Switching Control of Stochastic Linear Systems: Stability and Performance Results*

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Abstract

In presence of large uncertainty, traditional methodologies based on a single controller do not provide, in general, satisfactory performance when applied to any system in the uncertainty set, and may not even guarantee stability. A way to address this problem is to consider a set of candidate controllers and design a supervisor that appropriately orchestrates the switching among them, based on the data collected from the controlled system. The analysis of the resulting hybrid switched control system requires specialized tools.

In this paper, we consider the switching control of a class of stochastic linear system affected by a possibly unbounded noise. We study and compare two supervisory control schemes adopting respectively the dwell time and the hysteresis-based switching logic. We prove that they both stabilize the switched control system. We also show that self-optimality can be achieved in the dwell time switching scheme by adding an asymptotically vanishing dither noise to the control input.

1 Introduction

In this paper, we study the problem of controlling an unknown stochastic linear system.

*Research partly supported by the European Commission under the HYBRIDGE project, IST-2001-32460.

Suppose that a *candidate controllers set* is given. The performance of the control scheme composed by the system in closed-loop with a certain candidate controller can be measured by a cost criterion J: the lower the value for J, the more satisfactory the control performance (J can be, for instance, an H_2 or H_∞ cost). If the system were known, then the optimal controller according to the cost criterion J could be computed by minimizing J over the candidate controllers set.

Consider now the case of interest, i.e., the case when the system is not known. Suppose that a *parametric set of admissible models* is introduced to model the uncertainty on the system description. Then, the problem of selecting the best controller according to J can be addressed by introducing a state variable representing the unknown parameter vector, and then determining the optimal controller according to J for the so-obtained augmented state-space representation of the system. The resulting controller incorporates a self-adjusting mechanism, in that it selects a control input that realizes an appropriate compromise between the control and the identification objectives (dual action, [1]). However, such an optimal dual control problem is doable only in a few simple cases where computing the solution to the optimization problem is actually feasible.

A computationally feasible –though sub-optimal– approach to the design of self-adjusting controllers is the so-called *switching control* approach originally introduced in [2] and further developed in e.g. [3, 4, 5, 6]. A switching control scheme is typically composed of an inner loop where a candidate controller is connected in closed-loop with the system, and an outer loop where a *supervisor*, based on input-output data, decides which controller to place in feedback with the system and when to switch to a different one. Note that the *switched control system* constituting the inner loop of the switching control scheme is a hybrid system, which commutes between different modes when a switching event occurs. Its dynamics within each mode is that of the closed-loop system composed by the system and the controller operating between two consecutive switching events. The analysis of its behavior then requires specialized tools.

The candidate controller to switch to is typically selected through an *estimator-based* procedure ([3, 7]). Specifically, a parametric set of admissible models of the system is introduced, and each model is associated with the controller in the candidate controllers set that is the best for it according to the chosen control cost criterion J. A *monitoring signal* is computed for each admissible model parameter, based on the input-output data collected from the system. This signal is usually given by some integral norm of the output estimation error: the lower is its value, the more accurate is the corresponding model as a description of the system. At any switching time instant, the estimator-based supervisor then selects the controller that is associated to the model whose parameter minimizes the monitoring signal (the *estimated system*). The idea underlying the estimator-based approach to switching is that, as the amount of data collected from the system increases, the estimated system better resembles the behavior of the actual system, at least in closed-loop (*closed-loop identification property*). Hence, by imposing a specific desired behavior on the estimated system, one actually imposes that behavior on the underlying system (*self-tuning property*). If the estimated system is an accurate description of the true system, this ultimately results in applying to the underlying system the candidate controller that is optimal for it (*self-optimality property*).

As for the *switching times*, they are chosen so as to avoid that switching is too fast with respect to the system settling time, thus causing instability. In the *dwell time switching* method, the switching rate is slowed down by making a dwell time elapse between consecutive switching times, either by fixing it before implementing the switching controller ([3, 7]), or by selecting it on-line at each switching time ([8, 6]). In the *hysteresis-based switching* method, the switching rate is slowed down by changing controller only at those times t when the collected data reveal that the model used to select the currently operating controller is significantly worse than the estimated system (see, e.g., [9, 10, 11]).
The reader is referred to [12] for an overview on different switching logics.

In this paper, we study the estimator-based switching control of discrete time linear systems affected by a stochastic noise.
In Section 2, we introduce the admissible models set and the candidate controllers set, and then describe in details the estimator-based supervisor architecture. In Section 3, we then analyze the switching control scheme under the assumption that the true system belongs to the admissible models set. Our objective here is to present the results in [6, 11] in a unifying perspective. Specifically, we prove that when the least squares cost is used as monitoring signal, then both the dwell time and the hysteresis-based switching methods guarantee stability. Finally, in Section 4, we compare these two switching methods. The hysteresis-based switching approach generally leads to a better transient behavior than the dwell time switching approach. On the other hand, self-tuning is ensured only by adopting the dwell time switching approach. Also, we show that by adding an asymptotically vanishing dither noise to the dwell time switching control input, self-optimality is achieved.

2 Switching control scheme

Admissible models set

We introduce the following set of models to represent the uncertainty on the system description:

$$\mathcal{A}(\vartheta; q^{-1})\, y_{t+1} = \mathcal{B}(\vartheta; q^{-1})\, u_t + w_{t+1}, \tag{1}$$

where w_t is some stochastic white noise, $\mathcal{A}(\vartheta; q^{-1}) = 1 - \sum_{i=1}^{n_s} a_i q^{-i}$ and $\mathcal{B}(\vartheta; q^{-1}) = \sum_{i=1}^{m_s} b_i q^{-i+1}$ are polynomials in the unit-delay operator q^{-1}, and $\vartheta = [\, a_1\ a_2 \ldots a_{n_s}\ b_1\ b_2 \ldots b_{m_s}\,]^T$ is the model parameter vector. We denote by s_s the order of the model. We assume that

Assumption 1 *The admissible models parameter vector ϑ belongs to a compact set $\Theta \subset \Re^{n_s+m_s}$.*

Note that here we suppose that each model is linearly parameterized in ϑ, since it can in fact be rewritten as $y_{t+1} = \varphi_t^T \vartheta + w_{t+1}$, where $\varphi_t := [\, y_t\ y_{t-1}\ \ldots y_{t-n_s+1}\ u_t\ u_{t-1} \ldots u_{t-m_s+1}\,]^T$ is the regression vector. However, ϑ could be a nonlinear function of some parameter p without hampering any of the results presented in the paper. This is for example the case when ϑ is a continuous function of p belonging to a compact set. We decided to refer to the linear parameterization case only for ease of notation.

Candidate controllers set

For the sake of simplicity in the implementation of the controller and the supervisor, we assume that the candidate controllers set is finite. Specifically, the models parameter set Θ is partitioned into m compact sets Θ_k, $k \in \mathcal{K} := \{1, \ldots, m\}$, such that, for each $k \in \mathcal{K}$ the models in the set Θ_k are adequately controlled according to some cost criterion J by the controller

$$\mathcal{R}(k; q^{-1}) u_t = \mathcal{S}(k; q^{-1}) y_t, \tag{2}$$

of order s_c, where $\mathcal{R}(k; q^{-1}) = 1 - \sum_{i=1}^{m_c} r_i(k) q^{-i}$ and $\mathcal{S}(k; q^{-1}) = \sum_{i=0}^{n_c} s_i(k) q^{-i}$. This set is called a *finite controller cover* ([13, 14]) and has to satisfy the following stability condition.

Assumption 2 *For each $k \in \mathcal{K}$, the roots of the characteristic polynomial*

$$q^{s_s+s_c}\{\mathcal{A}(\vartheta; q^{-1})\mathcal{R}(k; q^{-1}) - \mathcal{B}(\vartheta; q^{-1})\mathcal{S}(k; q^{-1}) q^{-1}\}, \qquad \vartheta \in \Theta_k,$$

are within the open circle with radius $\lambda < 1$. We shall call λ the stability margin.

We then define $\Sigma : \Theta \to \mathcal{K}$ to be the map associating the parameter ϑ with the controller k which is optimal over the set Θ_k to which ϑ belongs (in the case when ϑ belongs to the frontier of two or more sets, some rule must be given to decide which controller to assign to ϑ).
Note that here we deal with a regulation problem. This is only for simplifying the presentation. All the results in the paper can be in fact extended to the case of a tracking problem with a deterministic and bounded reference signal.

Supervisor

We consider an *estimator-based* supervisor, which, based on the input signals u and y, generates the piecewise constant *switching signal* σ, whose value at any time t denotes the candidate controller that is placed in feedback with the system at that time t. The supervisor is composed of two blocks:

- a *monitoring signal generator*, which computes at each time instant $t \geq 0$ the least squares cost (LS): $V_t(\vartheta) = \frac{1}{t+1} \sum_{s=0}^{t} (y_s - \varphi_{s-1}^T \vartheta)^2 + \epsilon_J$, with $\epsilon_J > 0$;

- a *switching logic*, which decides at each time instant t whether or not a controller different from the one currently operating has to be switched in the loop, and which controller to switch to. This latter decision is based on the LS signal.
  Denote by $\{t_i\}$ the sequence of switching times and define
  $$\vartheta_t = \begin{cases} \hat{\vartheta}_t, & \text{if } t = t_i, \\ \vartheta_{t-1}, & otherwise, \end{cases} \tag{3}$$
  initialized with $\vartheta_{-1} = \bar{\vartheta} \in \Theta$, where $\hat{\vartheta}_t$ is the LS cost minimizer. Then, the switching signal σ_t, $t \geq 0$, can be expressed as
  $$\sigma_t = \Sigma(\vartheta_t). \tag{4}$$

As the amount of data collected from the system increases, the estimated system better describes the actual system behavior, at least in closed-loop. One can then stabilize the true system by stabilizing the time-varying estimated system. In order to do it, however, it is not sufficient to stabilize the "frozen" closed-loop estimated system with parameter ϑ_t at each time t. A possible solution to this issue is to update the parameter estimate at a slower rate than the updating of the system variables, so as to limit the estimated system time variability. By equation (3), this corresponds to slowing down the switching rate. We next describe how this is realized in the dwell time and hysteresis-based switching logics.

Dwell time switching logic. The switching rate is slowed down by making a dwell time elapse between two consecutive switching times. Specifically, the sequence of switching times is obtained by the recursive equation

$$t_{i+1} = t_i + \tau_D(\vartheta_{t_i}), \qquad\qquad i > 0 \tag{5}$$

initialized with $t_0 = 0$, with $\tau_D(\vartheta)$ denoting the *dwell time function* mapping the parameter ϑ in the corresponding dwell time. The dwell time function is defined next.

Consider the closed-loop system composed of the model with parameter ϑ and the controller with index k:

$$\begin{cases} \mathcal{A}(\vartheta; q^{-1})\, y_{t+1} = \mathcal{B}(\vartheta; q^{-1})\, u_t + w_{t+1}, \\ \mathcal{R}(k; q^{-1}) u_t = \mathcal{S}(k; q^{-1}) y_t. \end{cases} \tag{6}$$

By letting $x_t := [y_t \dots y_{t-n+1}\, u_{t-1} \dots u_{t-m+1}]^T$ where $n := \max\{n_s, n_c + 1\}$ and $m := \max\{m_s, m_c + 1\}$, the closed-loop system (6) can be given the state space representation

$$\begin{cases} x_{t+1} = A(\vartheta)\, x_t + B(\vartheta) u_t + C w_{t+1} \\ u_t = L(k) x_t, \end{cases}$$

where

$$A(\vartheta) = \left[\begin{array}{cccc|cccc} a_1 & \dots & a_{n-1} & a_n & b_2 & \dots & b_{m-1} & b_m \\ 1 & 0 & \dots & & 0 & \dots & & 0 \\ & \ddots & \ddots & & & \ddots & & 0 \\ & & 1 & 0 & & & & 0 \\ \hline 0 & \dots & \dots & 0 & 0 & \dots & \dots & 0 \\ 0 & \dots & \dots & 0 & 1 & 0 & & \\ & \ddots & \ddots & & & \ddots & \ddots & \\ & & 0 & 0 & & & 1 & 0 \end{array}\right], \quad B(\vartheta) = \left[\begin{array}{c} b_1 \\ 0 \\ \vdots \\ 0 \\ \hline 1 \\ 0 \\ \vdots \\ 0 \end{array}\right],$$

$$H = C^T = \left[\begin{array}{cccc|cccc} 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \end{array}\right],$$

$$L(k) = \left[\begin{array}{cccc|cccc} s_0(k) & \cdots & s_{n-2}(k) & s_{n-1}(k) & r_1(k) & \cdots & r_{m-2}(k) & r_{m-1}(k) \end{array}\right],$$

with $a_i = 0$ if $i > n_s$, $s_i(k) = 0$ if $i > n_c$, $b_i = 0$ if $i > m_s$, $r_i(k) = 0$ if $i > m_c$, thus leading to $x_{t+1} = F(\vartheta, k)\, x_t + C w_{t+1}$, where $F(\vartheta, k) = A(\vartheta) + B(\vartheta) L(k)$ is the closed-loop system dynamic matrix.
Note that the introduced state space representation of the model with parameter ϑ is nonminimal but, because of the block triangular matrix structure

of $A(\vartheta)$, the added eigenvalues are all identically equal to zero. This, jointly with the fact that the stability margin is λ, implies that if the model with parameter ϑ is controlled by the controller with parameter $k = \Sigma(\vartheta)$, then, its closed-loop dynamic matrix $F(\vartheta, \Sigma(\vartheta))$ satisfies

$$\max\{|\lambda_{\max}(F(\vartheta, \Sigma(\vartheta)))| : \vartheta \in \Theta\} < \lambda. \tag{7}$$

We are now in a position to define the dwell time switching function. Fix a positive constant $\mu < 1$. Then, $\tau_D : \Theta \to \Re$ is given by

$$\tau_D(\vartheta) := \inf\{\tau \in N : \|F(\vartheta, \Sigma(\vartheta))^\tau\| \leq \mu\}. \tag{8}$$

Hysteresis-based switching logic. The hysteresis-based switching logic continuously monitors the performance of the current controller σ_{t-1} and falsifies it as soon as data reveal that the model used to select it is significantly worse than the model whose parameter minimizes the LS signal. If and when the current controller is falsified, then, a switching occurs. In particular, the generic switching time t_{i+1} is defined as follows:

$$t_{i+1} = \min\left\{t > t_i : \ (1+h)V_t(\hat{\vartheta}_t) \leq V_t(\vartheta_{t-1})\right\}, \tag{9}$$

where $h > 0$ is the *hysteresis factor.*

In the next section, we analyze the introduced estimator-based switching control scheme and prove that stability is guaranteed when the true system is described by equation (1) with ϑ set equal to $\vartheta^\circ \in \Theta$ and the stochastic disturbance w_t satisfies the following assumption.

Assumption 3 *$\{w_t\}$ is a martingale difference sequence with respect to a filtration $\{\mathcal{F}_t\}$, satisfying the following conditions*

1. $\sup_t E[w_t^\beta / \mathcal{F}_{t-1}] < \infty$*, almost surely (a.s.), for some* $\beta > 2$*;*
2. $\lim_{t\to\infty} \frac{1}{t} \sum_{s=0}^{t-1} w_s^2 = \gamma^2 > 0$*, a.s.*

Note that Assumption 3 is e.g. satisfied when $\{w_t\}$ is an i.i.d. Gaussian sequence with zero mean and variance γ^2, but it includes many other situations. Also, differently from what is typically done in the switching control literature ([3, 7, 15, 5, 16]), w is not supposed to be bounded.

3 Stability analysis

The switched control system

$$\begin{cases} y_{t+1} = [1 - \mathcal{A}(\vartheta^\circ; q^{-1})]\, y_{t+1} + \mathcal{B}(\vartheta^\circ; q^{-1})\, u_t + w_{t+1} \\ u_t = \mathcal{S}(\sigma_t; q^{-1})\, y_t + [1 - \mathcal{R}(\sigma_t; q^{-1})]\, u_t, \end{cases}$$

with σ_t given by (4), can be represented as a variational system with respect to the closed-loop estimated system as follows:

$$\begin{cases} y_{t+1} = [1 - \mathcal{A}(\vartheta_t; q^{-1})] y_{t+1} + \mathcal{B}(\vartheta_t; q^{-1})\, u_t + w_{t+1} + e_t \\ u_t = \mathcal{S}(\Sigma(\vartheta_t); q^{-1})\, y_t + [1 - \mathcal{R}(\Sigma(\vartheta_t); q^{-1})]\, u_t, \end{cases} \tag{10}$$

where $e_t := \varphi_t^T[\vartheta^\circ - \vartheta_t]$ is the *perturbation term.*
The switched control system stability can then be proven based on the following two facts:

i) the closed-loop estimated system (10), where e_t is regarded as an exogenous input, is exponentially stable, uniformly in time;

ii) the internally generated perturbation term e_t is 'small'.

These two properties are proved next.
The stability result for the autonomous estimated system will be expressed in terms of the state space representation introduced in Section 2.

i) Uniform exponential stability

Theorem 1 *The autonomous estimated system $x_{t+1} = F(\vartheta_t, \sigma_t)x_t$, with ϑ_t and σ_t respectively given by (3) and (4), is a.s. exponentially stable, uniformly in time, i.e, there exists $\bar{\lambda} \in (\lambda, 1)$ and a (random) constant $\bar{k}$ such that, for all t^*, t, with $0 \le t^* \le t$,*

$$\|x_t\| \le \bar{k}\bar{\lambda}^{t-t^*}\|x_{t^*}\|, \qquad a.s. \tag{11}$$

Proof. We start by consider the time-invariant system

$$v_{t+1} = \frac{1}{\lambda} F(\vartheta, \Sigma(\vartheta))\, v_t. \tag{12}$$

By equation (7), we have that system (12) is exponentially stable $\forall\, \vartheta \in \Theta$. Moreover, $\mathcal{L}_t(\vartheta) := v_t^T P(\vartheta) v_t$, where $P(\vartheta)$ is the solution to the Lyapunov equation $\frac{1}{\lambda} F(\vartheta, \Sigma(\vartheta))^T\, P\, \frac{1}{\lambda} F(\vartheta, \Sigma(\vartheta)) - P = -I$, is a Lyapunov function for

system (12) satisfying $\mathcal{L}_{t+1}(\vartheta) - \mathcal{L}_t(\vartheta) = -\|v_t\|^2$, $\forall t$.
This implies that $\mathcal{L}_{t+1}(\vartheta) \leq \mathcal{L}_t(\vartheta)$, $\forall t$, from which it follows that

$$\|v_{\bar{t}+1}\| \leq \rho \, \|v_{t'}\|, \qquad t' \leq \bar{t}, \tag{13}$$

where we set $\rho := \sqrt{\sup_{\vartheta\in\Theta} \mathcal{K}_{\#}(P(\vartheta))}$, with $\mathcal{K}_{\#}(P)$ denoting the condition number with respect to the 2-norm of matrix P. It is important to note that ρ is bounded. In the case when Θ is finite, this property immediately follows from the fact that $\frac{1}{\lambda} F(\vartheta, \Sigma(\vartheta))$, $\vartheta \in \Theta$, is stable, hence $P(\vartheta)$ is positive define and $\mathcal{K}_{\#}(P(\vartheta)) < \infty$ for all $\vartheta \in \Theta$. In the case when $\Theta = \cup_{k=1}^{m} \Theta_k$, where Θ_k is a continuum of parameterizations and it is compact, one has to use also the property that, for every $k \in \mathcal{K}$, $F(\vartheta, \Sigma(\vartheta))$ is a continuous function of ϑ, $\vartheta \in \Theta_k$, and hence $P(\vartheta)$ (and $\mathcal{K}_{\#}(P(\vartheta))$) is also continuous on Θ_k (see [17]).

If we consider the time-invariant system $z_{t+1} = F(\vartheta, \Sigma(\vartheta))z_t$, setting $z_{t'} = v_{t'}$, by (13) we get the following bound on the state vector z_t:

$$\|z_{\bar{t}+1}\| \leq \rho\lambda^{\bar{t}+1-t'} \, \|z_{t'}\|, \qquad t' \leq \bar{t}'. \tag{14}$$

Consider now the time-varying autonomous estimated system

$$x_{t+1} = F(\vartheta_t, \sigma_t)x_t. \tag{15}$$

Denote by $\{t_i\}$ the switching time instant sequence, i.e., the time instants when σ_t (or equivalently ϑ_t) changes. Since during each time interval $[t_i, t_{i+1})$ system (15) is time invariant, by (14) we have

$$\|x_{\bar{t}+1}\| \leq \rho\lambda^{\bar{t}+1-t'} \, \|x_{t'}\|, \qquad t' \leq \bar{t}, \; t', \bar{t} \in [t_i, t_{i+1}), \tag{16}$$

In particular, for $t' = t_i$ and $\bar{t} = t_{i+1} - 1$,

$$\|x_{t_{i+1}}\| \leq \rho\lambda^{t_{i+1}-t_i}\|x_{t_i}\|. \tag{17}$$

Consider now the time interval $[t^*, t]$.
Let $\{t_{k_j}\}_{j=1}^{N_\sigma(t^*,t)}$, with $t^* < t_{k_1} < t_{k_2} < \ldots < t_{k_{N_\sigma(t^*,t)}} < t$, denote the $N_\sigma(t^*, t)$ consecutive switching times in (t^*, t). Suppose that $N_\sigma(t^*, t) > 0$ (if $N_\sigma(t^*, t) = 0$, then by (16) we have that (11) is satisfied for any $\bar{k} \geq \rho$).

Case 1: The dwell time switching logic.
By applying first (16) with $t' = t_{k_{N_\sigma(t^*,t)}}$ and $\bar{t} = t$, then recalling the fact that $\|x_{t_{k_{j+1}}}\| = \|F(\vartheta_{t_{k_j}}, \sigma_{t_{k_j}})^{\tau_D(\vartheta_{t_{k_j}})} x_{t_{k_j}}\| \leq \mu\|x_{t_{k_j}}\|$ (cf. equations (5) and (8)), and finally by applying (16) with $t' = t^*$ and $\bar{t} = t_{k_1}$, we obtain the following chain of inequalities

$$\begin{aligned}\|x_t\| &\leq \rho\lambda^{t-t_{k_{N_\sigma(t^*,t)}}}\|x_{t_{k_{N_\sigma(t^*,t)}}}\| \leq \rho\lambda^{t-t_{k_{N_\sigma(t^*,t)}}}\mu^{N_\sigma(t^*,t)-1}\|x_{t_{k_0}}\| \\ &\leq \rho^2\lambda^{t-t_{k_{N_\sigma(t^*,t)}}}\lambda^{t_{k_0}-t^*}\mu^{N_\sigma(t^*,t)-1}\|x_{t^*}\|.\end{aligned} \tag{18}$$

We next prove that the dwell time function $\tau_D(\vartheta)$, $\vartheta \in \Theta$, is uniformly bounded.
Set $\bar{\tau}_D = \inf\{\tau \in N : \rho\,\lambda^\tau \leq \mu\} < \infty$. Since from equation (14) with $\bar{t}+1-t' = \bar{\tau}_D$, it easily follows that $\|F(\vartheta,\Sigma(\vartheta))^{\bar{\tau}_D} z\| \leq \mu\|z\|$, $\forall\,\vartheta \in \Theta$, $\forall\, z$, then $\|F(\vartheta,\Sigma(\vartheta))^{\bar{\tau}_D}\| = \sup_{\|z\|\neq 0} \dfrac{\|F(\vartheta,\Sigma(\vartheta))^{\bar{\tau}_D}\, z\|}{\|z\|} \leq \mu$, $\forall\,\vartheta \in \Theta$. Therefore, the dwell time $\tau_D(\vartheta)$ defined in (8) satisfies

$$\sup_{\vartheta\in\Theta}\{\tau_D(\vartheta)\} \leq \bar{\tau_D} < \infty. \tag{19}$$

If we define $\bar{\lambda} := \max\{\lambda, \mu^{1/\bar{\tau}_D}\}$, by equation (18) we then have

$$\|x_t\| \leq \rho^2 \bar{\lambda}^{t-t_{k_{N_\sigma(t^*,t)}}} \bar{\lambda}^{t_{k_0}-t^*} \bar{\lambda}^{t_{k_{N_\sigma(t^*,t)}}-t_{k_0}} \|x_{t^*}\| = \rho^2 \bar{\lambda}^{t-t^*} \|x_{t^*}\|,$$

which is equation (11) with $\bar{k} = \rho^2$.

Case 2: The hysteresis-based switching logic.
By applying first (16) with $t' = t_{k_{N_\sigma(t^*,t)}}$ and $\bar{t} = t$, then, (17) repeatedly, and finally (16) with $t' = t^*$ and $\bar{t} = t_{k_1}$, we obtain the following inequality

$$\|x_t\| \leq \rho^{N_\sigma(t^*,t)+1}\ \lambda^{t-t^*} \|x_{t^*}\|. \tag{20}$$

By the scale-independent hysteresis switching theorem proven in [10, 9], we know that $N_\sigma(t^*,t)$ can be bounded as follows:

$$N_\sigma(t^*,t) \leq 1 + m + \frac{m}{\log(1+h)} \log \frac{\bar{V}_t(\vartheta)}{\min\{\bar{V}_{t^*}(\vartheta) :\ \vartheta \in \Theta\}}, \qquad \vartheta \in \Theta,$$

where $\bar{V}_t(\vartheta) := (t+1)V_t(\vartheta)$ is a rescaled version of the performance index V_t. The inequality above can be rewritten as follows:

$$\begin{aligned} N_\sigma(t^*,t) &\leq 1 + m + \frac{m}{\log(1+h)} \log\Big(\frac{1}{\epsilon_J}\frac{t+1}{t^*+1}\big[\frac{1}{t+1}\sum_{s=0}^{t} w_s^2 + \epsilon_J\big]\Big) \\ &= 1 + m + \frac{m}{\log(1+h)} \left(\log\big(\frac{t+1}{t^*+1}\big) + \log\big(\frac{1}{\epsilon_J}\big[\frac{1}{t+1}\sum_{s=0}^{t} w_s^2 + \epsilon_J\big]\big)\right). \end{aligned}$$

From Assumption 3, we have that with probability 1 there exists a random time instant $t' < \infty$ such that $\frac{1}{t+1}\sum_{s=0}^{t} w_s^2 \leq \gamma^2 + \epsilon_J$, $t \geq t'$. Therefore,

$$N_\sigma(t^*,t) \leq 1 + m + \frac{m}{\log(1+h)} \left(\log\big(\frac{t+1}{t^*+1}\big) + \log\big(\frac{1}{\epsilon_J}[\gamma^2 + 2\epsilon_J]\big)\right), \quad t \geq t'.$$

Now, observe that $\log\left(\frac{t+1}{t^*+1}\right) = o(t-t^*)$. This implies that, if we fix $\bar{\lambda} \in (\lambda, 1)$, there exists $\tau' > 0$ such that $\log\left(\frac{t+1}{t^*+1}\right) \leq \frac{\log(1+h)}{m}\big(\log_\rho(\bar{\lambda}) - \log_\rho(\lambda)\big)(t-t^*)$, $t-t^* \geq \tau'$, and therefore, $N_\sigma(t^*,t) \leq 1+m+\frac{m}{\log(1+h)}\log(\frac{\gamma^2}{\epsilon_J}+2)+\big(\log_\rho(\bar{\lambda}) - \log_\rho(\lambda)\big)(t-t^*)$, $t \geq t'$, $t-t^* \geq \tau'$. By replacing this bound in equation (20), we have that $\|x_t\| \leq \rho^{c_1+c_2(t-t^*)}\lambda^{t-t^*}\|x_{t^*}\|$, $t' \leq t^* \leq t,\ t-t^* \geq \tau'$, where $c_1 := 2+m+\frac{m}{\log(1+h)}\log(\frac{\gamma^2}{\epsilon_J}+2)$ and $c_2 := \log_\rho\left(\frac{\bar{\lambda}}{\lambda}\right)$. From this equation it then follows that

$$\|x_t\| \leq k\bar{\lambda}^{t-t^*}\|x_{t^*}\|, \qquad t' \leq t^* \leq t,\ t-t^* \geq \tau',$$

where $k := \rho^{c_1}$.

If $t-t^* < \tau'$ or $0 \leq t^* \leq t < t'$, then, $\|x_t\|$ can be bounded as follows

$$\|x_t\| \leq \big(\max_{\vartheta\in\Theta}\|F(\vartheta,\Sigma(\vartheta))\|\big)^{t-t^*}\|x_{t^*}\|. \tag{21}$$

Moreover, equation (21) still holds if $0 \leq t^* < t' < t$ and $t-t' < \tau'$. If $0 \leq t^* < t' < t$ and $t-t' \geq \tau'$, then $\|x_t\| \leq k\bar{\lambda}^{t-t'}\big(\max_{\vartheta\in\Theta}\|F(\vartheta,\Sigma(\vartheta))\|\big)^{t'-t^*}\|x_{t^*}\|$. Observe now that by (20) with $t^* = t-1$ and $x(t-1) = \bar{x}$, $\|x(t)\| = \|F(\vartheta,\Sigma(\vartheta))\,\bar{x}\| \leq \rho\,\lambda\|\bar{x}\|$, $\forall\bar{x},\ \vartheta \in \Theta$. Therefore, $\max_{\vartheta\in\Theta}\|F(\vartheta,\Sigma(\vartheta))\| \leq \rho\,\lambda$. Then, if we define the random constant $\bar{k} := \max\{k\rho^{t'}, \rho^{t'+\tau'}\}$, equation (11) is satisfied for all t^*, t such that $0 \leq t^* \leq t$.
This concludes the proof. □

ii) Bound on the perturbation term

Theorem 2 *Suppose that u_t is $\mathcal{F}_t$-measurable. If the hysteresis-based switching logic is used, then, ϑ_t given in (3) satisfies*

$$\sum_{s=0}^{t}(\varphi_{s-1}^T(\vartheta^\circ-\vartheta_t))^2 = o\Big(\sum_{s=0}^{t}\|\varphi_{s-1}\|^2\Big) + h\,O(t+1), \qquad a.s. \tag{22}$$

If the dwell time switching logic is used, then (22) holds with h set equal to 0, but only on the switching times sequence $\{t_i\}$.

Proof. In the hysteresis-based case, by (9) we have that, at each time t,

$$\begin{aligned}(t+1)V_t(\vartheta_t) &= \sum_{s=0}^{t}(\varphi_{s-1}^T(\vartheta^\circ-\vartheta_t)+w_s)^2 + (t+1)\epsilon_J \\ &\leq (1+h)(t+1)V_t(\vartheta^\circ) = (1+h)\Big(\sum_{s=0}^{t}w_s^2 + (t+1)\epsilon_J\Big). \end{aligned} \tag{23}$$

Then, by $\sum_{s=0}^{t} w_s^2 = O(t+1)$, a.s., (see point 2 in Assumption 3)

$$\sum_{s=0}^{t}(\varphi_{s-1}^T(\vartheta^\circ - \vartheta_t))^2 \leq h\,O(t+1) + 2\sum_{s=0}^{t}\varphi_{s-1}^T(\vartheta_t - \vartheta^\circ)w_s, \qquad a.s. \tag{24}$$

Observe that φ_{s-1} is $\mathcal{F}_{s-1}$-measurable whereas w_s satisfies Assumption 3. Then, by Theorem 2.8 in [18], $\sum_{s=0}^{t}\varphi_{s-1}^T(\vartheta_t - \vartheta^\circ)w_s = o\Big(\sum_{s=0}^{t}\|\varphi_{s-1}\|^2\Big)$, a.s. By plugging this bound in (24), equation (22) follows.
In the dwell time case, equation (22) holds only on $\{t_i\}$, since equation (23) is valid only at those ts which belong to $\{t_i\}$. □

The technical proof of the corollary below is obtained by a suitable manipulation of the sole result in Theorem 2, jointly with the uniform boundedness of ϑ_t. Its proof is basically the same in the two switching logic cases, here it is omitted due to space limitations (see [6] and [11]).

Corollary 1 *Suppose that u_t is $\mathcal{F}_t$-measurable. Then, the perturbation term $e_t = \varphi_t^T(\vartheta^\circ - \vartheta_t)$ satisfies the following equation*

$$\sum_{s=0,\ s\notin\mathcal{C}_t}^{t} e_s^2 = o\Big(\sum_{s=0}^{t}\|\varphi_s\|^2\Big) + h\,O(t+1), \qquad a.s., \tag{25}$$

with $h = 0$ in the dwell time logic case, where $\mathcal{C}_t$ is a set of instant points which depends on t, whose cardinality is bounded: $|\mathcal{C}_t| \leq K_{\mathcal{C}}, \forall t$.

Based on these results, stability of the switched control system can finally be proven (see [11] for a proof).

Theorem 3 *The switched closed-loop system is a.s. stable in the sense that*

$$\limsup_{t\to\infty}\frac{1}{t}\sum_{s=0}^{t-1}\left[y_s^2 + u_s^2\right] < \infty. \tag{26}$$

4 Performance analysis

Theorem 3 shows that both the dwell time and hysteresis-based switching logics ensure stability. However, the resulting switched control systems have a different performance in terms of short-term and long-term behavior.

The hysteresis-based switched control system is generally characterized by better transients than the dwell time switched control system. This is because in the hysteresis-based switching logic the control system behavior is continuously monitored, and a different controller is switched in the loop as soon as data reveal that the model used to select the currently operating controller is significantly worse than the model whose parameter minimizes the monitoring signal (cf. (9)). In the dwell time logic, instead, the controller switched in the loop at any switching time, say t_i, is kept in the loop for the whole dwell time interval $\tau(\vartheta_{t_i})$, irrespectively of the fact that the system behavior significantly deteriorates during such time interval (cf. (5)).

On the other hand, the dwell time switched control system is generally better performing in the long run, as explained next.
By combining the results in Theorem 3 and Corollary 1, one can easily prove that in the dwell time logic case $\frac{1}{t+1}\sum_{s=0}^{t} e_s^2 = o(1)$, a.s., i.e., the average square perturbation term asymptotically vanishes. This result can be interpreted as the dwell time switched system presenting a self-tuning property. This is not the case for the hysteresis-based switching logic, which allows for an average square perturbation term of the order of h times the noise variance to enter the system without causing any switching.

Note, however, that self-tuning does not imply self-optimality. Suppose, for example, that ϑ_t converges to some ϑ^*, which is indistinguishable from ϑ° in closed-loop when controller $\Sigma(\vartheta^*)$ is applied. Suppose also that $\Sigma(\vartheta^*) \neq \Sigma(\vartheta^\circ)$. Then, the switched control system is stabilized and self-tunes, but optimality is not achieved. In both the switching approaches, if ϑ_t converges to the true parameter vector ϑ°, and ϑ° is an interior point of $\Theta_{\Sigma(\vartheta^\circ)}$, then the optimal controller for the true system $\Sigma(\vartheta^\circ)$ is switched in the loop at some time, and then maintained indefinitely. Here, we show how the condition $\vartheta_t \to \vartheta^\circ$ can be ensured in the dwell time switching logic case by using the attenuating noise technique (cf. [18]).

Letting $\{d_t\}_{t\geq 0}$ be a sequence of i.i.d. random variables with continuous distribution, independent of $\{w_t\}$ and satisfying $E[d_t] = 0$, $E[d_t^2] = 1$, $|d_t| \leq c$, we introduce the asymptotically vanishing dither noise $\{n_t\}$ given by

$$n_t = \frac{d_t}{(t+1)^\zeta}, \quad \zeta \in (0, \frac{1}{4(s_s + n_s)}). \tag{27}$$

Without loss of generality, we may assume that $\{\mathcal{F}_t\}$ introduced in Assumption 3 is rich enough such that both w_t and n_t are $\mathcal{F}_t$-measurable. By the fact that $\{n_t\}$ satisfies $\frac{1}{t+1}\sum_{i=0}^{t} n_i^2 = 0$, and the independence between w_t and n_t, it can be shown that the stability result in Theorem 3 is still valid for the switched control system with dither noise:

$$\begin{cases} y_{t+1} = [1 - \mathcal{A}(\vartheta^\circ; q^{-1})]y_{t+1} + \mathcal{B}(\vartheta^\circ; q^{-1})\, u_t + w_{t+1} \\ u_t = \mathcal{S}(\sigma_t; q^{-1})\, y_t + [1 - \mathcal{R}(\sigma_t; q^{-1})] + n_t. \end{cases}$$

The proof of this fact follows exactly the same steps as that of the case when no dither noise is used, hence is omitted. Based on the stability property of the dwell time switching scheme with dither noise, we now prove the consistency of ϑ_t.

Theorem 4 *Suppose that the true system is controllable, i.e., $q^{s_s}\mathcal{A}(\vartheta^\circ; q^{-1})$ and $q^{s_s-1}\mathcal{B}(\vartheta^\circ; q^{-1})$ are coprime. Then, the parameter estimate ϑ_t computed in* (3) *based on the data collected from the dwell time switching scheme with dither noise is consistent, i.e., $\lim_{t\to\infty} \vartheta_t = \vartheta^\circ$, a.s.*

Proof. This theorem is proven by showing that $\hat{\vartheta}_t$ is a consistent estimate of ϑ°. The consistency of ϑ_t then easily follows from definition (3) and the fact that the dwell time function determining the switching times is uniformly bounded (see equation (19) in the proof of Theorem 1).
Given that the true system is controllable and the switched control system is stable, we can apply Theorem 6.2 in [18], thus getting $\lambda_{\min}(\sum_{s=0}^t \varphi_{s-1}\varphi_{s-1}^T) \geq \bar{c}\, t^{1-(s_s+n_s)2\zeta}$, $\forall t \geq \bar{t}$, a.s., where ζ is the constant introduced in (27).
As for the growing rate of $\lambda_{\max}(\sum_{s=0}^t \varphi_{s-1}\varphi_{s-1}^T)$, by the stability property $\lambda_{\max}(\sum_{s=0}^t \varphi_{s-1}\varphi_{s-1}^T) = O(\sum_{s=0}^t \|\varphi_{s-1}\|^2) = O(t)$, a.s. Since $\|\hat{\vartheta}_t - \vartheta^\circ\|^2 = O\left(\frac{\log(\lambda_{\max}(\sum_{s=0}^t \varphi_{s-1}\varphi_{s-1}^T))}{\lambda_{\min}(\sum_{s=0}^t \varphi_{s-1}\varphi_{s-1}^T)}\right)$ a.s. ([19]), this concludes the proof. □

5 Conclusions

The focus in the literature on switching control is generally on the stability issue, and mostly for deterministic systems subject to bounded noise. In this paper, we have studied switching control schemes for stochastic linear system affected by a possibly unbounded noise, and addressed for this class of systems not only stability, but also optimality issues.

References

[1] G. C. Goodwin, K. S. Sin, *Adaptive filtering prediction and control.* Englewood Cliffs, Prentice-Hall, 1984.

[2] R. H. Middleton, G. C. Goodwin, D. J. Hill, D. Q. Mayne, Design issues in adaptive control. IEEE Trans. on Automatic Control 33 (1988) 50–58.

[3] A. S. Morse, Supervisory control of families of linear set-point controllers—Part 1: Exact matching, IEEE Trans. on Automatic Control 41 (1996) 1413–1431.

[4] J. P. Hespanha, Logic-based switching algorithms in control. Ph.D. thesis, Dept. of Electrical Engineering, Yale University (1998).

[5] D. Borrelli, A. S. Morse, E. Mosca, Discrete-time supervisory control of families of 2-DOF linear set-point controllers. IEEE Trans. on Automatic Control 44 (1999) 178–181.

[6] M. Prandini, M. C. Campi, Logic-based switching for the stabilization of stochastic systems in presence of unmodeled dynamics. Proc. 40th Conf. on Decision and Control, 2001.

[7] A. S. Morse, Supervisory control of families of linear set-point controllers—Part 2: Robustness. IEEE Trans. on Automatic Control 42 (1997) 1500–1515.

[8] M. Prandini, M. C. Campi, Adaptive LQG control of input-output systems - A cost-biased approach. SIAM J. Control and Optim. 39 (5) (2001) 1499–1519.

[9] D. Liberzon, J. P. Hespanha, A. S. Morse, Hierarchical hysteresis switching. Proc. 39th Conf. on Decision and Control, 2000.

[10] J. P. Hespanha, D. Liberzon, A. S. Morse, Bounds on the number of switchings with scale-independent hysteresis: Applications to supervisory control. Proc. 39th Conf. on Decision and Control, 2000.

[11] M. Prandini, J. P. Hespanha, M. C. Campi, Hysteresis-based switching control of stochastic linear systems. Tech. rep., University of Brescia, Brescia, Italy (March 2002).

[12] A. S. Morse, Control using logic-based switching. in: A. Isidori (Ed.), *Trends in Control*, Springer-Verlag, 1995, pp. 69–113.

[13] B. Anderson, T. Brinsmead, F. de Bruyne, J. Hespanha, D. Liberzon, A. Morse, Multiple model adaptive control. I: Finite controller coverings. Int. J. of Robust and Nonlinear Control 10 (2000) 909–929.

[14] F. M. Pait, F. Kassab, On a class of switched, robustly stable, adaptive systems. Int. J. of Adapt. Control and Signal Proc. 15 (2001) 213–238.

[15] F. M. Pait, F. Kassab Jr, On a class of switched, robustly stable, adaptive systems. Proc. 38th Conf. on Decision and Control, 1999.

[16] J. P. Hespanha, D. Liberzon, A. S. Morse, B. D. O. Anderson, T. S. Brinsmead, F. de Bruyne, Multiple model adaptive control. II: Switching. Int. J. of Robust and Nonlinear Control 11 (2001) 479–496.

[17] D. F. Delchamps, Analytic feedback control and the algebraic Riccati equation. IEEE Trans. on Automatic Control 29 (1984) 1031–1033.

[18] H. F. Chen, L. Guo, *Identification and Stochastic Adaptive Control*, Birkhäuser, 1991.

[19] T. L. Lai, C. Z. Wei, Least squares estimates in stochastic regression models with applications to identification and control of dynamic systems. Ann. Statist. 10 (1982) 154–166.