrightarrow Y \triangleq \int_0^T \phi_t dW_t$, then

$$\begin{aligned}
\big|(\mathbb{E}^i[(Y^n)^2])^{1/2} - (\mathbb{E}^i[Y^2])^{1/2}\big| &\leq (\mathbb{E}^i[(Y^n - Y)^2])^{1/2} \\
&\leq \left(\frac{\mathbb{E}[(Y^n - Y)^2]}{\mathbb{P}(N(T) = i)}\right)^{1/2} \longrightarrow 0,\ n \longrightarrow \infty,
\end{aligned}$$

and

$$\begin{aligned}
&\big|(\mathbb{E}^i[\int_0^T (\phi_t^n)^2 dt])^{1/2} - (\mathbb{E}^i[\int_0^2 (\phi_t^n - \phi_t)^2 dt])^2\big| \\
&\qquad \leq \left(\frac{\mathbb{E}[\int_0^T (\phi_t^n - \phi_t)^2 dt]}{\mathbb{P}(N(T) = i)}\right)^{1/2} \longrightarrow 0,\ n \longrightarrow \infty,
\end{aligned}$$

Thus

$$\mathbb{E}^i[\int_0^T \phi_t dW_t]^2 = \mathbb{E}^i[\int_0^T (\phi_t)^2 dt].$$

□

Lemma 3.3. *Suppose functions $\lambda_{ij}(\cdot)$ $i, j = 1, \ldots, N$ satisfy the conditions of Proposition 3.1. Then there exist a constant C_c such that*

$$\int_{\mathbb{R}} |c(x, e_i, z) - c(y, e_k, z)|^2 dz \leq C_c(|x - y| + 1) \textit{ for } i \neq k \tag{3.14}$$

and

$$\int_{\mathbb{R}} |c(x,e_i,z)-c(y,e_k,z)|^2 dz \le C_c(|x-y|) \text{ for } i=k \tag{3.15}$$

for all $x, y \in \mathbb{R}^n$ *and* $e_i, e_k \in \mathbb{M}$.

Proof.

$$\begin{aligned}
&\int_{\mathbb{R}} |c(x,e_i,z)-c(y,e_k,z)|^2 dz \\
&\qquad = \int_{\mathbb{R}} |\sum_{\substack{j=1\\ j\neq i}}^{N} \mathbf{1}_{\Delta_{ij}(x)}(z)\cdot e_j - e_i - (\sum_{\substack{j=1\\ j\neq k}}^{N} \mathbf{1}_{\Delta_{kj}(y)}(z)\cdot e_j - e_k)|^2 dz \\
&\qquad = \int_{U_{\max}} |(e_k-e_i) + (\sum_{\substack{j=1\\ j\neq i}}^{N} \mathbf{1}_{\Delta_{ij}(x)}(z)\cdot e_j - \sum_{\substack{j=1\\ j\neq k}}^{N} \mathbf{1}_{\Delta_{kj}(y)}(z)\cdot e_j)|^2 dz \\
&\qquad \le 2\lambda_{\max}|e_k-e_i|^2 + 2\int_{U_{\max}} |\sum_{\substack{j=1\\ j\neq i}}^{N} \mathbf{1}_{\Delta_{ij}(x)}(z)\cdot e_j - \sum_{\substack{j=1\\ j\neq k}}^{N} \mathbf{1}_{\Delta_{kj}(y)}(z)\cdot e_j|^2 dz.
\end{aligned}$$

Let us consider two cases:
1) suppose $i=k$, then

$$\begin{aligned}
\int_{\mathbb{R}} |c(x,e_i,z)-c(y,e_k,z)|^2 dz &\le 2\int_{U_{\max}} |\sum_{\substack{j=1\\ j\neq i}}^{N} \mathbf{1}_{\Delta_{ij}(x)}(z)\cdot e_j - \sum_{\substack{j=1\\ j\neq k}}^{N} \mathbf{1}_{\Delta_{kj}(y)}(z)\cdot e_j|^2 dz \\
&= 2\int_{U_{\max}} |\sum_{\substack{j=1\\ j\neq i}}^{N} (\mathbf{1}_{\Delta_{ij}(x)}(z) - \mathbf{1}_{\Delta_{ij}(y)}(z))\cdot e_j|^2 dz \\
&\le 2N \int_{U_{\max}} \sum_{\substack{j=1\\ j\neq i}}^{N} |\mathbf{1}_{\Delta_{ij}(x)}(z) - \mathbf{1}_{\Delta_{ij}(y)}(z)|^2 dz \\
&= 2N \sum_{\substack{j=1\\ j\neq i}}^{N} \int_{U_{\max}} |\mathbf{1}_{\Delta_{ij}(x)}(z) - \mathbf{1}_{\Delta_{ij}(y)}(z)|^2 dz \\
&= 2N \sum_{\substack{j=1\\ j\neq i}}^{N} (\int_{\Delta_{ij}(x)\setminus\Delta_{ij}(y)} 1 dz + \int_{\Delta_{ij}(y)\setminus\Delta_{ij}(x)} 1 dz). \qquad (3.16)
\end{aligned}$$

(1a) suppose $\Delta_{ij}(x) \cap \Delta_{ij}(y) \neq \emptyset$. Then

$$\int_{\Delta_{ij}(x)\setminus\Delta_{ij}(y)} 1dz + \int_{\Delta_{ij}(y)\setminus\Delta_{ij}(x)} 1dz$$
$$= |\sum_{i'=1}^{i-1}\sum_{\substack{j'=1\\ j'\neq i'}}^{N} \lambda_{i'j'}(x) + \sum_{\substack{j'=1\\ j'\neq i}}^{j-1} \lambda_{ij'}(x) - \sum_{i'=1}^{i-1}\sum_{\substack{j'=1\\ j'\neq i'}}^{N} \lambda_{i'j'}(y) - \sum_{\substack{j'=1\\ j'\neq i}}^{j-1} \lambda_{ij'}(y)|$$
$$+ |\sum_{i'=1}^{i-1}\sum_{\substack{j'=1\\ j'\neq i'}}^{N} \lambda_{i'j'}(x) + \sum_{\substack{j'=1\\ j'\neq i}}^{j} \lambda_{ij'}(x) - \sum_{i'=1}^{i-1}\sum_{\substack{j'=1\\ j'\neq i'}}^{N} \lambda_{i'j'}(y) - \sum_{\substack{j'=1\\ j'\neq i}}^{j} \lambda_{ij'}(y)|$$
$$\leq 2N^2 C_\lambda |x-y|.$$

(1b) now suppose $\Delta_{ij}(x) \cap \Delta_{ij}(y) = \emptyset$. We denote by $\Delta_{ij}^{x,y}$ the interval that is contiguous to intervals $\Delta_{ij}(x)$ and $\Delta_{ij}(y)$. Then

$$\int_{\Delta_{ij}(x)} 1dz + \int_{\Delta_{ij}(y)} 1dz \leq \int_{\Delta_{ij}(x)\cup\Delta_{ij}^{x,y}} 1dz + \int_{\Delta_{ij}(y)\cup\Delta_{ij}^{x,y}} 1dz$$
$$= |\sum_{i'=1}^{i-1}\sum_{\substack{j'=1\\ j'\neq i'}}^{N} \lambda_{i'j'}(x) + \sum_{\substack{j'=1\\ j'\neq i}}^{j-1} \lambda_{ij'}(x) - \sum_{i'=1}^{i-1}\sum_{\substack{j'=1\\ j'\neq i'}}^{N} \lambda_{i'j'}(y) - \sum_{\substack{j'=1\\ j'\neq i}}^{j-1} \lambda_{ij'}(y)|$$
$$+ |\sum_{i'=1}^{i-1}\sum_{\substack{j'=1\\ j'\neq i'}}^{N} \lambda_{i'j'}(x) + \sum_{\substack{j'=1\\ j'\neq i}}^{j} \lambda_{ij'}(x) - \sum_{i'=1}^{i-1}\sum_{\substack{j'=1\\ j'\neq i'}}^{N} \lambda_{i'j'}(y) - \sum_{\substack{j'=1\\ j'\neq i}}^{j} \lambda_{ij'}(y)|$$
$$\leq 2N^2 C_\lambda |x-y|.$$

Now we can proceed with expression (3.16):

$$2N\sum_{\substack{j=1\\ j\neq i}}^{N}(\int_{\Delta_{ij}(x)\setminus\Delta_{ij}(y)} 1dz + \int_{\Delta_{ij}(y)\setminus\Delta_{ij}(x)} 1dz)$$
$$\leq 2N\sum_{\substack{j=1\\ j\neq i}}^{N}(2N^2 C_\lambda|x-y|) \leq 4N^4 C_\lambda |x-y|.$$

2) suppose $i \neq k$, then

$$\int_{\mathbb{R}} |c(x,e_i,z) - c(y,e_k,z)|^2 dz$$
$$\leq 4\lambda_{\max} + 2\int_{U_{\max}} |\sum_{\substack{j=1\\ j\neq i}}^{N} \mathbf{1}_{\Delta_{ij}(x)}(z)\cdot e_j - \sum_{\substack{j=1\\ j\neq k}}^{N} \mathbf{1}_{\Delta_{kj}(y)}(z)\cdot e_j)|^2 dz$$
$$\leq 4\lambda_{\max} + 2\cdot 4N^2\lambda_{\max} \leq \lambda_{\max}(4+8N^2) + |x-y|.$$

From the above estimations follows that there exists a constant C_c such that

$$\int_{\mathbb{R}} |c(x,e_i,z) - c(y,e_k,z)|^2 dz \leq C_c(|x-y|+1) \text{ for } i \neq k,$$

and

$$\int_{\mathbb{R}} |c(x,e_i,z)-c(y,e_k,z)|^2 dz \le C_c(|x-y|) \text{ for } i=k.$$

□

Lemma 3.4. *Suppose all conditions of Proposition 3.1 are satisfied. Then*

$$\sup_{s\le T} \mathbb{E}^i[|X_s^h - X_s|^2 + |\theta_s^h-\theta_s|^2] \le K^{i+2}h^{2^{-i}},$$

as $h \longrightarrow 0$, *for some constant* K *depending only on* T.

Proof. Let estimate the difference of the coefficients. For all $h>0$ take $s \in [t_k, t_{k+1})$, then applying conditions (3.6)-(3.7) and lemma 3.3, and since $\theta_s = \theta_{t_k}$ for any $t_k \in I$, and $\theta_t^h, \theta_t \in \mathbb{M}$ (unit vectors), we obtain

$$\begin{aligned}
&|a^h(s,X^h,\theta^h)-a(X_s,\theta_s)|^2 + |b^h(s,X^h,\theta^h)-b(X_s,\theta_s)|^2 \\
&\quad = |a(X_{t_k}^h,\theta_{t_k}^h)-a(X_s,\theta_s)|^2 + |b(X_{t_k}^h,\theta_{t_k}^h)-b(X_s,\theta_s)|^2 \\
&\quad \le C_{ab}(|X_{t_k}^h - X_s|^2 + |\theta_{t_k}^h-\theta_s|^2) \\
&\quad \le 2C_{ab}(|X_{t_k}^h - X_{t_k}|^2 + |X_{t_k}-X_s|^2 + |\theta_{t_k}^h - \theta_{t_k}|^2)
\end{aligned}$$

$$\begin{aligned}
&\int_{\mathbb{R}} |c^h(t_k-,X^h,\theta^h,z)-c(X_{t_k},\theta_{t_k-},z)|^2 dz \\
&\quad = \int_{\mathbb{R}} |c(X_{t_k}^h,\theta_{t_k-}^h,z)-c(X_{t_k},\theta_{t_k-},z)|^2 dz \\
&\quad \le C_c(|X_{t_k}^h - X_{t_k}| + |\theta_{t_k-}^h - \theta_{t_k-}|^2).
\end{aligned}$$

Denote $C \triangleq \max(2C_{ab}, C_c)$. Taking conditional expectation we obtain

$$\begin{aligned}
(3.17)\quad &\mathbb{E}^i[|a^h(s,X^h,\theta^h)-a(X_s,\theta_s)|^2 + |b^h(s,X^h,\theta^h)-b(X_s,\theta_s)|^2] \\
&\quad \le C(\mathbb{E}^i[|X_{t_k}^h - X_{t_k}|^2] + \mathbb{E}^i[|X_{t_k}-X_s|^2] + \mathbb{E}^i[|\theta_{t_k}^h-\theta_{t_k}|^2]),
\end{aligned}$$

$$\begin{aligned}
(3.18)\quad &\mathbb{E}^i[\int_{\mathbb{R}} |c^h(t_k-,X^h,\theta^h,z)-c(X_{t_k},\theta_{t_k-},z)|^2 dz] \\
&\quad \le C(\mathbb{E}^i[|X_{t_k}^h - X_{t_k}|] + \mathbb{E}^i[|\theta_{t_k-}^h - \theta_{t_k-}|^2]).
\end{aligned}$$

Denote

$$(3.19)\qquad \begin{cases}
\varphi_i^h(t) \triangleq \sup_{u\le t} \mathbb{E}^i[|X_u^h - X_u|^2], & t\in[0,T], \\
\psi_i(h) \triangleq \sup_{|s-u|\le h} \mathbb{E}^i[|X_s - X_u|^2], & h>0, \\
\gamma_i^h(t) \triangleq \sup_{u\le t} \mathbb{E}^i[|\theta_u^h - \theta_u|^2], & t\in[0,T].
\end{cases}$$

Then, using (3.19) and the inequality (3.17) we have

$$\begin{aligned}
&\mathbb{E}^i[|a^h(s,X^h,\theta^h)-a(X_s,\theta_s)|^2 + |b^h(s,X^h,\theta^h)-b(X_s,\theta_s)|^2] \\
&\quad \le 2C(\varphi_i^h(s) + \psi_i(h) + \gamma_i^h(s)),
\end{aligned}$$

$s \in [0,T]$, $h > 0$.
Thus, for $t < \tau_{k+1}$, $k < i$ (i.e. less then $(k+1)$-th jump time)

$$\begin{aligned}
\varphi_i^h(t) &= \sup_{s\le t} \mathbb{E}^i[|X_s^h - X_s|^2] \\
&= \sup_{s\le t} \mathbb{E}^i[(\int_0^s (a^h(u,X^h,\theta^h) - a(X_u,\theta_u))du + \int_0^s (b^h(u,X^h,\theta^h) - b(X_u,\theta_u))dW_u)^2] \\
&\le \sup_{s\le t} \mathbb{E}^i[2T\int_0^s (a^h(u,X^h,\theta^h) - a(X_u,\theta_u))^2 du + \int_0^s (b^h(u,X^h,\theta^h) - b(X_u,\theta_u))^2 du] \\
&\le K_1 \int_0^t (\varphi_i^h(u) + \psi_i(h) + \gamma_i^h(u))du \\
&\le K_1 \int_0^t \varphi_i^h(u)du + K_1 T(\psi_i(h) + \gamma_i^h(\tau_k))
\end{aligned}$$

here $K_1 = \max(1, 2T, 2C)$.
By Gronwall's lemma:

$$\varphi_i^h(t) \le K_1 T(\psi_i(h) + \gamma_i^h(\tau_k))e^{K_1 t} \le K_1 T(\psi_i(h) + \gamma_i^h(\tau_k))e^{K_1 T}, \text{ for } t < \tau_{k+1}.$$

Note, that

$$\begin{aligned}
\mathbb{E}^i[|X_s - X_u|^2] &= \mathbb{E}^i[|\int_u^s a(X_v,\theta_v)dv + \int_u^s b(X_v,\theta_v)dW_v|^2] \\
&\le 2(s-u)\int_u^s \mathbb{E}^i[|a(X_v,\theta_v)|^2]dv + 2\int_u^s \mathbb{E}^i[|b(X_v,\theta_v)|^2]dv \\
&\le K_2(s-u), \quad 0 \le u \le s,
\end{aligned}$$

and thus

$$\psi_i(h) = \sup_{|s-u|\le h} \mathbb{E}^i[|X_s - X_u|^2] \le K_2 h, \ h \to 0.$$

From here

$$\varphi_i^h(t) \le K_1 T(K_2 h + \gamma_i^h(\tau_k))e^{K_1 T}, \tag{3.20}$$

for $t < \tau_{k+1}$, $1 \le k < i$, and using the fact that $(X_s^h - X_s) = (X_{s-}^h - X_{s-})$, (i.e. continuity from the left) we get

$$\varphi_i^h(\tau_{k+1}) \le K_1 T(K_2 h + \gamma_i^h(\tau_k))e^{K_1 T}, \ 1 \le k < i. \tag{3.21}$$

Denote

$$q(dt,dz) = p(dt,dz) - dtdz.$$

Now we will derive the similar recurrent formula for $\gamma_i^h(\tau_{k+1})$:

$$
\begin{aligned}
\gamma_i^h(\tau_{k+1}) &= \sup_{s\le\tau_{k+1}} \mathbb{E}^i[|\theta_s^h-\theta_s|^2]\\
&= \sup_{s\le\tau_{k+1}} \mathbb{E}^i\Big[\Big|\int_0^s\int_{U_{\max}}(c^h(u-,X^h,\theta^h,z)-c(X_u,\theta_{u-},z))p(du,dz)\Big|^2\Big]\\
&= \sup_{s\le\tau_{k+1}} \mathbb{E}^i\Big[\Big|\int_0^s\int_{U_{\max}}(c^h(u-,X^h,\theta^h,z)-c(X_u,\theta_{u-},z))q(du,dz)\\
&\qquad\qquad+\int_0^s\int_{U_{\max}}(c^h(u-,X^h,\theta^h,z)-c(X_u,\theta_{u-},z))dudz)\Big|^2\Big]\\
&\le \sup_{s\le\tau_{k+1}} 2\mathbb{E}^i\Big[\int_0^s\int_{U_{\max}}|c^h(u-,X^h,\theta^h,z)-c(X_u,\theta_{u-},z)|^2dudz\\
&\qquad\qquad+(\int_0^s\int_{U_{\max}}(c^h(u-,X^h,\theta^h,z)-c(X_u,\theta_{u-},z))dudz)^2\Big]\\
&\le \sup_{s\le\tau_{k+1}} 2\mathbb{E}^i\Big[\int_0^s\int_{U_{\max}}|c^h(u-,X^h,\theta^h,z)-c(X_u,\theta_{u-},z)|^2dudz\\
&\qquad\qquad+T\lambda_{\max}\int_0^s\int_{U_{\max}}|c^h(u-,X^h,\theta^h,z)-c(X_u,\theta_{u-},z)|^2dudz\Big]\\
&\le 2C(1+T\lambda_{\max})\int_0^{\tau_{k+1}}\mathbb{E}^i[|X_u^h-X_u|+|\theta_{u-}^h-\theta_{u-}|^2]du\\
&\le 2CT(1+T\lambda_{\max})\big(\sqrt{\varphi_i^h(\tau_{k+1})}+\gamma_i^h(\tau_k)\big),\quad\text{for } 1\le k<i. \qquad (3.22)
\end{aligned}
$$

Assume $\tau_0 \triangleq 0$. Then $\gamma_i^h(\tau_0)=0$ and $\varphi_i^h(\tau_0)=0$. Define

$$K_3 \triangleq 2e^{K_1T}K_1K_2CT(1+T\lambda_{\max}).$$

Then, using the recurrent formulas (3.20), (3.21) and (3.22) for $\varphi_i^h(\tau_{k+1})$ and $\gamma_i^h(\tau_{k+1})$ we obtain:

$$
\begin{aligned}
&\varphi_i^h(\tau_1)\le 2K_3^2h &&\gamma_i^h(\tau_1)\le 2K_3^2\sqrt{h}\\
&\varphi_i^h(\tau_2)\le 4K_3^3\sqrt{h} &&\gamma_i^h(\tau_2)\le 4K_3^3h^{\frac14}\\
&\qquad\dots &&\qquad\dots\\
&\varphi_i^h(\tau_k)\le K_3(2K_3)^kh^{2^{1-k}} &&\gamma_i^h(\tau_k)\le K_3(2K_3)^kh^{2^{-k}}\\
&\qquad\dots &&\qquad\dots\\
&\varphi_i^h(T)\le K_3(2K_3)^{i+1}h^{2^{-i}} &&\gamma_i^h(T)\le K_3(2K_3)^ih^{2^{-i}}
\end{aligned}
$$

Denote $K=2K_3$. From the above estimates follows that

$$
\begin{aligned}
\sup_{s\le T}\mathbb{E}^i[|X_s^h-X_s|^2+|\theta_s^h-\theta_s|^2] &\le \varphi_i^h(T)+\gamma_i^h(T)\\
&\le K_3(2K_3)^{i+1}h^{2^{-i}}+K_3(2K_3)^ih^{2^{-i}}\\
&\le K^{i+2}h^{2^{-i}},\ h\longrightarrow 0.
\end{aligned}
$$

□

Proof of Proposition 3.1. Using the results of lemma 3.4 we have

$$\begin{aligned}\sup_{s\le T}\mathbb{E}[|X_s^h-X_s|^2+|\theta_s^h-\theta_s|^2] &= \sup_{s\le T}\sum_{k=0}^{\infty}\mathbb{E}^k[|X_s^h-X_s|^2+|\theta_s^h-\theta_s|^2]\cdot\mathbb{P}(N(T)=k)\\ &\le \sum_{k=0}^{\infty}(\sup_{s\le T}\mathbb{E}^k[|X_s^h-X_s|^2+|\theta_s^h-\theta_s|^2])\cdot\mathbb{P}(N(T)=k)\\ &\le e^{-\lambda_{\max}T}K^2\sum_{k=0}^{\infty}h^{2^{-k}}\cdot\frac{(\lambda_{\max}TK)^k}{k!}.\end{aligned}$$

Denote

$$S_m(h)\triangleq\sum_{k=0}^{m}h^{2^{-k}}\cdot\frac{(\lambda_{\max}TK)^k}{k!},$$

and

$$S(h)\triangleq\lim_{m\to\infty}S_m(h)=\sum_{k=0}^{\infty}h^{2^{-k}}\cdot\frac{(\lambda_{\max}TK)^k}{k!}.$$

Since

$$\left|h^{2^{-k}}\cdot\frac{(\lambda_{\max}TK)^k}{k!}\right|\le\frac{(\lambda_{\max}TK)^k}{k!},\ h\in[0,1]$$

and

$$\sum_{k=0}^{\infty}\frac{(\lambda_{\max}TK)^k}{k!}=e^{\lambda_{\max}TK}<\infty,$$

then, by Weierstrass M-Test, $S(h)<\infty$ and the convergence is uniform on $[0,1]$, furthermore, function $S(h)$ is continuous on $[0,1]$. Thus

$$\lim_{h\to0}S(h)=\lim_{h\to0}\lim_{m\to\infty}S_m(h)=\lim_{m\to\infty}\lim_{h\to0}S_m(h)=0.$$

Hence, we have proven that

$$\sup_{s\le T}\mathbb{E}[|X_s^h-X_s|^2+|\theta_s^h-\theta_s|^2]\le e^{-\lambda_{\max}T}K^2\sum_{k=0}^{\infty}h^{2^{-k}}\cdot\frac{(\lambda_{\max}TK)^k}{k!},\tag{3.23}$$

and

$$\sup_{s\le T}\mathbb{E}[|X_s^h-X_s|^2+|\theta_s^h-\theta_s|^2]\longrightarrow0,\ \text{as } h\longrightarrow0.$$

Using Jensen's inequality we obtain

$$\mathbb{E}[\sup_{s\le T}|X_s^h-X_s|]\le\Big(\mathbb{E}[\sup_{s\le T}|X_s^h-X_s|^2]\Big)^{1/2}.\tag{3.24}$$

Next, using Doob's maximal martingale inequality, conditions (3.6)-(3.7) and (3.23) we show that

$$\begin{aligned}
\mathbb{E}[\sup_{s\le T}|X_s^h - X_s|^2] &= \mathbb{E}\Big[\sup_{s\le T}\Big|\int_0^s (a^h(u,X^h,\theta^h) - a(X_u,\theta_u))du \\
&\qquad + \int_0^s (b^h(u,X^h,\theta^h) - b(X_u,\theta_u))dW_u\Big|^2\Big] \\
&\le 2\mathbb{E}\Big[\sup_{s\le T}\Big(T\int_0^s |a^h(u,X^h,\theta^h) - a(X_u,\theta_u)|^2 du \\
&\qquad + \Big|\int_0^s (b^h(u,X^h,\theta^h) - b(X_u,\theta_u))dW_u\Big|^2\Big)\Big] \\
&\le 2\mathbb{E}\Big[T\int_0^T |a^h(u,X^h,\theta^h) - a(X_u,\theta_u)|^2 du \\
&\qquad + 4\Big|\int_0^T (b^h(u,X^h,\theta^h) - b(X_u,\theta_u))dW_u\Big|^2\Big] \\
&\le 2(T+4)C_{ab}\mathbb{E}\Big[\int_0^T (|X_u^h - X_u|^2 + |\theta_u^h - \theta_u|^2)du\Big] \\
&\le 2T(T+4)C_{ab}\sup_{s\le T}\mathbb{E}[|X_s^h - X_s|^2 + |\theta_s^h - \theta_s|^2] \\
&\le 2T(T+4)C_{ab}\cdot e^{-\lambda_{\max}T}K^2\sum_{k=0}^{\infty} h^{2^{-k}}\cdot\frac{(\lambda_{\max}TK)^k}{k!}.
\end{aligned}$$

Hence

$$\mathbb{E}[\sup_{s\le T}|X_s^h - X_s|] \le \Big(2T(T+4)C_{ab}\cdot e^{-\lambda_{\max}T}K^2\sum_{k=0}^{\infty} h^{2^{-k}}\cdot\frac{(\lambda_{\max}TK)^k}{k!}\Big)^{1/2},$$

and

$$\mathbb{E}[\sup_{s\le T}|X_s^h - X_s|] \longrightarrow 0, \quad \text{as } h \longrightarrow 0.$$

□

4. Approximation of first passage times

Our aim is to show that first passage times of discretized switching diffusion converge in distribution to first passage times of original process, i.e. $P(\tau \le T) - P(\tau^h \le T) \longrightarrow 0$ as the maximal discretization step tends to zero. We follow the approach of [5] and [6].

We are interested in computing

$$P(\tau \le T) = 1 - \mathbb{E}[\mathbf{1}_{\{T<\tau\}}].$$

where $\tau = \inf\{t > 0 : x_t \in D\}$ is a first passage time to a given closed connected set D. So, we have to evaluate $\mathbb{E}[1_{\{T<\tau\}}]$, which looks very similar to the problem considered by Gobet.

Proposition 4.1. *Suppose all conditions of Proposition 3.1 are satisfied. Assume that a and b are globally Lipschitz functions, and set $D = \mathbb{R}^n \backslash D^c$ is defined by $D^c = \{x \in \mathbb{R}^n : F(x) > 0\}$, $\partial D^c = \{x \in \mathbb{R}^n : F(x) = 0\}$ for some globally Lipschitz function F. Provided that the condition (C) below is satisfied*

$$\text{(C): } P(\exists t \in [0,T]\ x_t \notin D^c;\ \forall t \in [0,T]\ x_t \in \bar{D}^c) = 0,$$

then we have

$$\lim_{N\to+\infty} |P(\tau \le T) - P(\tau^h \le T)| = 0,$$

Remark 4.2. If D is of class C^2 with a compact boundary, the existence of such a function F holds.

Remark 4.3. Condition (C) rules out the pathological situation where the paths may reach ∂D without leaving D^c. If this condition is not satisfied, then the approximation may not converge to the exact solution.

Remark 4.4. If a and b are bounded and D^c is of class C^3 with a compact boundary, then an uniform ellipticity condition on the diffusion implies condition (C).[2]

Proof. Obviously, we have

$$|P(\tau \le T) - P(\tau^h \le T)| = |P(T < \tau^h) - P(T < \tau)| \le \mathbb{E}[|\mathbf{1}_{\{T<\tau^h\}} - \mathbf{1}_{\{T<\tau\}}|].$$

Fix $\delta > 0$. Then, elementary arguments lead to

$$\begin{aligned}|\mathbf{1}_{\{T<\tau^h\}} - \mathbf{1}_{\{T<\tau\}}| &= |\mathbf{1}_{\inf_{t\in[0,T]} F(X_t^h)>0} - \mathbf{1}_{\inf_{t\in[0,T]} F(X_t)>0}| \\ &\quad \times (\mathbf{1}_{|\inf_{t\in[0,T]} F(X_t)|<\delta} + \mathbf{1}_{|\inf_{t\in[0,T]} F(X_t)|\ge\delta}) \\ &\le \mathbf{1}_{|\inf_{t\in[0,T]} F(X_t)|<\delta} + \mathbf{1}_{|\inf_{t\in[0,T]} F(X_t)-\inf_{t\in[0,T]} F(X_t^h)|\ge\delta}.\end{aligned}$$

Thus,

$$\begin{aligned}(4.1)\quad &\mathbb{E}[|\mathbf{1}_{\{T<\tau^h\}} - \mathbf{1}_{\{T<\tau\}}|] \\ &\le P(|\inf_{t\in[0,T]} F(X_t)| < \delta) + P(|\inf_{t\in[0,T]} F(X_t) - \inf_{t\in[0,T]} F(X_t^h)| \ge \delta).\end{aligned}$$

Set

$$\delta \propto \Big(2T(T+4)C_{ab}\cdot e^{-\lambda_{\max}T}K^2\sum_{k=0}^{\infty} h^{2^{-k}} \cdot \frac{(\lambda_{\max}TK)^k}{k!}\Big)^{\frac{1}{4}}.$$

Using the proposition (3.1), the first term in the r.h.s. of (4.1) converges to

$$P(\inf_{t\in[0,T]} F(x_t) = 0),$$

which equals 0 using condition (C). The second one can be easily bounded by

$$\Big(2T(T+4)C_{ab}\cdot e^{-\lambda_{\max}T}K^2\sum_{k=0}^{\infty} h^{2^{-k}} \cdot \frac{(\lambda_{\max}TK)^k}{k!}\Big)^{\frac{1}{4}}.$$

using the Markov inequality and the estimate (3.10). This proves that $\lim_{h\longrightarrow+0} |P(\tau \le T) - P(\tau^h \le T)| = 0$. □

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[2]For more detail on this remarks see [5].

5. Concluding Remarks

The aim of this work was to study the approximation of first passage times of switching diffusion process. We developed a strong Euler-type discretization scheme for switching diffusion process with state-dependent switching rates and proved its convergence. Then, following the approach of Gobet we have shown that the first passage time of discretized switching diffusion converges in distribution to the first passage time of continuous time switching diffusion process. One can easily simulate the discretized switching diffusion using Monte Carlo simulation approaches and obtain estimates of moments of first passage time or obtain an estimate of probability to hit a target set before some fixed time.

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